JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1994), p. 1-9
<http://www.numdam.org/item?id=JEDP_1994____A8_0>

Divisor of the Selberg Zeta Function for Kleinian Groups

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June 23, 1994

Abstract

This note describes joint work with S. J. Patterson. Let Γ be a discrete group of isometries of hyperbolic $n+1$-dimensional space $\mathbf{H}^{n+1}$ having no elements of finite order, no parabolic elements, and infinite covolume, i.e, $\mathrm{vol}(\Gamma \setminus \mathbf{H}^{n+1})$ is infinite. The orbit space $\mathbf{H}^{n+1}$ is isometric to a non-compact Riemannian manifold M of infinite volume which is the union of a convex compact manifold with boundary, N, and finitely many ends diffeomorphic to cylinders. The manifold N has infinitely many closed geodesics whose length spectrum is described by the Selberg zeta function $Z_\Gamma(s)$. We show that scattering poles for the Laplace operator on M together with the Euler characteristic of M determine the zeros of $Z_\Gamma(s)$, and prove a version of the Selberg trace formula for this class of groups. A by-product of our analysis is a sharp bound on the distribution of scattering poles.

1 Main Results

The relationship between scattering poles and closed geodesics has attracted great interest in the case of obstacle scattering, where Gérard and Ikawa [7, 10, 11] has shown how closed geodesics (trapped rays) "generate" poles of the scattering operator. An important element in Ikawa's analysis is the dynamical zeta function associated to the geodesics. Here we would like to describe an analogous situation in geometric analysis where the scattering poles are among the zeros of an associated dynamical zeta function. The results to be described were obtained in joint work with S. J. Patterson.

Let Γ be a discrete group of isometries of hyperbolic $n+1$-dimensional space $\mathbf{H}^{n+1}$ so chosen that (i) Γ has no elements of finite order, (ii) Γ has no parabolic elements, and (iii) the orbit space $\Gamma \setminus \mathbf{H}^{n+1}$ has infinite hyperbolic volume. Under these assumptions, the orbit space $\Gamma \setminus \mathbf{H}^{n+1}$ is a Riemannian manifold M; the first assumption rules out conical singularities of M and the second assumption rules out cuspidal singularities of M. The third assumption guarantees that the Laplacian Δ on M will have absolutely continuous spectrum in $[n^2/4, \infty)$ together with finitely many discrete eigenvalues in $[0, n^2/4)$ (see Lax and Phillips [12, 13, 14]).

Thus the $L^2(M)$-resolvent operator $R(s) = (\Delta - s(n-s))^{-1}$ is meromorphic in the half-plane $\Re(s) > n/2$ with finitely many poles along the real s-axis corresponding to eigenvalues of Δ. The critical line $\Re(s) = n/2$ corresponds to the continuous spectrum of Δ. It is a deep result of scattering theory (see [1, 6, 15, 17, 22] and references therein) that the resolvent admits a meromorphic continuation to the complex plane whose poles are, morally at least, poles of the scattering operator for Δ. The 'spectral' data for Δ thus consist of eigenvalues and scattering poles.

Here we will define the scattering poles by reference to the *scattering operator* $S(s)$ for Δ, viewed as a map between incoming and outgoing generalized eigenfunctions for Δ. To define it, consider the upper half-space model $\mathbf{R}^n \times \mathbf{R}^+ \ni (x, y)$ of $\mathbf{H}^{n+1}$, so that $y = 0$ corresponds to the boundary at infinity of $\mathbf{H}^{n+1}$. The manifold M admits boundary charts isometric to a neighborhood of infinity in $\mathbf{H}^{n+1}$, so that the geometric boundary B of M corresponds to $y = 0$. A smooth solution of the eigenvalue equation $(\Delta - s(n-s))u = 0$ admits an asymptotic expansion of the form (see for example [2])

$$u(x,y) \sim \sum_{j\geq 0} a_j^+(x) y^{s+2j} + a_j^-(x) y^{n-s+2j}$$

near the boundary $y = 0$; the coefficients $a_j^\pm(x)$ are smooth functions uniquely determined by $a_0^-(x)$. The scattering operator $S(s)$ is the linear map $S(s)$: $a_0^- \mapsto a_0^+$. This definition is not invariant under coordinate changes but $S(s)$ is well-defined as a map between certain line bundles over the boundary B. It can be shown [1, 15, 17, 22] that

Theorem 1.1 *For $\Re(s) > n/2$, the operator $S(s)$ is a meromorphic family of elliptic pseudodifferential operators whose poles in $\Re(s) > n/2$ consist of trival poles at $s = n/2 + k$ and simple poles for $s(n-s)$ an eigenvalue of Δ. Moreover, $S(s)$ admits a meromorphic continuation the the complex plane whose poles are smoothing operators.*

Remarks. 1. The meromorphic continuation of $S(s)$ is proven by showing that $S(s)$ factors into $c(s)P(s)(I + T(s))P(s)$ where $P(s)$ is an entire family of pseudodifferential operators, $c(s)$ is a multiplicative factor, and $T(s)$ is a compact operator. One then shows that $(I + T(s))$ is invertible on the line $\Re(s) = n/2$ and applies the meromorphic Fredholm theorem. 2. The operator $T(s)$ is sufficiently regular that a renormalized determinant $\det(I + T(s))$ exists. The 'poles of the scattering operator' are defined as zeros of this regularized determinant.

The *Selberg zeta function* for Γ, denoted $Z_\Gamma(s)$, is an analytic function that encodes information about the distribution of lengths of closed geodesics on M, much as the Riemann zeta function encodes information about the distribution of prime numbers. If γ is a closed geodesic of M, associated to it are its length $\ell(\gamma)$ and an $SO(n)$ matrix $K(\gamma)$ that describes the rotation of nearby closed geodesics under the Poincaré once-return map. If $\{\gamma\}$ is a listing of the primitive closed geodesics of M, the formula

$$Z_\Gamma(s) = \exp\left(-\sum\nolimits_{\{\gamma\}} \sum\nolimits_{m=0}^{\infty} \frac{e^{-s\ell(\gamma^m)}}{k} \det\left(I - e^{-\ell(\gamma^m)} K(\gamma^m)\right)^{-1}\right) \qquad (1)$$

holds for $\Re(s) > n$, and defines an analytic function in this half-plane. Using results of Ruelle and Fried (see [5]), one can show:

Theorem 1.2 *The function* $Z_\Gamma(s) = F_\Gamma(s)/G_\Gamma(s)$ *where* F_Γ *and* G_Γ *are entire functions obeying the estimates*

$$\begin{aligned} |F_\Gamma(s)| &\leq c_1 \exp\left(c_2|s|^{n+1}\right) \\ |G_\Gamma(s)| &\leq c_1 \exp\left(c_2|s|^{n+1}\right) \end{aligned}$$

for positive constants c_1 *and* c_2.

The idea of the proof is to represent geodesic flow on M as a symbolic dyanmical system. This representation depends on the construction of a Markov partition for geodesic flow as in Bowen [4]. Although Bowen assumes that the ambient manifold is compact, this poses no essential difficulty (see the remarks in [21]). One can then express the Selberg zeta function in terms of Fredholm determinants of transfer operators defined using the symbolic dynamics as in Ruelle [25]; a detailed argument, using [16] and [25], is given in Fried's paper [5]. Estimates on the resulting determinants give the dimension-dependent estimates in Theorem 1.2; These estimates show that F_Γ and G_Γ are entire functions of order at most $n+1$ and finite type (see Boas [3]).

It is now natural to consider the spectral and topological data that determine these zeros. We have proved [21]:

Theorem 1.3 *Suppose that* $n+1 = 2m$ *is even. Then* $Z_\Gamma(s)$ *is an entire function with the following zeros:*

(i) The finite set of real numbers s *with* $s > n/2$ *and* $s(n-s)$ *an eigenvalue of the Laplacian on* M.

(ii) The set of complex numbers s *such that* s *is a pole of the scattering operator for* Δ.

(iii) The set $s = -k$, $k = 0, 1, 2, \cdots$, *where zeros occur with multiplicity*

$$(-2)\frac{2m+2k+1}{2m-1}\begin{pmatrix} 2m+k-2 \\ k \end{pmatrix}\chi(N)$$

where $\chi(M)$ *is the Euler characteristic of* M.

(iv) The point $s = n/2$.

Remarks. 1. The multiplicity of type (i) zeros is the multiplicity of the corresponding eigenvalue. 2. The scattering poles all have $\Re(s) < n/2$, and their multiplicities are determined by the multiplicities of zeros of a renormalized Fredholm determinant. 3. It is not known whether scattering poles also occur at the points $s = -k$, $k = 0, 1, 2, \cdots$. In this case the multiplicities of zeros of types (ii) and (iii) add. 4. The Euler characteristic of M is defined by viewing M as a manifold with boundary. 5. The expression multiplying the Euler characteristic in (iii) is the dimension of the space of spherical harmonics of degree k on S^{2m+1}, and is therefore integral. 6. We cannot at present compute the multiplicity of the zero at $s = n/2$. It may be zero!

The proof relies on Patterson's [19] observation that for $\Re(s)$ large and positive, the Selberg zeta function $Z_\Gamma(s)$ obeys the identity

$$\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} = (2s-n)\int_{\mathcal{F}} G(x,x;s) - G_0(x,x;s)\,\mathrm{dvol}$$

where $\mathcal{F}$ is a fundamental domain for Γ in $\mathbf{H}^{n+1}$, $G(x,x;s)$ is the integral kernel of the operator $(\Delta - s(n-s))^{-1}$, and $G_0(x,x;s)$ is the integral kernel for the corresponding operator on $\mathbf{H}^{n+1}$. Using this identity one can derive a functional equation for $Z'_\Gamma(s)/Z_\Gamma(s)$ of the form

$$\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} + \frac{Z'_\Gamma(n-s)}{Z_\Gamma(n-s)} = G_1(s) + G_2(s) \tag{2}$$

where

$$\begin{aligned} G_1(s) &= \int_{B\times B} k(b,b';n-s)\partial_s k(b,b';s) \\ G_2(s) &= -4\pi\frac{(-1)^m}{(2m-1)!}\frac{\Gamma(s)\Gamma(n-s)}{\Gamma(s-n/2)\Gamma(n/2-s)}\chi(M) \end{aligned}$$

where $\chi(M)$ is the Euler characteristic of M. Here B is the boundary at infinity of the manifold M and $k(\cdot,\cdot;s)$ is the integral kernel of the scattering operator for Δ on M. It is shown that the poles of $G_1(s)$ are eigenvalues and scattering poles of Δ while those of $G_2(s)$ can be explicitly calculated. Using this result and the known analyticity properties of $Z_\Gamma(s)$ when $\Re(s) > n/2$, we obtain Theorem 1.3.

Together with Theorem 1.2 and Jensen's inequality, this implies a sharp bound on the distribution function for scattering poles:

Theorem 1.4 *Let*

$$N(r) = \#\{s : \Re(s) < n/2, s \text{ is a pole of the scattering operator for } \Delta\}.$$

Then

$$N(r) \leq Cr^{n+1}.$$

Remarks. 1. This improves an earlier bound of Perry [24]. 2. Explicit examples (see [8] or [24]) show that this bound is optimal. 3. Recent work of Guillopé and Zworski [8, 9] proves polynomial bounds on the distribution of resolvent resonances for asymptotically hyperbolic manifolds.

The Euler product representation (1) together with Theorem 1.3 and a short calculation give the following Selberg-type trace formula (compare [26]):

Theorem 1.5 *Let $h \in C_0^\infty(0,\infty)$ and let*

$$\hat{h}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)e^{itx}\, dx.$$

Then

$$2\pi\sum_{\{\gamma\}}\sum_{m=1}^{\infty}\ell(\gamma)\frac{e^{mn\ell(\gamma)/2}h(m\ell(\gamma))}{\det\left(I - K(\gamma)^m e^{-m\ell(\gamma)}\right)} =$$

$$-2\pi i N\hat{h}(0) - \int c(t)\hat{h}(t)\, dt - \sum_\sigma 2\pi i\hat{h}(r_\sigma) - \sum_\rho 2\pi i\hat{h}(r_\rho).$$

Here the sum over γ runs over primitve closed geodesics, N the multiplicity of the zero of $Z(s)$ at $s = n/2$ and $c(t)$ is an explicit logarithmic derivative of Gamma functions.

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