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Error estimate in the generalized Szegö theorem


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1. Let $A$ be a positive selfadjoint elliptic pseudodifferential operator of order 1 on a smooth compact manifold $M$ without boundary, $\dim M = n \geq 2$. The spectrum of the operator $A$ consists of infinite number of eigenvalues $\lambda_k \to +\infty$, $k \to \infty$. By $N(\lambda)$ we denote a counting function of the spectrum of operator $A$,

$$N(\lambda) = \# \{ k : \lambda_k < \lambda \}$$

(we take into account the multiplicity of the eigenvalues). Let $\Pi_\lambda$ be the spectral projectors of operator $A$ corresponding to the intervals $(0, \lambda)$. We consider a family of operators

$$B_\lambda = \Pi_\lambda B \Pi_\lambda,$$

where $B$ is a selfadjoint pseudodifferential operator of order zero. The rank of the operator $B_\lambda$ is finite, so it has a finite number of eigenvalues $\mu_i(\lambda)$ lying in the interval

$$K = [-\|B\|, \|B\|] \subset \mathbb{R}^1.$$

The number of these eigenvalues is infinitely increasing when $\lambda \to +\infty$.

Let $\rho_\lambda$ be a measure on $K$ which is equal to the sum of the Dirac measures at the points $\mu_i(\lambda)$, i.e.

$$\rho_\lambda(f) = \sum_j f(\mu_j(\lambda)) = \text{Tr} f(\Pi_\lambda B \Pi_\lambda)$$

for any function $f \in C(K)$. We shall study the asymptotic behaviour of $\rho_\lambda$ when $\lambda \to +\infty$.

It is well known [1, theorem 29.1.7] that the measures $\lambda^{-n} \rho_\lambda$ converge weakly to the measure $\rho_0$ which is defined by the following formula...
\[ \rho_0(f) = (2\pi)^{-n} \int_{a_0(x,\xi) < 1} f(b_0(x,\xi)) \, dx \, d\xi, \]

where \( f \in C(K) \) and \( a_0, b_0 \) are the principal symbols of the operators \( A \) and \( B \). By other words, for any \( f \in C(K) \)

\[ \rho_\lambda(f) = \rho_0(f) \lambda^n + o(\lambda^n). \quad (1) \]

This result is considered as a generalization of the classical Szegö theorem [2] on the contraction of a multiplication operator to the space of trigonometrical polynomials. It dues to Guillemin [3].

We prove that for sufficiently smooth function \( f \) the remainder in (1) is \( 0(\lambda^{-n-1}) \). Our main results are the following theorems.

**Theorem 1.** There exist an integer \( r \) and a positive constant \( C \) such that for any function \( f \in C^r(K) \) the following inequality holds

\[ |\varphi_\lambda(f) - \rho_0(f) \lambda^n| \leq C(\lambda^{n-1} + 1) \|f\|_{C^r(K)}. \quad (2) \]

**Theorem 2.** If \( B \) is a multiplication by sufficiently smooth function \( b_0(x) \) then the estimate (2) is valid for \( r = 2 \).

2. Let \( \varphi_j(x) \) be eigenfunctions of the operator \( A \) corresponding to the eigenvalues \( \lambda_j \) and \( (\varphi_j, \varphi_k) = \delta_{jk} \). The proof of the generalized Szegö theorem is based on the following well known result (see [1, §29.1]).

**Theorem 3.** For any pseudodifferential operator \( H \) of order zero

\[ \sum_{\lambda_j < \lambda} \frac{\varphi_j(x)}{H \varphi_j(x)} = \]

\[ = (2\pi)^{-n} \int_{a_0(x,\xi) < 1} h_0(x,\xi) \, d\xi \lambda^n + o(\lambda^{n-1}) \]

uniformly with respect to \( x \in M \), where \( h_0 \) is the principal symbol of the operator \( H \).
The theorem 3 (with \( H = I \)) immediately implies that for any bounded function \( h(x) \) and corresponding multiplication operator \((h)\)

\[
|\text{Tr} \Pi_{\lambda}(h) \Pi_{\lambda} - (2\pi)^{-n} \int_{a_0(x,\xi) < 1} h(x) \, dx \, d\xi \lambda^n| \lesssim \\
\lesssim C \lambda^{n-1} \sup_x |h(x)|,
\]

where \( \lambda \geq 1 \), and the constant \( C \) does not depend on \( h \). In particular,

\[
N(\lambda) = (2\pi)^{-n} \int_{a < 1} dx d\xi \lambda^n + o(\lambda^{n-1}).
\]

If \( f \) is a smooth function then \( f(B) \) is a pseudodifferential operator and its principal symbol is \( f(b_0) \). Therefore according to the theorem 3, for \( f \in C^\infty(K) \) we have

\[
\text{Tr} \Pi_{\lambda} f(B) \Pi_{\lambda} = \rho_0(f) \lambda^n + o(\lambda^{n-1}),
\]

where remainder somehow depends on \( f \). It is easy to see from the proof of the theorem 3 [1, §29.1] that this remainder term is estimated for \( \lambda \geq 1 \) by

\[
C \lambda^{n-1}\|f\|_{C^r(K)}
\]

where the constant \( C \) and the integer \( r \) are independent of \( f \).

**Remark 4.** We suppose that this estimate holds for \( r = 2 \). If it is true then the theorem 1 is valid for \( r = 2 \) as well.

3. Now we shall prove the following abstract theorem.

**Theorem 5.** Let \( A \) be a positive selfadjoint operator and \( B \) be a bounded selfadjoint operator in a Hilbert space. Suppose that spectrum of the operator \( A \) consists of eigenvalues, and let \( \Pi_{\lambda} \) be the spectral projectors corresponding to the
intervals $([0,\lambda])$, $N(\lambda)$ be the counting eigenvalues function, and 
$N^O(\lambda) = \sup_{\mu \in \lambda} [N(\mu) - N(\mu - \varepsilon)]$.

Assume that the commutator $B = [A, B]$ is a bounded operator. Then for any $\varepsilon > 0$ and for any function $f \in C^2(K)$ the following inequality holds

$$|\text{Tr} \Pi_\lambda f(B)\Pi_\lambda - \text{Tr} f(\Pi_\lambda B\Pi_\lambda)|$$

$$\leq (2\|B\|^2 + C_\varepsilon \|\tilde{B}\|^2) N^O(\lambda) \max_{K} |f''|,$$

where $K = [-\|B\|, \|B\|]$, and the constant $C_\varepsilon$ depends on $\varepsilon$ only.

On account of (3) the theorems 1 and 2 follow from the results mentioned in the section 2.

We deduce (3) from the following well known Berezin's inequality.

**Theorem 6.** Let $B$ be a bounded self adjoint operator in a Hilbert space, $K = [-\|B\|, \|B\|]$, and $\Pi$ be a selfadjoint projector, rank $\Pi < \infty$. Then for any convex function $\varphi \in C(K)$

$$\text{Tr} \Pi \varphi(B)\Pi \geq \text{Tr} \varphi(\Pi B\Pi).$$

**Corollary 7.** Let $\varphi \in C^2(K)$ is a strictly convex function. Then for any $f \in C^2(K)$

$$|\text{Tr} \Pi f(B)\Pi - \text{Tr} f(\Pi B\Pi)| \leq$$

$$\leq \left( \max_{\varphi''} \left| \frac{f''}{\varphi''} \right| \right) (\text{Tr} \varphi(B)\Pi - \text{Tr} \varphi(\Pi B\Pi)).$$

In particular (if $\varphi(t) = t^2$),

$$|\text{Tr} \Pi f(B)\Pi - \text{Tr} f(\Pi B\Pi)| \leq \frac{1}{2} (\max_{k} |f''|) \| (I - \Pi) B\Pi \|^2_2,$$

where $\| \cdot \|_2$ is the Hilbert–Schmidt norm.
Proof. Applying the Berezin's inequality to the convex functions

\[ \psi_\pm = \left( \max_k \left| \frac{f(x)}{\varphi_n(x)} \right| \right) \varphi \pm f \]

we obtain exactly (4).

In view of (5), to prove the theorem 5 it is sufficient to estimate

\[ \|(1 - \Pi_\lambda)B\Pi_\lambda\|_2 \]

by \( 2\|B\|^2 + C_\epsilon \|B\|^2 \) \( N_\epsilon(\lambda) \). Note that

\[ \|(1 - \Pi_\lambda)B\Pi_\lambda\|_2 \leq 2 \left( \|(1 - \Pi_\lambda)B\Pi_\lambda - \epsilon\|_2 + \|(1 - \Pi_\lambda)B(\Pi_\lambda - \Pi_\lambda - \epsilon)\|_2 \right) \]

and \( \|(1 - \Pi_\lambda, B(\Pi_\lambda - \Pi_\lambda - \epsilon))\|_2 \leq \|B\|^2 N_\epsilon(\lambda) \). So it remains to estimate

\[ \|(1 - \Pi_\lambda)B\Pi_\lambda - \epsilon\|_2 \]

only. According to the definition

\[ \|(1 - \Pi_\lambda)B\Pi_\lambda - \epsilon\|_2 = \sum_{\lambda_k \geq \lambda} \sum_{\lambda_j < \lambda - \epsilon} \|B\varphi_j, \varphi_k\|^2 \]

where \( \varphi_j \) are the eigenfunctions of the operator corresponding to the eigenvalues \( \lambda_j \).

Since \( (B\varphi_j, \varphi_k) = (\lambda_k - \lambda_j)^{-1} (\tilde{B}\varphi_j, \varphi_k) \), we obtain that

\[ \|(1 - \Pi_\lambda)B\Pi_\lambda - \epsilon\|_2 = \sum_{\lambda_k \geq \lambda} \sum_{\lambda_j < \lambda - \epsilon} \|B\varphi_j, \varphi_k\|^2 \]

\[ \leq \frac{1}{2} \sum_{\lambda_k \geq \lambda} \sum_{\lambda_j < \lambda - \epsilon} (\lambda - \lambda_j)^{-2} \|(\tilde{B}\varphi_j, \varphi_k)\|^2 \leq \]

\[ \leq \|\tilde{B}\|^2 \sum_{\lambda_j < \lambda - \epsilon} (\lambda - \lambda_j)^{-2} = \|\tilde{B}\|^2 \int_0^{\lambda - \epsilon} (\lambda - \mu)^{-2} dN(\mu) \]

\[ \leq \|\tilde{B}\|^2 \sum_{\lambda_j < \lambda - \epsilon} \|B\|^2 N_{\epsilon/2}(\lambda) \sum_{k = 0}^{k_0} (\lambda - k\epsilon/2)^{-2} \]

where \( (\lambda - \epsilon/2) > k^* \epsilon/2 > (\lambda - \epsilon) \). The sum in the right hand side is estimated by some constant \( C_\epsilon \) not depending on \( \lambda \). Therefore

\[ \|(1 - \Pi_\lambda)B\Pi_\lambda - \epsilon\|_2 \leq C_\epsilon \|B\|^2 N_{\epsilon/2}(\lambda) \]

It completes the proof of the theorem 5 and of the theorems 1 and 2.

Remark 8. Under some additional assumptions one can obtain a two-term asymptotic formula for \( \text{Tr} \Pi_\lambda f(B) \Pi_\lambda \). However, even under these assumptions the difference
\[ \text{Tr } \Pi_s f (B) \Pi_h - \text{Tr } f (\Pi_s B \Pi_h) \]
can really have the order 0 \((\lambda^{n-1})\). So the second term in (1) (if it exists) can be
different one.

**Remark 9.** The theorem 5 can be applied in various different problems as
well. For example, it allows to improve some results from [4].
References


