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Error estimate in the generalized Szegö theorem


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1. Let \( A \) be a positive selfadjoint elliptic pseudodifferential operator of order 1 on a smooth compact manifold \( M \) without boundary, \( \dim M = n \geq 2 \). The spectrum of the operator \( A \) consists of infinite number of eigenvalues \( \lambda_k \to +\infty, k \to \infty \). By \( N(\lambda) \) we denote a counting function of the spectrum of operator \( A \),

\[
N(\lambda) = \# \{ k : \lambda_k < \lambda \}
\]

(we take into account the multiplicity of the eigenvalues). Let \( \Pi_\lambda \) be the spectral projectors of operator \( A \) corresponding to the intervals \( (0, \lambda) \). We consider a family of operators

\[
B_\lambda = \Pi_\lambda B \Pi_\lambda,
\]

where \( B \) is a selfadjoint pseudodifferential operator of order zero. The rank of the operator \( B_\lambda \) is finite, so it has a finite number of eigenvalues \( \mu_j(\lambda) \) lying in the interval

\[
K = [-\|B\|, \|B\|] \subset \mathbb{R}.
\]

The number of these eigenvalues is infinitely increasing when \( \lambda \to +\infty \).

Let \( \rho_\lambda \) be a measure on \( K \) which is equal to the sum of the Dirac measures at the points \( \mu_j(\lambda) \), i.e.

\[
\rho_\lambda(f) = \sum_j f(\mu_j(\lambda)) = \text{Tr } f (\Pi_\lambda B \Pi_\lambda)
\]

for any function \( f \in \mathcal{C}(K) \). We shall study the asymptotic behaviour of \( \rho_\lambda \) when \( \lambda \to +\infty \).

It is well known [1, theorem 29.1.7] that the measures \( \lambda^{-n} \rho_\lambda \) converge weakly to the measure \( \rho_0 \) which is defined by the following formula.
\[ \rho_0(f) = (2\pi)^{-n} \int_{a_0(x,\xi) < 1} f(b_0(x,\xi)) \, dx \, d\xi, \]

where \( f \in C(K) \) and \( a_0, b_0 \) are the principal symbols of the operators \( A \) and \( B \). By other words, for any \( f \in C(K) \)
\[ \rho_\lambda(f) = \rho_0(f) \lambda^n + o(\lambda^n). \quad (1) \]
This result is considered as a generalization of the classical Szegő theorem [2] on the contraction of a multiplication operator to the space of trigonometrical polynomials. It due to Guillemin [3].

We prove that for sufficiently smooth function \( f \) the remainder in (1) is \( o(\lambda^{n-1}) \). Our main results are the following theorems.

**Theorem 1.** There exist an integer \( r \) and a positive constant \( C \) such that for any function \( f \in C^r(K) \) the following inequality holds
\[ |\rho_\lambda(f) - \rho_0(f) \lambda^n| \leq C(\lambda^{n-1}+1) \|f\|_{C^r(K)}. \quad (2) \]

**Theorem 2.** If \( B \) is a multiplication by sufficiently smooth function \( b_0(x) \) then the estimate (2) is valid for \( r = 2 \).

2. Let \( \varphi_j(x) \) be eigenfunctions of the operator \( A \) corresponding to the eigenvalues \( \lambda_j \), and \( \langle \varphi_j, \varphi_k \rangle = \delta_j^k \). The proof of the generalized Szegő theorem is based on the following well known result (see [1, §29.1]).

**Theorem 3.** For any pseudodifferential operator \( H \) of order zero
\[ \sum_{\lambda_j < \lambda} \frac{\varphi_j(x)}{H \varphi_j(x)} = \]
\[ = (2\pi)^{-n} \int_{a_0(x,\xi) < 1} h_0(x,\xi) \, d\xi \, \lambda^n + o(\lambda^{n-1}) \]
uniformly with respect to \( x \in M \), where \( h_0 \) is the principal symbol of the operator \( H \).
The theorem 3 (with \( H = I \)) immediately implies that for any bounded function \( h(x) \) and corresponding multiplication operator \((h)\)

\[
|\text{Tr} \, \Pi_\lambda \, (h) \, \Pi_\lambda - (2\pi)^{-n} \int_{a_0(x, \xi) < 1} h(x) \, dx \, d\xi \, \lambda^{-n}| \leq C \lambda^{-n-1} \sup_x |h(x)|,
\]

where \( \lambda \geq 1 \), and the constant \( C \) does not depend on \( h \). In particular,

\[
N(\lambda) = (2\pi)^{-n} \int_{a < 1} dx \, d\xi \, \lambda^n + O(\lambda^{n-1}).
\]

If \( f \) is a smooth function then \( f(B) \) is a pseudodifferential operator and its principal symbol is \( f(b_0) \). Therefore according to the theorem 3, for \( f \in C^\infty_0(K) \) we have

\[
\text{Tr} \, \Pi_\lambda \, f(B) \, \Pi_\lambda = \rho_0(f) \lambda^n + O(\lambda^{n-1}),
\]

where remainder somehow depends on \( f \). It is easy to see from the proof of the theorem 3 [1, §29.1] that this remainder term is estimated for \( \lambda \geq 1 \) by

\[
C \lambda^{n-1} \|f\|_{C^r(K)}
\]

where the constant \( C \) and the integer \( r \) are independent of \( f \).

**Remark 4.** We suppose that this estimate holds for \( r = 2 \). If it is true then the theorem 1 is valid for \( r = 2 \) as well.

3. Now we shall prove the following abstract theorem.

**Theorem 5.** Let \( A \) be a positive selfadjoint operator and \( B \) be a bounded selfadjoint operator in a Hilbert space. Suppose that spectrum of the operator \( A \) consists of eigenvalues, and let \( \Pi_\lambda \) be the spectral projectors corresponding to the
intervals \( ([0,\lambda]) \), \( N(\lambda) \) be the counting eigenvalues function, and
\[
N_\varepsilon(\lambda) = \sup_{\mu \in \lambda} [N(\mu) - N(\mu - \varepsilon)].
\]

Assume that the commutator \( \tilde{B} = [A, B] \) is a bounded operator. Then for any \( \varepsilon > 0 \) and for any function \( f \in C^2(K) \) the following inequality holds
\[
|\text{Tr} \Pi f(B) \Pi - \text{Tr} f(\Pi B \Pi)|
\]
\[
\leq (2\|B\|^2 + C_\varepsilon \|B\|^2) N_\varepsilon(\lambda) \max_K |f''|.
\]
where \( K = [-\|B\|, \|B\|] \), and the constant \( C_\varepsilon \) depends on \( \varepsilon \) only.

On account of (3) the theorems 1 and 2 follow from the results mentioned in the section 2.

We deduce (3) from the following well known Berezin's inequality.

**Theorem 6.** Let \( B \) be a bounded self adjoint operator in a Hilbert space, \( K = [-\|B\|, \|B\|] \), and \( \Pi \) be a selfadjoint projector, rank \( \Pi < \infty \). Then for any convex function \( \phi \in C(K) \)
\[
\text{Tr} \Pi \phi(B) \Pi \geq \text{Tr} \phi(\Pi B \Pi).
\]

**Corollary 7.** Let \( \phi \in C^2(K) \) is a strictly convex function. Then for any \( f \in C^2(K) \)
\[
|\text{Tr} \Pi f(B) \Pi - \text{Tr} f(\Pi B \Pi)| \leq \left( \max_{\phi''} \left| f'' \right| \right) (\text{Tr} \phi(B) \Pi - \text{Tr} \phi(\Pi B \Pi)).
\]
In particular (if \( \phi(t) = t^2 \)),
\[
|\text{Tr} \Pi f(B) \Pi - \text{Tr} f(\Pi B \Pi)| \leq \frac{1}{2} \left( \max_k |f''(k)| \right) \| (I - \Pi) B \Pi \|^2,
\]
where \( \| \cdot \|_2 \) is the Hilbert–Schmidt norm.
Proof. Applying the Berezin’s inequality to the convex functions

\[ \psi_\pm = \left( \max_k \frac{|f^n|}{|\varphi^n|} \right) \varphi \pm f \]

we obtain exactly (4).

In view of (5), to prove the theorem 5 it is sufficient to estimate

\[ \|(1-\Pi_\lambda)B\Pi_\lambda\|_2^2 \] 

by \((2\|B\|_2^2 + C_\varepsilon \|B\|_2^2) N_\varepsilon(\lambda) \). Note that

\[ \|(1-\Pi_\lambda)B\Pi_\lambda\|_2^2 \leq 2 \left( \|(1-\Pi_\lambda)B\Pi_{\lambda-\varepsilon}\|_2^2 + \|(1-\Pi_\lambda)B(\Pi_{\lambda-\varepsilon} - \Pi_{\lambda})\|_2^2 \right), \]

and \( \|(1-\Pi_\lambda, B(\Pi_{\lambda-\varepsilon} - \Pi_{\lambda}))\|_2^2 \leq \|B\|^2 N_\varepsilon(\lambda) \). So it remains to estimate

\[ \|(1-\Pi_\lambda)B\Pi_{\lambda-\varepsilon}\|_2^2 \]

only. According to the definition

\[ \|(1-\Pi_\lambda)B\Pi_{\lambda-\varepsilon}\|_2^2 = \sum_{\lambda_k \geq \lambda} \sum_{\lambda_j < \lambda-\varepsilon} |(B\varphi_j, \varphi_k)|^2, \]

where \( \varphi_j \) are the eigenfunctions of the operator corresponding to the eigenvalues \( \lambda_j \).

Since \( (B\varphi_j, \varphi_k) = (\lambda_k-\lambda_j)^{-1} (B\varphi_j, \varphi_k) \), we obtain that

\[ \|(1-\Pi_\lambda)B\Pi_{\lambda-\varepsilon}\|_2^2 = \sum_{\lambda_k \geq \lambda} \sum_{\lambda_j < \lambda-\varepsilon} |(B\varphi_j, \varphi_k)|^2, \]

\[ \leq \sum_k \sum_{\lambda_j < \lambda-\varepsilon} (\lambda-\lambda_j)^{-2} |(\tilde{B}\varphi_j, \varphi_k)|^2 \leq \]

\[ \leq \|\tilde{B}\|^2 \sum_{\lambda_j < \lambda-\varepsilon} (\lambda-\lambda_j)^{-2} = \|B\|^2 \int_0^{\lambda-\varepsilon} (\lambda-\mu)^{-2} dN(\mu) \]

\[ \leq \|\tilde{B}\|^2 N_{\varepsilon/2}(\lambda) \sum_{k=0}^{k^*} (\lambda-k\varepsilon/2)^{-2} \]

where \( \lambda-\varepsilon \geq k^* \varepsilon/2 > (\lambda-\varepsilon) \). The sum in the right hand side is estimated by some constant \( C_\varepsilon \) not depending on \( \lambda \). Therefore

\[ \|(1-\Pi_\lambda)B\Pi_{\lambda-\varepsilon}\|_2^2 \leq C_\varepsilon \|B\|^2 N_{\varepsilon/2}(\lambda). \]

It completes the proof of the theorem 5 and of the theorems 1 and 2.

Remark 8. Under some additional assumptions one can obtain a two-term asymptotic formula for \( \text{Tr} \Pi_\lambda f(B) \Pi_\lambda \). However, even under these assumptions the difference
Tr \Pi f (B) \Pi - Tr f (\Pi B \Pi) 

can really have the order 0 (\lambda^{n-1}). So the second term in (1) (if it exists) can be different one.

**Remark 9.** The theorem 5 can be applied in various different problems as well. For example, it allows to improve some results from [4].
References


4. D. Robert, Remarks on a paper of S. Zelditch: