Waichiro Matsumoto
Hideo Yamahara

Necessary conditions for strong hyperbolicity of first order systems


<http://www.numdam.org/item?id=JEDP_1989____A8_0>
Necessary conditions for strong hyperbolicity of first order systems

by

Waichiro MATSUMOTO and Hideo YAMAHARA

§0. Introduction, definitions and theorems.

On higher order scalar equations, the strong hyperbolicity is well characterized. (See O. A. Oleinik [13], V. Ja. Ivrii and V. M. Petkov [3], V. Ja. Ivrii [2], L. Hörmander [1], N. Iwasaki [4], [5], [6], etc.) On the other hand, on first order systems, if their coefficients are constant, we also have a complete result. (See K. Kasahara and M. Yamaguti [7]). In case of first order systems with variable coefficients, we have some results, but they are not satisfactory. (See, for example, N. D. Koutev and V. M. Petkov [8], T. Nishitani [10], [11], [12], H. Yamahara [14], [15] etc.).

In this note, we give some necessary conditions for the strong hyperbolicity of first order systems with variable coefficients, assuming that coefficients depend only on the time variable. This is a further developed results of H. Yamahara [14] and [15]. On the other hand, these become sufficient under a reasonable supplementary condition.

Let us consider the following Cauchy problem.

\[ \begin{aligned}
    & Pu + (P_p - B)u = (D_t - \sum_{i=1}^{\xi} A_i(t,x) D_{x_i} - B(t,x))u = f(t,x), \\
    & u(t_0,x) = u_0(x),
\end{aligned} \]

(1)

where \( u(t,x), u_0(x), f(t,x) \) are vectors of dimension \( N \) and \( A_i(t,x), B(t,x) \) are square matrices of order \( N \) with elements in \( C^\infty(\Omega) \), \( \Omega \) is an open set in \( \mathbb{R}^{t,x}_t \). We say
that the Cauchy problem (1) is uniformly well-posed in $\Omega$ if the following holds:

\[ \forall K = [T_1, T_2] \times K_0, \forall K' \subset K \]

(2) \exists \omega: a lens-shaped neighborhood of the origin,
\[
\forall (t_0, x_0) \in K', \forall u_0 \in C^{\infty}(K_0), \forall f \in C^{\infty}(K), \exists \text{ } u \text{ solution of } (1) \text{ in } (t_0, x_0) + \omega.
\]

**Proposition 0.1.** If (1) is uniformly well-posed in $\Omega$, the following holds:

\[
\left\{ \begin{array}{l}
\forall (\tilde{t}_0, \tilde{x}_0) \in K', \forall M \in \mathbb{N}, \exists M' \in \mathbb{N}, \exists \delta > 0, \exists C > 0 \\
\forall (t_0, x_0) \in K' \text{ s.t. } |t_0 - \tilde{t}_0| \leq \delta, \forall U_0 \in C^M(K) \\
\forall f \in C^{M-1}(K), \exists \text{ solution of } (1) \text{ in } K'', (K'' = \{ |t - t_0| \leq \delta \} \times \{ |x - x_0| \leq \delta \})
\end{array} \right.
\]

and $u$ satisfies

\[ |u|_{M,K} \leq C(|u_0|_{M',K_0} + |f|_{M'-1,K}), \]

where

\[ |u|_{M,K} = \sum_{|\alpha| \leq M} \max_{(t,x) \in K} |D^\alpha_{t,x} u(t,x)|. \]

By the estimate (4), we have the following theorem.

**Theorem 0.** (P. D. Lax and S. Mizohata)

If (1) is uniformly well posed, all characteristic roots of $P$ are real in $\Omega \times \mathbb{R}^p \setminus 0$.

From now on, we always suppose the conclusion of the above theorem.

**Definition 0.1.** (Strong hyperbolicity)

We say that $P$ is strongly hyperbolic when the Cauchy problem (1) of $P + B$ is uniformly well posed in $\Omega$ for arbitrary choice of $B(t,x)$.

Throughout this note, we assume the following:

**Assumption.** $A_i$ depends only on $t$, $1 \leq i \leq \ell$.

Let $\{\lambda_i\}_{i=1}^d$ be the different characteristic roots of $P$ at $t = t_0$ and $\xi = \xi_0 \neq 0$.

We set

\[ A^{(0)} = \sum_{i=1}^\ell A_i(t_0) \xi_{0i}. \]
\[ A^{(i)} = \sum_{i=1}^{\ell} \frac{\partial}{\partial t} A_i(t_0) \xi_{oi}, \]

\( \mathcal{P}_j \): the projection to the generalized eigenspace of \( \lambda_j \),

\[ A_j^{(i)} = A^{(i)}_{\mathcal{P}_j}, \quad (0 \leq i \leq 1, 1 \leq j \leq d). \]

**Theorem 1.** If \( P \) is strongly hyperbolic in \( \Omega \), the following holds

\[ \mathcal{P}_j(A_j^{(o)} - \lambda_j I_N)(A_j^{(t)})^k (A_j^{(o)} - \lambda_j I_N) = 0 \]

for \( 1 \leq j \leq d \) and \( k \in \mathbb{Z}_+ = \{0,1,...\} \).

**Remark.** Let \( m^j \) be the multiplicity of \( \lambda_j \). At least for \( k \geq m^j \), Condition (5) becomes trivial.

**Corollary 2.** The lengths of Jordan chains of \( A^{(o)} \) are at most 2.

By virtue of Bronshtein–Mandai's theorem, the characteristic roots \( \lambda_{(i)}(t) \) \( (1 \leq j \leq N) \) of \( P_p(t: \xi^o_0) \) belong to \( C^\infty \). (See T. Mandai [17] and M. D. Bronshtein [16]). Let us set

\[ \lambda_{(i)}(t) = \lambda_{(i)}(t_0) + (t-t_0) \frac{\partial}{\partial t} \lambda_{(i)}(t_0). \]

**Theorem 3.** If \( \ell = 1 \) and \( \{\lambda_{(i)}(t)\}_{j=1}^{N} \) are distinct for \( 0 \leq \|t - t_0\| \leq \delta_0 \), condition (5) is sufficient for the strong hyperbolicity of \( P_p \) near \( t_0 \).

**Remark.** \( \lambda_{(i)}(t) \) is obtained by \( \sum_i \left(\frac{\partial}{\partial t}\right)^k A_i(t_0) \xi_{oi} \) with \( 0 \leq k \leq 2 \).

In the following sections 1, 2 and 3, we give a proof of Theorem 1 for \( k = 0 \) and 1. The proof of Theorem 1 for \( k \geq 2 \) and that of Theorem 3 will be given in the forthcoming paper [19].

§I. Reduction.

We may assume \( t_0 = 0 \). We take \( B \) as constant matrix and \( f = 0 \). Let us take Fourier image of (1) on the variable \( x \);
\[
\begin{aligned}
\left\{ \begin{array}{l}
(D_t - \sum_{i=0}^{\ell} A_i(t) \xi_i - B(t)) \bar{u} = 0 \\
\bar{u}(0, \xi) = \bar{u}_0(\xi).
\end{array} \right.
\end{aligned}
\]

(1.1)

Setting \( \xi = n \xi_0 \), we expand \( \sum_{i=1}^{\ell} A_i(t) \xi_0 \) as \( A^{(0)} + t A^{(1)} + t^2 A^2(t) \). Further, we transform \( A^{(0)} \) to Jordan's normal form \( \Lambda : \)

\[
\Lambda = \left( \begin{array}{cccc}
\Lambda_1 & & & \\
& \ddots & & \\
& & \Lambda_d & \\
& & & \Lambda_d
\end{array} \right), \quad \Lambda_j = \lambda_j I_m + J^j,
\]

\[
J^j = \left\{ \begin{array}{c}
0 \\
1 \\
& \ddots \\
& & 1 \\
& & & \ddots \\
& & & & 0
\end{array} \right\}; \quad k \times k, \quad (1 \leq j \leq d).
\]

Thus, we arrive at

\[
\left\{ \begin{array}{l}
(D_t - n(\Lambda + t A^{(1)} + t^2 A^{(2)} - B(t))) \bar{u}_1 = 0, \\
\bar{u}_1(0) = \bar{u}_{10}.
\end{array} \right.
\]

(1.2)

Corresponding to \( \Lambda \), we can transform (1.2) by the similar transformation by \( N(t) = I + t N_1 \) :

\[
\left\{ \begin{array}{l}
\tilde{P} \bar{u} = (D_t - n(\Lambda + t \tilde{A}^{(1)} + t^2 \tilde{A}^{(2)}(t)) - \tilde{B}(t)) \bar{u}_2 = 0, \\
\bar{u}_2(0) = \bar{u}_{20},
\end{array} \right.
\]

(1.3)

where, decomposing in blocks \( \tilde{A}^{(1)} \) and \( \tilde{A}^{(1)} \) corresponding to \( \Lambda \), say, \( (\tilde{A}^{(1)}(j,j'))_{1 \leq i, j \leq d} \) and \( (\tilde{A}^{(1)}(j,j'))_{1 \leq i, j \leq d} \), it holds that \( \tilde{A}(j,j) = \tilde{A}(j,j) \) and \( \tilde{A}(j,j') = 0 \) for \( j \neq j' \).

Ex.

\[
\Lambda_1 \quad A(1,1) \quad A(1,2)
\]

\[
\Lambda = \Lambda_2 \quad A(2,1) \quad A(2,2)
\]

\( \text{VIII-4} \)
As our consideration becomes independent of the part which has the factor $t^2 n$, from now on, we take out $(j,j)$ block and omit the subscript "j". We may assume $\lambda = 0$. Further, we set $t = n^{-\sigma} s (\sigma > 0)$. Thus, we arrive at

$$
\begin{align*}
\mathbf{P}_0 \mathbf{v} &= (n^\sigma D_s - (n J + n^{1-\sigma} s A_1 + n^{1-2\sigma} s^2 A_2(s) + B)\mathbf{v} \\
&= 0, \\
\mathbf{v}(0) &= \mathbf{v}_0,
\end{align*}
$$

where $\mathbf{v}, \mathbf{v}_0$ are vectors of dimension $m$, $J, A_1, A_2, B$ are square matrix of order $m$ and

$$
J = J(r,1) \oplus \cdots \oplus J(r,m) \oplus J(r-1,1) \oplus \cdots \oplus J(r-1,m_{r-1}) \oplus J(r-2,1) \oplus \cdots \oplus J(1,m),
$$

$$
\sum_{i=1}^r j m_j = m.
$$

Here, condition (5) for $j$ in §0 is equivalent to

$$
(1.5) \quad J(A_j)^k J = 0 \text{ for } k \in \mathbb{Z}_+.
$$

**Proposition 1.1.** We assume that (1) is uniformly well posed in $\Omega$. If, for $\mathbf{P}_0$ in (1.4), there exists an invertible matrix $N(s,n)$ for $0 < |s| \leq \delta$ and $\ell \geq 2, (\ell \in \mathbb{N})$ such that

$$
\widetilde{\mathbf{P}} = N^{-1} \mathbf{L} N = n^\sigma D_s - n^\mu (\tilde{J}(s) + \tilde{K}(s)) - n^{\mu'} C(s,n),
$$

$\mu > \mu', \mu > \sigma, C(s,n)$ is bounded,

$$
\tilde{J} = \oplus_{1 \leq k \leq R} \tilde{J}(k,h), \tilde{J}(k,h) = \begin{pmatrix} 0 & k,h \\ \mathbf{a}_k \end{pmatrix},
$$

$\mathbf{a}_k^{k,h}$ is not identically zero, and is analytic for $s \neq 0$,

$$
\tilde{K} = (K(k,h,k',h'))_{1 \leq k,k' \leq R}; \text{ block decomposed with respect to } \tilde{J},
$$

with

VIII-5
then, we have the following:

1) If \( \ell \geq 3 \), \( \tilde{J} + \tilde{K} \) is nilpotent.

2) Let \( \det(\lambda_1 - (\tilde{J} + \tilde{K})) \) be \( \sum_{i=0}^{m} c_i(s) \lambda^{m-i} \).

If \( \ell = 2 \) and \( C_{2i}(s) \) (\( C_{2i+1}(s) \), resp.) is even function (odd function, resp.), 
\( \tilde{J} + \tilde{K} \) is nilpotent.

Now, we assume (5) does not hold for \( k = 0 \) or \( k = 1 \).

In order to make \( B \) stronger than \( n^{1-2\alpha} s^2 A_2(s) \), we take \( 1 - 2\alpha < 0 \) if \( \sigma > \frac{1}{2} \).

§2. Maximal connection.

Let us consider

\[
\mathcal{J} = \bigoplus_{1 \leq k \leq R} \mathcal{J}(k,h), \quad \mathcal{J}(k,h) \text{ is that in Prop. 1.1, } M = \sum_{j=1}^{R} j M_j.
\]

Corresponding to the blocks of \( \mathcal{J} \), we decompose \( M \times M \) matrix \( K \) to

\[
(K(k,h,k',h'))_{1 \leq k,k' \leq R} \quad 1 \leq h \leq M_k \quad 1 \leq h' \leq M_{k'}.
\]

Ex.

\[
\begin{align*}
\tilde{J}(2,1) & \quad \tilde{J}(2,1,2,1) \quad \tilde{J}(2,1,2,2) \quad \tilde{J}(2,1,1,1) \\
\tilde{J}(2,2) & \quad \tilde{J}(2,2,2,1) \quad \tilde{J}(2,2,2,2) \quad \tilde{J}(2,2,1,1) \\
\tilde{J}(1,1) & \quad \tilde{J}(1,1,2,1) \quad \tilde{J}(1,1,2,2) \quad \tilde{J}(1,1,1,1)
\end{align*}
\]

We call \( (K(k,h,k',h')) \) the block decomposition of \( K \) with respect to \( \tilde{J} \).

The following notions are important.
Definition 2.1. (Maximal connection of Jordan chain).

Let \((K(k,h,k',h'))\) be the block decomposition of \(K\) with respect to \(\widetilde{J}\). If

1) \[
\begin{align*}
K(R,h,k',h') &= \begin{cases}
0 & (k < R) \\
\alpha & (R, h, k', h') \neq 0
\end{cases}
\end{align*}
\]

for arbitrary \(h, k'\) and \(h'\),

2) \(K \neq 0\),

3) \((\alpha^{R,h,h'})_{1 < h, h' < \mathbb{M}}\) is nilpotent,

\(\widetilde{J} + K\) is again nilpotent. We say that in \(\widetilde{J} + K\), the Jordan chains of \(\widetilde{J}\) are maximally connected by \(K\), or that \(K\) brings a maximal connection (of Jordan chain) to \(\widetilde{J}\).

Definition 2.2. (Self similar matrix).

Let us take \(1 < R_0 < R_1 < \ldots < R_p < R_{p+1}\), such that

\[R_{j+1} = k_j R_j + R_j^0, \quad k_j \geq 1, \quad 0 \leq R_j^0 < R_j, \quad k_j, R_j \in \mathbb{N}.
\]

We set

\[
A_0 = \begin{pmatrix}
0 & 1 \\
& 0 & \ddots \\
& & & 1 \\
& & & 0
\end{pmatrix} ; \quad R_0 \times R_0,
\]

\[
A_{j+1} = A_j \oplus \ldots \oplus A_j \oplus A_j^0 + K_j : \quad R_{j+1} \times R_{j+1}
\]

where \(A_j^0\) is the first \(R_j^0\) rows and \(R_j^0\) columns part of \(A_j\),

\[
K_j = (K_j(k,k')) ; \text{ block decomposition w.r.t. } A_{j+1}
\]

\[
K_j(h,h+1) = \begin{pmatrix}
0 & 0 \\
& 0 & \ddots \\
& & & 0 \\
& & & 0
\end{pmatrix}, \quad 1 \leq h \leq k_j,
\]

\(\hat{J}(i) : i \times i = \text{the first } i \text{ rows and } i \text{ columns part of } A_{p+1}^0\).

We call
\[
J = \bigoplus_{1 \leq i \leq R_{p+1}^M} (J(i) \oplus \cdots \oplus J(i))
\]
a self similar matrix of step \(p+1\) and \(A_j\) and \(A_j^0\) the factors of step \(j\).

Let \(J\) be \(M \times M\) selfsimilar matrix and \(K\) is a \(M \times M\) matrix block decomposed w.r.t. \(J\). Let an element of a block of \(K\) belong to \(q^{(r)}_p\)-th \(A_p\) in the direction of row and to \(q^{(c)}_p\)-th \(A_p\) in the direction of column. ((\(k_p+1\))-th \(A_p = A_p^0\)). Further, in \(A_p\), let it belong to \(q^{(r)}_{p-1}\)-th \(A_{p-1}\) in the direction of row and to \(q^{(c)}_{p-1}\)-th \(A_{p-1}\) in the direction of column. We continue this procedure up to \(q^{(s)}_o\). At last, let it be the \((q^{(r)}_{-1}, q^{(c)}_{-1})\) element of \(A_o\). We set \(q_h = q^{(r)}_h - q^{(c)}_h + 1 (1 \leq h \leq p)\).

**Definition 2.3. (Address)**
We call \(q = (q_p, q_{p-1}, \ldots, q_1)\) the address of the element.

To the set of addresses, we give the dictionary order.

**Definition 2.4. (Acceptable matrix).**
Let us take \(v, v_o > \ldots > v_p > 0, v = \sum_{j=1}^{p} v_j, \) and \(\sigma > 0 (v_j, \sigma \in \mathbb{R}_+)\). For a block decomposed matrix \(K\) w.r.t. a selfsimilar matrix \(J\), if the address \(q\) of its element has a \(q_j\) such that \(q_j = k_j + 1\) and \(\sum_{h=0}^{j-1} (q_h - 1)R_h + q_{-1} = R^0_j\) (that is, the element is found at the left-down corner of \(R_{j+1} \times R\) matrix in step of \(j+1\), \(R(\leq R_{j+1})\); free), the element has the form \(c(s) n^{1-v'}, v' = v'(q) = 2\sigma - \sum_{j=1}^{p} (q_j - 1)v_j\) and otherwise, it has the form \(c n^{1-v'}, v' = v'(q) = \sigma - \sum_{j=1}^{p} (q_j - 1)v_j\) and \(c\) is constant. Further, if all \(v(q)\) are greater than \(v\), we say that \(K\) is acceptable w.r.t. \(n^{1-v}\). We call \(v(q) = (v' - v)/\sum_{j=1}^{p} (q_j - 1)R_j + q_{-1}\) the descent index of the element with the address \(q\).

When the descent index is smaller, we say that it is more effective.

**Remark.** Corresponding to the above \(J\), we take a shearing operator with weight \(\varepsilon\)
\[
W = \bigoplus_{1 \leq k \leq R_{p+1}} W(k, h, \varepsilon),
\]
\[
W(k, h, \varepsilon) = \text{diag}(1, n^\varepsilon, n^{2\varepsilon}, \ldots, n^{(k-1)\varepsilon}), \varepsilon = v(q).
\]
Then, the element with the adress \(q\)
obtain the order $1 - \nu - \varepsilon$ of $W^{-1}(n^{1-\nu})W$ by the shearing transformation $W^{-1}K W$.

Now, we return to the equation (1.4). We assume that (i) is uniformly well-posed.

Let us set $W = \bigoplus W(k,h,-)$. Let $w_i = W^{-1}w$, $w_i$ satisfies $P_i w_i = 0$.

$P_i = W^{-1}P_0 W = n^\sigma D_s - (n^{1-\sigma/r}) J + n^{1-\sigma/r} s K_i + s A_i(n) + s^2 A_2(s;n) + B'(n)$,

where $s(n^{1-\sigma/r} K_i + A_i(n))$ is brought from $n^{1-\sigma/s} A_i$ and the order of $A_i(n)$ is less than $1-\sigma/r$. $s A_i(n) + s^2 A_2(s;n)$ is acceptable w.r.t. $n^{1-\sigma/r} J$.

In $K_i = (K_i(k,h,k',h'))$, $K_i(k,k',h') = \left( \begin{array}{cc} 0 & \alpha(h,k') \\ 0 & h,k' \end{array} \right)$

and $K_i(k,h,k',h') = 0$ for $k < r$. By virtue of Proposition 1.1, $(\alpha_{h,h'})$ must be nilpotent, and then, $K_i$ brings a maximal connection to $J$ if $K_i \neq 0$. We can take each Jordan chain in $J + K_i$ composed by vectors of $s^\mu v, v :$ constant vector. Replacing $s^\mu v$ by $v n$ we can have a constant matrix $N$ which transform $J$ to $J_1$, a selfsimilar matrix. We have

$P_i = N^{-1} P_i N = n^\sigma D_s - (n^{1-\sigma/r}) J^1(s) + s A_i^1(n) + s^2 A_2^1(s;n) + B^1(n))$.

Let us set the length of the longest Jordan chain of $J_i(s)$ as $R_i = k_0 R_0 + \ell, R_0 = r, 0 \leq \ell < R_0$. In $s A_i^1(n) + s^2 A_2^1(s;n)$, the highest order on $n$ is given only by the elements with the address $(k_0+1, r-1)$ if $\ell \geq R_0 - 1$, and by those with the adress $(k_0, r-1)$ (and also by those with $(k_0+1, r-2)$ in case of $k_0 = 1$) if $\ell < R_0 - 1$. In the former case, if an element with the address $(k_0+1, r-1)$ does not vanish, after the shearing transformation with weight $\frac{\sigma}{R_0 R_1}$, a maximal connection occurs by virtue of Proposition 1.1. In the latter case, no maximal connection occurs. Continuing this procedure, we arrive at the following proposition.

VIII-9
Proposition 2.1. Let us set \( R_{-1} = 1, R_0 = R, R_{j+1} = k_j R_j + R_j - R_{j-1} \) (0 \( \leq j \leq p+1 \)) and \( \hat{R} = k_p R_p + \ell, (k_j \in \mathbb{N} = \{1,2,\ldots\}, 0 \leq \ell < R_p) \).

1. In the above procedure, if \( p \) times maximal connections occur, the highest order part must be the selfsimilar matrix of step \( p+1 \) replacing \( R_{p+1} \) by \( \hat{R} \) and has the order \( 1-v \) on \( n, v = \sum_{j=0}^{p} v_j \) where \( v_j = \frac{\sigma}{R_{j-1} R_j} \).

2. The operator \( \tilde{P}_{p+1} \) has the following form:

\[
\tilde{P}_{p+1} = n^\sigma D_s \cdot (n^{\frac{1}{1-v}} J_{p+1}(s) + s A^{p+1}_1(n) + s^2 A^{p+1}_2(s; n) + B^{p+1}),
\]

where \( s A^{p+1}_1(n) + s^2 A^{p+1}_2(s; n) \) is acceptable w.r.t. \( n^{1-v} \).

3. In \( s A^{p+1}_1(n) + s^2 A^{p+1}_2(s; n) \), if \( \ell \geq R_p - R_{p-1} \), the highest order is given only by the elements with address \( (k_{p+1}, k_{p-1}, \ldots, k_0, r-1) \) and if \( \ell < R_p - R_{p-1} \), it is given by those with the address \( (k_p, k_{p-1}, \ldots, k_0, r-1) \) (and also by those with \( (1,\ldots,1,2,\ell-1,\ell-1,\ldots,\ell,0,1, \ldots,0, r-2) \) in case of \( k_{\ell+1} = \ldots = k_p = 1 \) and \( k_{\ell} \geq 2 \) and also by those with \( (1,\ldots,1,2,1,\ell-1,\ell-1,\ldots,\ell,0,1, \ldots,0, r-2) \) in case of \( k_0 = k_1 = \ldots = k_p = 1 \).

Proof By the induction on \( p \).

§3 Proof of Theorem 1, case of \( k \leq 1 \).

The maximal connections can occur at most \( \lfloor \frac{m-c}{r-1} \rfloor \) times. Let no maximal connection occur on \( \tilde{P}_{p+1} \), that is, in \( \hat{R} = k_p R_p + \ell, \ell = R_p - R_{p-1} \) or \( \ell = R_p - R_{p-1} \) but all elements with the address \( (k_{p+1}, k_{p-1}, \ldots, k_0, r-1) \) vanish.

Let \( W \) be the shearing operator corresponding to \( \tilde{J}_{p+1} \) in (2.5) with weight \( \varepsilon \) (\( \varepsilon = \frac{\sigma}{R_p} \) in case of \( \ell > R_p - R_{p-1} \) and \( \varepsilon = \frac{\sigma}{R_{p+1} - R_p} \) in case of \( \ell < R_p - R_{p-1} \)).

We set

\[
\tilde{P}_{p+2} = W^{-1} \tilde{P}_{p+1} W =
\]

\[
n^\sigma D_s \cdot (n^{1-v-\varepsilon} (\tilde{J}_{p+1}(s) + s K_{p+2}) + s A^{p+2}_1(n) + s^2 A^{p+2}_2(s; n) + B^{p+2}(n)),
\]

where the orders of \( A^{p+2}_1 \) and \( A^{p+2}_2 \) are less than \( 1-v-\varepsilon \). Here the highest order in \( B^{p+2}(n) \) is \( \sigma \) and it is given by the elements with the address \( (k_{p+1}, k_{p-1}, \ldots, k_0, r) \) in
case of $\ell > R_p - R_{p-1}$ and $(k_p, k_{p-1}, \ldots, k_0, r)$ or case of $R_p - R_{p-1}$.

By a suitable choice of $B$ in the original operator $P$, we can take $B^{p+2}$ such that it has only one non-zero element, $(g, l)$-element $c_0 n^\sigma$ ($c_0$ is a large constant), where $g = k_p R_p + \sum_{j=0}^{p-1} (k_j - 1) R_j + r$ if $\ell > R_p - R_{p-1}$ and $g = \sum_{j=0}^{p-1} (k_j - 1) R_j + r$ if $\ell \leq R_p - R_{p-1}$.

We consider the characteristic polynomial of the full operator $\tilde{P}^{p+2}$:

$$\det(\lambda I_n - (n^{1-v-\varepsilon}(J_{p+1} + s K_{p+2}) + s A_1^{p+2}(n) + s^2 A_2^{p+2} + B^{p+2}(n)))$$

$$= \sum_{j=0}^{m} \alpha_j(s; n) \lambda^{m-j}.$$  

$\alpha_j(s; n)$ has the form $c_0 n^\delta s^\mu (1 + o(1))$, $\delta = (g - 1)(1 - v - \varepsilon) + \sigma$ and $\mu \in \mathbb{Z}^+$. Here, we cannot find Jordan chains which are composed the vector of type $s^\mu v$, $v$: constant vector.

By virtue of Proposition 1.1, $J_{p+1} + K_{p+2}$ is nilpotent. Let us take $N(s)$ which transforms $J_{p+1} + K_{p+2}$ to Jordan's normal form and set

$$\tilde{P}^{p+2} = N^{-1} \cdot \tilde{P}^{p+2} \cdot N = n^\sigma D_s - n^{1-v-\varepsilon} J_{p+2} - C(s; n).$$

Here, the commutator $n^\sigma N^{-1}(s) D_s N(s)$ has the same order $\sigma$ as $B^{p+2}(n)$ and it can give an influence on $\alpha_j(s; n)$. That is, setting

$$\det(\lambda I - n^{1-v-\varepsilon} J_{p+2} - C(s; n)) = \sum_{j=0}^{m} \alpha_j'(s; n) \lambda^{m-j},$$

$\alpha_j'(s; n)$ may have the form $(c_0 + c_0'(s)) n^\delta (1 + o(1))$. However, $c_0'(s)$ is decided by the principal part part of the original operator $P$ and independent of $B^{p+2}(n)$. Thus, $\alpha_j'(s; n) \neq 0$ and it has the order $\sigma$, if we take $c_0$ sufficiently large.

Let $\sigma$ be $i_1/i_0$ ($i_0, i_1 \in \mathbb{N}$). $1 - v - \varepsilon$ is also expressed as $i_2/i_0$ ($i_2 \in \mathbb{N}$). If $1 - v - \varepsilon > \sigma$, we can find a matrix $N' \sim I + \sum_{h \in \mathbb{N}} \frac{n^{h/i_0}}{n^{h/i_0}} N_h(s)$ such that

$$Q = N'^{-1} \circ \tilde{P}^{p+2} \circ N' = n^\sigma D_s - n^{1-v-\varepsilon} J_{p+2} - C(s; n),$$

where $C(s; n) = (C(k, h, k', h')(s; n))$; block decomposition w.r.t. $J_{p+2}$,

$$C(k, h, k', h') = \begin{pmatrix} \gamma_{k}^h k' h' & 0 \\ \vdots & 0 \\ \gamma_{k}^h k' h' & 0 \end{pmatrix}$$

and

VIII-11
We say that a matrix which has the form as \( C \) is admissible to \( n^{1-v-\varepsilon} J \). By this transformation, the principal part of \( \alpha_s'(s ; n) \) is preserved. From now on, we assume that \( 1-v-\varepsilon > \sigma \).

We introduce a notion:

**Definition 3.1. (Stable coefficient of characteristic polynomial)**

Let \( \tilde{C} \) be admissible to \( n^\sigma J \). We set

\[
\det(\lambda I - n^\sigma J - \tilde{C}(s ; n)) = \Sigma_{j=0}^{m} \alpha_j(s ; n)\lambda^{m-j}.
\]

When the principal part of \( \tilde{\alpha}_j \) is preserved by any perturbation of order at most \( \sigma \), we say that \( \alpha_j \) is a stable coefficient of the characteristic polynomial of full operator.

On the stable coefficients, the following proposition was obtained by W. Matsumoto [9].

**Proposition 3.1.** *If the original Cauchy problem is uniformly well-posed, the characteristic polynomial of full operator has no stable coefficient.*

We transform \( Q \) by shearing operator \( W' \) corresponding to \( J \) with weight \( \varepsilon_0 > 0, \varepsilon_0 : \) very small.

\[
(3.4) \quad Q' = W'^{-1} Q W' = n^\sigma D_s - n^{1-v-\varepsilon-\varepsilon_0} J - C'(s ; n),
\]
where the order of \( C'(s ; n) \) is less than \( 1-v-\varepsilon-\varepsilon_0 \) and \( C'(s,n) \) is admissible to \( n^{1-v-\varepsilon-\varepsilon_0} J \). Further, the elements which concern \( \alpha'_g(s ; n) \) has the order \( \sigma + (g-1)\varepsilon_0 \) in \( C'(s,n) \). This implies that \( \alpha'_g(s ; n) \) is stable in the characteristic polynomial of the full operator \( Q' \), if we can find \( \sigma \) such that \( 1-v-\varepsilon > \sigma > \frac{1}{2} \). Then, when we can find a \( \sigma \) such that \( 1-v-\varepsilon > \sigma > \frac{1}{2} \), we arrive at a contradiction. Here,
the existence of such $\sigma$ is equivalent to "$g \geq 3$" and further equivalent to "$r \geq 2$ and if $r=2$,

$$A_1 = (A_1 (k,h,k',h'))_{1 \leq k,k' \leq 2} \text{ in } (1.4) \left( A_1 (2,h,2,h') = \begin{pmatrix} \alpha(h,h') & \ast \\ \ast & \ast \end{pmatrix} \right),$$

($\alpha_{hh'}_{1 \leq h,h' \leq m_k}$ vanishes". "$r \leq 2$" is equivalent to condition (1.5) with $k = 0$ (Corollary 2) and the rest is equivalent to (1.5) with $k = 1$.

Q.E.D.

References

1. L. Hörmander
   The Cauchy problem for differential equations with double characteristics,

2. V. Ja. Ivrii
   Sufficient conditions for regular and completely regular hyperbolicity,

3. V. Ya. Ivrii and V.M. Petkov
   Necessary conditions for the Cauchy problem for non-strictly hyperbolic
   equations to be well-posed,

4. N. Iwasaki
   The Cauchy problem for effectively hyperbolic equations, (a special case),
5. N. Iwasaki
The Cauchy problem for effectively hyperbolic equations, (a standard type),

6. N. Iwasaki
The Cauchy problem for effectively hyperbolic equations, (general cases),

7. K. Kasahara and M. Yamaguti
Strong hyperbolic systems of linear partial differential equations with constant coefficients,

8. N.D. Koutev and V.M. Petkov
Sur les systèmes régulièrement hyperboliques du premier ordre,

9. W. Matsumoto

10. T. Nishitani
Système effectivement hyperbolique, Calcul opér. fronts d’ondes, Travaux en Cours 29,

11. T. Nishitani
On strong hyperbolicity of systems, Hyperbolic Equations, Res. Notes Math. 158

VIII–14
12. T. Nishitani

Strong hyperbolic systems with transverse propagation cone,
(to appear in Note of J. Vaillant Seminar).

13. O.A. Oleinik

On the Cauchy problem for weakly hyperbolic equations,

14. H. Yamahara

On the strong hyperbolic systems,

15. H. Yamahara

On the strong hyperbolic systems, II,

16. M.D. Bronshtein

Smoothness of roots of polynomials depending on parameters,

17. T. Mandai

Smoothness of roots of hyperbolic polynomials with respect to one-dimensional parameter,

18. V.M. Petkov

Microlocal forms for hyperbolic systems,
W. Matsumoto

Conditions for strong hyperbolicity of first order systems.

Waichiro MATSUMOTO
Ryukoku University
Faculty of Science and Technology
Department of Applied Mathematics and Informatics
520 - 21 Seta, Otsu,
JAPAN

and

Hideo YAMAHARA
Osaka Electro-Communication University
Faculty of Engineering
572 Hatsucho, Neyagawa
JAPAN