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Some remarks on the multi-dimensional Borg-Levinson theorem

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1.1-dim-case. The Borg-Levinson theorem is a uniqueness theorem in inverse eigenvalue problems. We first recall the 1-dim case. Consider the Dirichlet problem:
\[-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq 1,\]
\[y(0) = y(1) = 0,\]
q(x) being a real function. Let
\[\lambda_1 < \lambda_2 < ...\]
be the eigenvalues. Now suppose for two potentials \(q_1, q_2\),
\[\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geq 1.\]
One can then easily see that it does not necessarily imply \(q_1 = q_2\). To derive the uniqueness of the potential, we must add some auxiliary conditions.

Let \(y = y(x, \lambda) = y(x, \lambda, q)\) be the solution of the Cauchy problem:
\[-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq 1,\]
\[y(0) = 0, \quad y'(0) = 1.\]
Then we have

THEOREM (Borg-Levinson). Suppose that
\[\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geq 1,\]
\[y'(1, \lambda_i, q_1) = y'(1, \lambda_i, q_2) \text{ for all } i \geq 1.\]
Then \(q_1 = q_2\).

This is a starting point of 1-dim. inverse problems ([1],[2]). The recent
article of Pöschel–Trabowitz [3] gives a deep insight. It is proved that the map
\[ q \rightarrow \{\lambda_i\}_{i=1}^{\infty} \times \{\log|y'(1,\lambda_i, q)|\}_{i=1}^{\infty} \]
defines an analytic isomorphism from \( L^2(0,1) \) to a Hilbert space of infinite sequences.
And also, for any fixed potential \( p \), the set defined by
\[ M(p) = \{ q : \lambda_i(q) = \lambda_i(p) \text{ for } \forall i \geq 1 \} \]
is a real analytic manifold (isospectral manifold) with the system of coordinates \( \{\log|y'(1,\lambda_i, q)|\}_{i=1}^{\infty} \).

Since \( y(x,\lambda, q) \) is an eigenfunction of \( -\frac{d^2}{dx^2} + q(x) \) associated with the
eigenvalue \( \lambda_i \), one can see that there is a one to one correspondence between the
potential and the eigenvalues and the normal derivatives of eigenfunctions.

2. \( n \)-dim. case. Next we turn to the \( n \)-dim. case \( (n \geq 2) \). Let \( \Omega \) be a bounded
domain in \( \mathbb{R}^n \) with smooth boundary \( S \). Consider the Dirichlet problem:
\[
\begin{cases}
(-\Delta + q)u = \lambda u \text{ in } \Omega, \\
u|_S = 0.
\end{cases}
\]
Although we treat the Dirichlet problem here, all of the arguments below also hold for
the Neumann or Robin boundary conditions by a suitable modification.

Let \( \lambda_1 < \lambda_2 \leq ... \) be the eigenvalues. To derive the uniqueness theorem
corresponding to the 1-dim. case, we consider the normal derivatives of eigenfunctions.
However, one must be careful to choose a system of eigenfunctions, since in the
multi-dimensional case eigenvalues are not simple in general.

Let \( m \) be the multiplicity of \( \lambda_i \) and \( u_1, ..., u_m \) be a real-valued orthonormal
eigenfunctions system associated with \( \lambda_i \). We set
\[ E_i = \left\{ \left( \frac{\partial u_1}{\partial v}, ..., \frac{\partial u_m}{\partial v} \right) \right\}, \]
v being the outer unit normal to \( S \). One can then see that for two such systems of
eigenfunctions \( \{u_1, ..., u_m\}, \{v_1, ..., v_m\} \), there exists an orthogonal matrix \( T \in O(m) \) such that
Now, this defines an equivalent relation in the space of functions on the boundary $S$. Further, it shows that for the set $\{E_i\}$, the totality of $E_i$, there corresponds only one equivalence class, which we denote by $W_i$:

$$W_i = \{E_i\}/\simeq.$$

Then, we have the following theorem due to Nachman–Sylvester–Uhlmann [4].

**THEOREM A.** Let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Suppose that

$$\lambda_i(q_1) = \lambda_i(q_2) \quad \text{for} \quad \forall i \geq 1,$$

$$W_i(q_1) = W_i(q_2) \quad \text{for} \quad \forall i \geq 1.$$

Then $q_1 = q_2$.

This theorem seems to be a direct generalization of the 1-dimensional Borg–Levinson theorem. So, it is natural to ask: does the map $q \rightarrow (\lambda_i) \times (W_i)$ define an isomorphism? Can $(W_i)$ be the coordinates of the isospectral set of potentials? The answer is always negative. In fact, we have

**THEOREM B.** Let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Suppose that there exists an $N > 0$ such that

$$\lambda_i(q_1) = \lambda_i(q_2) \quad \text{for} \quad \forall i > N,$$

$$W_i(q_1) = W_i(q_2) \quad \text{for} \quad \forall i > N.$$

Then $q_1 = q_2$.

In other words, if $\lambda_i$ and $W_i$ are equal except for a finite number of indices $i$, the potentials are equal. It also shows that the totality of $\lambda_i$ and $W_i$ is too much to determine the potential. It is a common belief that, in contrast to the 1-dim. case, the multi-dimensional inverse eigenvalue problem has a sort of rigidity. Here one can find XVII-3
3. Proof of Theorem B. We sketch the proof of theorem B. Let $N(\lambda)$ be the Neumann operator, namely,

$$N(\lambda)f = \frac{\partial v}{\partial n},$$

where $v$ satisfies

$$\begin{cases}
(-\Delta + q)v = \lambda v, \\
v|_{\partial\Omega} = f.
\end{cases}$$

We introduce the following notation:

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} \, dx,$$

$$\langle f, g \rangle_S = \int_{\partial\Omega} f(x) \overline{g(x)} \, dS,$$

$$\phi_{\lambda,\omega}(x) = e^{i\sqrt{\lambda}\omega \cdot x}, \quad \omega \in S^{n-1}, \quad \lambda \in \mathbb{C}.$$ 

Let $S(\lambda, \theta, \omega)$ be defined by

$$S(\lambda, \theta, \omega) = \langle N(\lambda)\phi_{\lambda,\omega}, \phi_{\lambda,-\theta} \rangle.$$

The crucial fact is the following lemma.

**Lemma C.** If $\lambda$ is not an eigenvalue,

$$S(\lambda, \theta, \omega) = -\frac{\lambda}{2} (\theta - \omega)^2 \int_{\Omega} e^{-i\sqrt{\lambda}(\theta - \omega)x} \, dx$$

$$+ \int_{\Omega} e^{-i\sqrt{\lambda}(\theta - \omega)x} q(x) dx$$

$$- \langle R(\lambda) q \phi_{\lambda,\omega}, \phi_{\lambda,-\theta} \rangle,$$

where $R(\lambda) = (-\Delta_D + q - \lambda)^{-1}$.

Note that the above expression is similar to the $S$-matrix in scattering theory.

Now, we recall the Born approximation.

Let $\mathbb{R}^n \ni \xi \neq 0$ be arbitrarily fixed. Take $\eta \in S^{n-1}$ such that $n \cdot \xi$. For a a large parameter $N$, we define
\[
\begin{align*}
\theta_N &= C_N \eta + \xi/2N, \quad C_N = (1-|\xi|^2/4N^2)^{1/2}, \\
\omega_N &= C_N \eta - \xi/2N, \\
\sqrt{t_N} &= N + i.
\end{align*}
\]

They have the following properties:
\[
\begin{align*}
\theta_N, \omega_N &\in S^{n-1}, \\
\sqrt{t_N} (\theta_N - \omega_N) &\to \xi, \text{ as } N \to \infty, \\
\text{Im } t_N &\to \infty \text{ as } N \to \infty, \\
\text{Im } \sqrt{t_N} \theta_N, \text{ Im } \sqrt{t_N} \omega_N \text{ are bounded.}
\end{align*}
\]

Invoking these properties, one can easily show

**THEOREM D.**

\[
\lim_{N \to \infty} S(t_N, \theta_N, \omega_N) = -\frac{|\xi|^2}{2} \int_{\Omega} e^{-ix_1} \, dx + \int_{\Omega} e^{-ix_1} q(x) \, dx.
\]

So, one can reconstruct the potential from \(S(\lambda, \theta, \omega)\).

Now, we prove Theorem B. \(N(\lambda)\) has, formally, the integral kernel:

\[
\sum_{i=1}^{\infty} \frac{1}{\lambda_i - \lambda} \left( \frac{\partial \phi_i}{\partial \nu}(x) \frac{\partial \phi_i}{\partial \nu}(y) \right),
\]

where \(\phi_i\) is the eigenfunction. In view of this expression one can show that the assumption of Theorem B implies

\[
\|N(\lambda, q_1) - N(\lambda, q_2)\|_{B(L^2(S^{n-1}))} \leq C/|\lambda|
\]

for large \(|\lambda|\). Theorem B then follows from this inequality and theorem D.

From the very proof, one can see that the potential is uniquely determined by the asymptotic properties of the eigenvalues and eigenfunctions.

4. Variable coefficient case. We briefly mention the variable coefficients case.
Consider the operator
\[ H = -\sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + q(x). \]
Assume that for \(|\alpha| \leq N\), \(N\) is chosen large enough,
\[ \sup_{x \in \Omega} (|a_{ij}(x) - \delta_{ij}|) \equiv \delta < 1. \]
Then the above Theorem B, Lemma C, Theorem D also hold in this case. The proof relies on the method of asymptotic solutions and Fourier integral operators. Note that we are fixing \(a_{ij}\) and seeking \(q(x)\).

The above results may be extended to higher order elliptic operators and elliptic systems.

References