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Some remarks on the multi-dimensional
Borg-Levinson theorem

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1.1-dim-case. The Borg-Levinson theorem is a uniqueness theorem in inverse
eigenvalue problems. We first recall the 1-dim case. Consider the Dirichlet problem:

\[-y'' + q(x)y = \lambda y, \ 0 \leq x \leq 1,\]
\[y(0) = y(1) = 0,\]

$q(x)$ being a real function. Let

$$\lambda_1 < \lambda_2 < \ldots$$

be the eigenvalues. Now suppose for two potentials $q_1, q_2$,

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geq 1.$$

One can then easily see that it does not necessarily imply $q_1 = q_2$. To derive the
uniqueness of the potential, we must add some auxiliary conditions.

Let $y = y(x, \lambda) = y(x, \lambda, q)$ be the solution of the Cauchy problem:

\[-y'' + q(x)y = \lambda y, \ 0 \leq x \leq 1,\]
\[y(0) = 0, \ y'(0) = 1.\]

Then we have

**THEOREM** (Borg-Levinson). Suppose that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geq 1,$$
$$y'(1, \lambda_i, q_1) = y'(1, \lambda_i, q_2) \text{ for all } i \geq 1.$$

Then $q_1 = q_2$.

This is a starting point of 1-dim. inverse problems ([1],[2]). The recent
article of Pöschel–Trabowitz [3] gives a deep insight. It is proved that the map
\[ q \rightarrow (\lambda_i)_{i=1}^{\infty} \times \{\log y'(1,\lambda_i, q)\}_{i=1}^{\infty} \]
defines an analytic isomorphism from \( L^2(0,1) \) to a Hilbert space of infinite sequences. And also, for any fixed potential \( p \), the set defined by
\[ M(p) = \{ q : \lambda_i(q) = \lambda_i(p) \text{ for } \forall i \geq 1 \} \]
is a real analytic manifold (isospectral manifold) with the system of coordinates \( \{\log y'(1,\lambda_i, q)\}_{i=1}^{\infty} \).

Since \( y(x,\lambda_i, q) \) is an eigenfunction of \( -\frac{d^2}{dx^2} + q(x) \) associated with the eigenvalue \( \lambda_i \), one can see that there is a one to one correspondence between the potential and the eigenvalues and the normal derivatives of eigenfunctions.

2. n-dim. case. Next we turn to the n-dim. case \( (n \geq 2) \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( S \). Consider the Dirichlet problem:
\[
\begin{align*}
(-\Delta + q)u &= \lambda u \text{ in } \Omega, \\
u|_S &= 0.
\end{align*}
\]
Although we treat the Dirichlet problem here, all of the arguments below also hold for the Neumann or Robin boundary conditions by a suitable modification.

Let \( \lambda_1 < \lambda_2 \leq \ldots \) be the eigenvalues. To derive the uniqueness theorem corresponding to the 1-dim. case, we consider the normal derivatives of eigenfunctions. However, one must be careful to choose a system of eigenfunctions, since in the multi-dimensional case eigenvalues are not simple in general.

Let \( m \) be the multiplicity of \( \lambda_i \) and \( u_1, \ldots, u_m \) be a real-valued orthonormal eigenfunctions system associated with \( \lambda_i \). We set
\[
E_i = \left\{ \frac{\partial u_1}{\partial v}, \ldots, \frac{\partial u_m}{\partial v} \right\},
\]
v being the outer unit normal to \( S \). One can then see that for two such systems of eigenfunctions \( \{u_1, \ldots, u_m\}, \{v_1, \ldots, v_m\} \), there exists an orthogonal matrix \( T \in O(m) \) such that

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\[
\frac{\partial u}{\partial v} = \frac{\partial^m u}{\partial v^m}
\]

Now, this defines an equivalent relation - in the space of functions on the boundary \(S\). Further, it shows that for the set \(\{E_i\}\), the totality of \(E_i\), there corresponds only one equivalence class, which we denote by \(W_i\):

\[
W_i = \{E_i\}/\ldots
\]

Then, we have the following theorem due to Nachman–Sylvester–Uhlmann [4].

**THEOREM A.** Let \(q_1, q_2 \in C^\infty(\Omega)\). Suppose that

\[
\lambda_i(q_1) = \lambda_i(q_2) \quad \text{for} \quad \forall i \geq 1,
\]

\[
W_i(q_1) = W_i(q_2) \quad \text{for} \quad \forall i \geq 1.
\]

Then \(q_1 = q_2\).

This theorem seems to be a direct generalization of the 1-dimensional Borg–Levinson theorem. So, it is natural to ask: does the map \(q \to (\lambda_i) \times (W_i)\) define an isomorphism? Can \(W_i\) be the coordinates of the isospectral set of potentials? The answer is always negative. In fact, we have

**THEOREM B.** Let \(q_1, q_2 \in C^\infty(\Omega)\). Suppose that there exists an \(N > 0\) such that

\[
\lambda_i(q_1) = \lambda_i(q_2) \quad \text{for} \quad \forall i > N,
\]

\[
W_i(q_1) = W_i(q_2) \quad \text{for} \quad \forall i > N.
\]

Then \(q_1 = q_2\).

In other words, if \(\lambda_i\) and \(W_i\) are equal except for a finite number of indices \(i\), the potentials are equal. It also shows that the totality of \(\lambda_i\) and \(W_i\) is too much to determine the potential. It is a common belief that, in contrast to the 1-dim. case, the multi-dimensional inverse eigenvalue problem has a sort of rigidity. Here one can find...
3. Proof of Theorem B. We sketch the proof of theorem B. Let $N(\lambda)$ be the Neumann operator, namely,

$$N(\lambda)f = \frac{\partial v}{\partial n},$$

where $v$ satisfies

$$
\begin{cases}
(-\Delta + q)v = \lambda v, \\
v|_S = f.
\end{cases}
$$

We introduce the following notation:

$$(f, g) = \int_{\Omega} f(x) \overline{g(x)} \, dx,$$

$$<f, g> = \int_{S} f(x)\overline{g(x)} \, dS,$$

$$\varphi_{\lambda, \omega}(x) = e^{i \sqrt{\lambda} \cdot \omega \cdot x}, \omega \in S^{n-1}, \lambda \in \mathbb{C}.$$

Let $S(\lambda, \theta, \omega)$ be defined by

$$S(\lambda, \theta, \omega) = <N(\lambda)\varphi_{\lambda, \omega}, \varphi_{\lambda, -\theta}>$$

The crucial fact is the following lemma.

**LEMMA C.** If $\lambda \neq$ eigenvalue,

$$S(\lambda, \theta, \omega) = -\frac{\lambda}{2} (\theta - \omega)^2 \int_{\Omega} e^{-i \sqrt{\lambda} (\theta - \omega) x} \, dx$$

$$+ \int_{\Omega} e^{-i \sqrt{\lambda} (\theta - \omega) x} q(x) \, dx$$

$$- (R(\lambda) q \varphi_{\lambda, \omega}, q \varphi_{\lambda, -\theta}),$$

where $R(\lambda) = (-\Delta + q - \lambda)^{-1}$.

Note that the above expression is similar to the $S$-matrix in scattering theory.

Now, we recall the Born approximation.

Let $\mathbb{R}^n \ni \xi \neq 0$ be arbitrarily fixed. Take $\eta \in S^{n-1}$ such that $n^\perp \xi$. For a a large parameter $N$, we define
\[
\begin{aligned}
\theta_N &= C_N \eta + \xi / 2N, \quad C_N = (1 - |\xi|^2 / 4N^2)^{1/2}, \\
\omega_N &= C_N \eta - \xi / 2N, \\
\sqrt{t_N} &= N + i.
\end{aligned}
\]

They have the following properties:
\[
\begin{aligned}
\theta_N, \quad \omega_N &\in S^{n-1}, \\
\sqrt{t_N} (\theta_N - \omega_N) &\to \xi, \text{ as } N \to \infty, \\
\text{Im} \ t_N &\to \infty \text{ as } N \to \infty, \\
\text{Im} \sqrt{t_N} \theta_N, \text{ Im} \sqrt{t_N} \omega_N &\text{ are bounded.}
\end{aligned}
\]

Invoking these properties, one can easily show

**THEOREM D.**

\[
\lim_{N \to \infty} S(t_N, \theta_N, \omega_N) = -\frac{1}{2} \int_\Omega e^{-ix\xi} d\mathbf{x} + \int_\Omega e^{-ixq(x)} d\mathbf{x}.
\]

So, one can reconstruct the potential from \(S(\lambda, \theta, \omega)\).

Now, we prove Theorem B. \(N(\lambda)\) has, formally, the integral kernel:

\[
\sum_{i=1}^\infty \frac{1}{\lambda_i - \lambda} \frac{\partial \phi_i}{\partial \nu}(x) \frac{\partial \phi_i}{\partial \nu}(y),
\]

where \(\phi_i\) is the eigenfunction. In view of this expression one can show that the assumption of Theorem B implies

\[
\|N(\lambda, q_1) - N(\lambda, q_2)\|_{B(L^2(S^{n-1}))} \leq C/|\lambda|
\]

for large \(|\lambda|\). Theorem B then follows from this inequality and theorem D.

From the very proof, one can see that the potential is uniquely determined by the asymptotic properties of the eigenvalues and eigenfunctions.

**4. Variable coefficient case.** We briefly mention the variable coefficients case.
Consider the operator
\[ H = -\sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + q(x). \]

Assume that for \(|x| \leq N\), \(N\) is chosen large enough,
\[ \sup_{x \in \Omega} |\partial^\alpha_{x} (a_{ij}(x) - \delta_{ij})| \leq \delta < 1. \]

Then the above Theorem B, Lemma C, Theorem D also hold in this case. The proof relies on the method of asymptotic solutions and Fourier integral operators. Note that we are fixing \(a_{ij}\) and seeking \(q(x)\).

The above results may be extended to higher order elliptic operators and elliptic systems.

References