ARNE JENSEN

Stark hamiltonians with periodic potentials


<http://www.numdam.org/item?id=JEDP_1989___A11_0>
1. Introduction

Let \( H_0 = -\nabla^2 + Fx \) denote the free Stark Hamiltonian on \( L^2(\mathbb{R}^n) \). It is essentially selfadjoint on the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \). Let \( V \) be a real-valued bounded function. Then \( H = H_0 + V \) is selfadjoint with domain \( \mathcal{D}(H) = \mathcal{D}(H_0) \). The time-dependent Schrödinger equation \[ i \frac{d}{dt} \psi = H \psi, \quad \psi(0) = \psi_0, \] has the solution \( \psi(t) = e^{-itH} \psi_0 \). The questions we want to consider here are the following:

1° Describe the asymptotic behavior of \( \psi(t) = e^{-itH} \psi_0 \) as \( t \to \pm \infty \). This is in a general form the basic question in scattering theory.

2° Describe the spectrum \( \sigma(H) \) of \( H \) in detail, i.e. classify it according to the usual categories: point spectrum, continuous spectrum, absolutely continuous and singular continuous spectrum.

For the one-dimensional case we obtain fairly complete results, see section 4. For the higher dimensional case we obtain some general results, see section 3, and for the case of a half-crystal we obtain some interesting new results, see section 5.

This presentation is a preliminary report on \([J]\^1\). Concerning previous papers on Stark effect Hamiltonians with decaying potentials, we refer to the references given in \([J]\^2\).
2. Periodic potentials and lattices

A discrete subset of \( \mathbb{R}^n \) is called a lattice, if it can be represented in the form

\[
T = \{ k_1 a_1 + k_2 a_2 + \ldots + k_n a_n \mid k_1, \ldots, k_n \in \mathbb{Z} \},
\]

where \( a_1, \ldots, a_n \) are linearly independent vectors in \( \mathbb{R}^n \). A function \( V \) on \( \mathbb{R}^n \) is said to be periodic with the period lattice \( T \), if for all \( x \in \mathbb{R}^n \) and all \( \tau \in T \) we have

\[
V(x + \tau) = V(x).
\]

The position of the lattice \( T \) relative to the \( x_i \)-axis plays an important role in our study. We introduce the following definitions. Let \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \). The inner product on \( \mathbb{R}^n \) is denoted \( \langle \cdot, \cdot \rangle \).

**Definition 2.1.** (i) The lattice \( T \) is said to be irrational with respect to \( e_1 \), if the set \( \{ \langle e_1, \tau \rangle \mid \tau \in T \} \) is dense in \( \mathbb{R} \).
(ii) The lattice \( T \) is said to be rational with respect to \( e_1 \), if the set \( \{ \langle e_1, \tau \rangle \mid \tau \in T \} \) is discrete in \( \mathbb{R} \).

This is a classification, since it is easy to see that these are the only possibilities.

The translation group associated to the lattice is given by \( (U(\tau)f)(x) = f(x - \tau) \).

Assume that the potential \( V \) above is periodic with period lattice \( T \). Then we have the important relation

\[
(2.1) \quad U(\tau)HU(\tau)^{-1} = H - F \langle e_1, \tau \rangle.
\]

3. General spectral results

Throughout this section we assume that the potential \( V \) is a real-valued function with period lattice \( T \).

**Proposition 3.1.** Assume that \( T \) is irrational with respect to \( e_1 \). Then \( \sigma(H) = \mathbb{R} \).

**Proof:** By (2.1) \( \sigma(H) = \sigma(H) - F \langle e_1, \tau \rangle \). Since \( \sigma(H) \neq \emptyset \) and \( \{ F \langle e_1, \tau \rangle \mid \tau \in T \} \) is dense in \( \mathbb{R} \), the result follows. \( \square \)
Proposition 3.2. Assume that $T$ is rational with respect to $e_i$. Assume that $(e \in T \mid <e, e_i> = 0)$ is a sublattice of dimension $n-1$. Assume that

$$\sigma(-d^2/dx_i^2 + Fx_i + V(x_i, \bar{x})) = \mathbb{R}$$

for a dense set of $\bar{x} \in \mathbb{R}^{n-1}$. Then $\sigma(H) = \mathbb{R}$.

Remark 3.3. A sufficient condition for $\sigma(-d^2/dx_i^2 + Fx_i + V(x_i, \bar{x})) = \mathbb{R}$ is $V(x_i, \bar{x}) = (\partial/\partial x_i)W(x_i, \bar{x})$ for some bounded function $W$ with two bounded derivatives, see [[J]].

Proof: We use a direct integral decomposition with respect to the sublattice in the proposition and the the variable $\bar{x}$. The proof is somewhat long, so the details are omitted. See also section 5. □

Propositions 3.1 and 3.2 cover all cases for $n = 2$. For $n > 2$ not all cases are covered. We expect to find $\sigma(H) = \mathbb{R}$ in all cases. For a strong electric field it is easy to obtain a result on the type of spectrum.

Theorem 3.4. Assume $V$, $\partial V/\partial x_i$ and $\partial^2 V/\partial x_i^2$ continuous real-valued bounded functions on $\mathbb{R}^n$ and $\alpha_0 = \inf(F + (\partial V/\partial x_i)(x) \mid x \in \mathbb{R}^n) > 0$. Assume $\sigma(H) = \mathbb{R}$. Then the spectrum is purely absolutely continuous.

Proof: This result is an immediate consequence of Mourre's commutator method [[M]]. We use the conjugate operator $A = i\partial/\partial x_i$. The assumption implies that we have the Mourre commutator estimate

$$i[H, A] = F + \partial V/\partial x_i(x) \geq \alpha_0 I.$$ 

Furthermore, the second commutator $i[i[H, A], A] = \partial^2 V/\partial x_i^2$ is a bounded operator on $L^2(\mathbb{R}^n)$ by our assumption. Thus all the essential conditions for applying Mourre result are verified. The remaining technical conditions are easily verified. □

4. One-dimensional Stark Hamiltonians

In the one-dimensional case there are fairly complete answers to questions 1* and 2* in section 1. We shall briefly recall these results from [[J]]. Let us recall that the basic objects in the scattering theory for the pair of operators $H$ and $H_0$ are the wave operators $W_{\pm}(H, H_0) = s-lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$. One asks whether these
operators exist and are complete, i.e. \( \text{Ran}(W) = \mathcal{H}_p(H)^\perp \), the orthogonal complement to the closed subspace \( \mathcal{H}_p(H) \) spanned by the \( L^2 \)-eigenfunctions of \( H \). The point spectrum of \( H \) is denoted \( \sigma_p(H) \).

**Theorem 4.1.** \((n=1)\) Assume \( V \in C^2(\mathbb{R}) \), \( V \) periodic with period \( a \), and 
\[
\int_0^a V(x) \, dx = 0.
\]
Then \( W(H, H_0) \) exist and are unitary.

**Theorem 4.2.** \((n=1)\) Assume \( V = V_1 + V_2 \), where \( V_1 \) satisfies the assumptions of the previous theorem and \( V_2 \) satisfies 
\[
V_2(x) = O(|x|^{-1/2-\varepsilon}) \quad \text{as } x \to \infty,
\]
\[
V_2(x) = O(|x|^{-1/2-\varepsilon}) \quad \text{as } x \to -\infty \quad \text{for some } \varepsilon > 0.
\]
Then \( W(H, H_0) \) exist and are complete. Furthermore, \( \sigma_p(H) \) is discrete in \( \mathbb{R} \).

**Theorem 4.3.** \((n=1)\) Assume \( V = W'' \), where \( W \) is a real-valued bounded function with four bounded derivatives. Then \( W(H, H_0) \) exist and are unitary.

Theorem 4.1 is of the expected type. It shows that the crystal becomes "transparent" with respect to the time evolution, if one waits a long time. Theorem 4.2 shows that we can add "impurities" (in the form of \( V_2 \)) and retain the same result, except the possible occurrence of a discrete set of embedded eigenvalues.

Theorem 4.3 shows that the same result holds, even for sums of periodic potentials and for a large class of almost-periodic functions. For example, one can take
\[
V(x) = \int_{\mathbb{R}} e^{i\omega x} \, d\mu(\omega)
\]
where \( \mu \) is a Borel measure satisfying
\[
\int_{\mathbb{R}} (\omega^{-2} + \omega^2) \, d|\mu|(\omega) < \infty.
\]
As a special case we can take
\[
V(x) = \sum_{k=1}^{\infty} a_k \sin(\omega_k x)
\]
with
5. The half-crystal model

We now consider the case where the crystal fills up half the space. Half-solids have been briefly considered in [S]. Here we add a constant electric field orthogonal to the surface directed into the empty part of space. The results below show that after a long time an electron will eventually move freely, irrespective of the initial position.

Let $V_i$ be a periodic function on $\mathbb{R}^n$ with period lattice $T = \mathbb{Z} \times \tilde{T}$, where $\tilde{T}$ is a lattice in $\mathbb{R}^{n-1}$. We assume $V_i \in C^2(\mathbb{R}^n)$. Let $\chi$ be a cutoff function, i.e. $\chi \in C^\infty(\mathbb{R})$ realvalued, $0 \leq \chi(x_i) \leq 1$, $\chi(x_i) = 0$ for $x_i < -\delta$, and $\chi(x_i) = 1$ for $x_i > \delta$, where $\delta > 0$ is a fixed parameter. We take as our potential

$$ V(x) = \chi(x_i)V_i(x). $$

The main result is the following

**Theorem 5.1.** ($n \geq 2$) Let $V$ satisfy the assumptions above. Then $W_\pm(H, H_0)$ exist and are unitary. Consequently, $\sigma(H) = \sigma_{ac}(H) = \mathbb{R}$.

The proof of this theorem will only be sketched. Let $F_T$ denote a fundamental region for the lattice $\tilde{T}$, chosen diffeomorphic to the $n-1$-dimensional torus $\mathbb{T}^{n-1}$. The dual lattice is denoted $\tilde{T}^*$ and a fundamental region $F^*_T$, again chosen diffeomorphic to $\mathbb{T}^{n-1}$. We now use the Floquet–Bloch reduction, see for example [Sk] for details. There exists a unitary operator $W_T$ from $L^2(\mathbb{R}^n)$ to the direct integral space $\mathcal{H} = \int_{\tilde{T}} \oplus \mathcal{H}(k)dk$, where $k$ varies over $F_T^*$. The operator $H$ is transformed into $W_T HW_T^{-1} = \int_{\tilde{T}} \oplus H(k)dk$. In our case we do not reduce in $x_i$, so we have $\mathcal{H}(k) = L^2(\mathbb{R}) \otimes L^2(F_T)$ and $H(k) = \hat{\mathcal{R}}_0 \otimes I_2 + I_1 \otimes Q(k) + V(x_i, \tilde{x})$ with

$$ \hat{\mathcal{R}}_0 = -(d^2/dx_i^2) + Fx_i \text{ on } L^2(\mathbb{R}) \text{ and } Q(k) = (-i\nabla_{\tilde{x}} - k)^2 \text{ on } L^2(F_T) \text{ with periodic boundary conditions. Here } k \in F_T^*. $$

The main step is the following lemma.

$\sum_{k=1}^{\infty} |a_k| (\omega^{-2} + \omega^2) < \infty.$
Lemma 5.2. The wave operators $\mathcal{U}_k (\mathcal{H}(k), H_0(k))$ exist and are unitary on $\mathcal{H}(k)$, $k \in \mathbb{F}^*$.

To prove this lemma, we verify the conditions in the abstract theorems in [J]$_2$. The proof of absence of embedded eigenvalues requires a separate argument. Details can be found in [J]$_3$.

References


