JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

NICOLAS LERNER

# Sufficiency of condition ($\psi$) for local solvability in two dimensions

*Journées Équations aux dérivées partielles* (1987), p. 1-23

<http://www.numdam.org/item?id=JEDP_1987____A2_0>

# Sufficiency of Condition ($\psi$) for Local Solvability in Two Dimensions

NICOLAS LERNER[1]
Purdue University

## Introduction

In this paper, we establish the existence of local solutions of the equation

$$Pu = f \quad ,$$

when $P$ is a classical pseudo-differential operator in two dimensions, of principal type, of order $m$, which satisfies condition ($\psi$): the imaginary part $p_2$ of the principal symbol of $P$ does not change sign from – to + along any oriented bicharacteristic of the real part $p_1$ of the principal symbol. Let us recall briefly part of the well-known history of this problem. Nirenberg and Treves [8] proved the sufficiency of condition ($P$) (i.e. $p_2$ does not change sign along the bicharacteristic of $p_1$, which is equivalent to ($\psi$) for differential operators) for local solvability in the analytic case. The analyticity assumption was removed by R. Beals and C. Fefferman [2] who proved local existence of $H^{s+m-1}$ solutions for $H^s$ right-hand side. Using a propagation of singularities argument, Hörmander [4] proved local existence of $C^\infty$ solutions for $C^\infty$ right-hand sides and obtained also semi-global existence results ([4], Chapter 26 in [5]).

Nirenberg and Treves [8] also conjectured the necessity of condition ($\psi$) for local solvability and proved its invariance (by multiplication by an elliptic factor). They proved the necessity of condition ($P$) in the differential case. Later on, Moyer [7] gave a proof of the necessity of condition ($\psi$) for local solvability in two dimensions. Hörmander (Corollary 26.4.8 in [5]) fully proved the necessity of condition ($\psi$).

Summing up, Nirenberg-Treves' conjecture for local solvability of pseudo- differential equations of principal type (that is, condition ($\psi$) is equivalent to local solvability) is proved for differential operators. On the other hand, the necessity of condition ($\psi$) is established

[1]This research was partially supported by NSF Grant DMS 8601755

for pseudo- differential equations, but the sufficiency remains open. Here we prove that sufficiency holds in *two* dimensions.

Our proof relies at a first level on a generalization of a Nirenberg-Treves' energy estimate (cf. e.g. th. 28.6.1 in [5]). Let us say briefly that these authors proved an estimate for an operator,

$$\frac{d}{dt} + A(t)B \qquad ,$$

where $A(t), B$ were bounded operators in a Hilbert space $H$, with $A(t) \leq 0$. We use here the fact that it is possible to derive an estimate for

$$\frac{d}{dt} + A(t)B(t) \qquad ,$$

provided that the sign of the operator $B(t)$ (in the spectral sense) is *non-decreasing* (and $A(t) \leq 0$). At a second level, the specificity of the two-dimensional case allows us to use a factorization of our operator.

As it is clear through the title of this paper, we focused our attention on the local solvability problem (Th. 1.2.1 below). Nevertheless, we think that the energy estimates (lemmas 2.3.5 and 3.3.1 below) may have their own interest.

The paper is organized as follows.

1. Statement of the main results

 1.1 Notations

 1.2 Theorem

 1.3 Remarks

2. Some Hilbertian Lemmas

 2.1 Notations

 2.2 The Nirenberg-Treves Estimate

 2.3 A New Energy Estimate

3. Pseudo-differential Operators

 3.1 Factorization

 3.2 Non Homogeneous Reduction

 3.3 Summarizing the Reductions

4. Proof's End

 4.1 The $(\overline{\psi})$ Condition

 4.2 Specificity of the Factorization

 4.3 Final $L^2$ Estimate

## §1. Statement of the Main Results

### 1.1 Notations.

Let $\Omega$ be a $C^\infty$ $n$-dimensional manifold and $P$ be a properly supported (see e.g. def. 18.1.21 in [5]) pseudo-differential operator on $\Omega$ (see e.g. def 18.1.20 in [5]) with an homogeneous principal symbol of degree $m, p = p_1 + ip_2 (p_1, p_2$ real-valued). Assume moreover that $P$ is of principal type, i.e. $H_p \wedge L \neq 0$, where $L$ is the Liouville vector field. Our main assumption will be that $p$ satisfies condition $(\psi)$ (see def. 26.4.6 in [5]).

### 1.2 Results.

THEOREM 1.2.1. *Let $\Omega, P$ as above (section 1.1) with $n = 2$ and $x_0 \in \Omega$. Then for each $s$, there is a neighborhood $\Omega_s$ of $x_0$ such that*

$$Pu = f$$

*has a solution $u \in H^{s+m-1}_{loc}(\Omega_s)$ for every $f \in H^s_{loc}(\Omega_s)$.*

It was shown by Nirenberg and Treves [8] that this theorem can be reduced by localization and homogeneous canonical transformation to an analogous statement for a first-order pseudo-differential operator of the following form:

$$\frac{1}{i}\frac{\partial}{\partial t} + iq(t, x, D_x) \tag{1.1}$$

where $q(t, x, \xi) \in C^\infty(\mathbf{R}_t \times \mathbf{R}^{n-1}_x \times \mathbf{R}^{n-1}_\xi)$ is real-valued, positively homogeneous of degree one for $|\xi| \geq 1$, i.e.

$$\text{if } \rho \geq 1 \text{ and } |\xi| \geq 1, q(t, x, \rho\xi) = \rho q(t, x, \xi), \tag{1.2}$$

and such that

$$|(D_t^\kappa D_x^\alpha D_\xi^\beta q)(t, x, \xi)| \leq C_{\alpha\beta\kappa}(1 + |\xi|)^{1-|\beta|}. \tag{1.3}$$

Moreover, condition $(\psi)$ in that framework can be expressed as follows:

for each fixed $(x, \xi) \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ the function $t \mapsto q(t, x, \xi)$ does not change sign from $-$ to $+$ as $t$ increases. These non-trivial reductions are now classical and we refer the reader to the theorem 21.3.6 in [5] (which follows from the Malgrange preparation theorem ([6], th. 7.5.6 in [5]) and to the Egorov theorem ([3], th. 25.3.5 in [5]). Using this background we are reduced to prove the next theorem, dealing with an estimate for the transposed operator.

Let us first state a definition.

DEFINITION 1.2.2. *We shall say that $q$ is a normalized $(\overline{\psi})_M$ function if $q \in C^\infty(\mathbf{R}_t \times \mathbf{R}_x^{n-1} \times \mathbf{R}_\xi^{n-1})$, is real-valued, satisfies (1.2), is supported in*

$$\Gamma_0 = \mathbf{R}_t \times \{x, |x| \leq 1\} \times \left[\{\xi \neq 0, |\frac{\xi}{|\xi|} - \xi_0| \leq 1\} \cup \{\xi, |\xi| \leq 1\}\right], \tag{1.4}$$

*with $\xi_0 = (0, ..., 0, 1) \in \mathbf{R}^{n-1}$, is such that*

$$\max_{\substack{|\xi| \leq 2 \\ t, x}} |(D_\tau^\kappa D_x^\alpha D_\xi^\beta q)(t, x, \xi)| \leq C_{\kappa\alpha\beta},$$

*with*

$$\max_{|\alpha|+|\beta| \leq M} C_{0\alpha\beta} \leq 1. \tag{1.6}$$

*Moreover we assume that*

$$q(t, x, \xi) > 0 \textit{ implies } q(s, x, \xi) \geq 0 \textit{ if } s \geq t. \tag{1.7}$$

THEOREM 1.2.3. *Assume $n = 2$. There exist $C_0, M_0, T_0$ positive ("universal") constants, such that, if $q$ is a normalized $(\overline{\psi})_M$ function (definition 1.2.2) with $M \geq M_0$, if $u \in S(\mathbf{R}_t \times \mathbf{R}_x^{n-1}), u(t, x) = 0$ when $|t| \geq T$ and $0 < T \leq T_0$, we have*

$$C_0 \|D_t u + iq(t, x, D_x)u\|_{L^2(\mathbf{R}^n)} \geq T^{-1} \|u\|_{L^2(\mathbf{R}^n)}. \tag{1.8}$$

*1.3 Remarks*: a. The symbol $\tau + iq$ satisfies condition $(\overline{\psi})$ (see def. 26.4.6 in [5]). The estimate stated in theorem 1.2.3 implies a local solvability result for an operator with principal symbol $\tau - iq$; the lower order terms are unimportant because of the "large" constant $T^{-1}$ in (1.8).

b. It would have been possible to state the result in the theorem 1.2.3 by saying that the constant $T_0$ depends only on a finite fixed number of semi-norms of $q(t,\cdot,\cdot)$ (i.e. the $C_{0\alpha\beta}$). Let us first remark that if $q(t,x,\xi) \in C^\infty(\mathbf{R}_t \times \mathbf{R}_x^{n-1} \times \mathbf{R}_\xi^{n-1})$ is real-valued, satisfies (1.2) and (1.7) is supported in $\Gamma_0$ (1.4) and is such that

$$(1.9) \qquad |(D_t^\kappa D_x^\alpha D_\xi^\beta q)(t,x,\xi)| \le C_{\kappa\alpha\beta}(1+|\xi|)^{1-|\beta|},$$

then, setting $\omega_0 = 3 \max\limits_{|\alpha|+|\beta|\le M_0} C_{0\alpha\beta}$ and $Q(t,x,\xi) = \omega_0^{-1} q(\omega_0^{-1}t,x,\xi)$, we obtain that $Q$ is a normalized $(\overline{\psi})_{M_0}$ function. Then, if $v(t,x) \in \mathcal{S}(\mathbf{R}_t \times \mathbf{R}_x^{n-1})$ is 0 for $|t| \ge \widetilde{T}$, with $0 < \widetilde{T} \le \omega_0^{-1} T_0$, the function $u(t,x) = v(t\omega_0^{-1},x)\omega_0^{-\frac{1}{2}}$ is in $\mathcal{S}(\mathbf{R}^n)$ and is 0 for $|t| \ge \omega_0 \widetilde{T}$, with $\omega_0 \widetilde{T} \le T_0$. Applying theorem 1.2.3, to $Q$ and $u$, we get

$$C_0 \|(D_t v)(t\omega_0^{-1},x)\omega_0^{-\frac{3}{2}} + i\omega_0^{-1}(q(t,x,D_x)v)(\omega_0^{-1}t,x)\omega_0^{-\frac{1}{2}}\|_{L^2} \\ \ge \omega_0^{-1}\widetilde{T}^{-1}\|v\|_{L^2},$$

that is

$$(1.10) \qquad C_0\|D_t v + iq(t,x,D_x)v\|_{L^2} \ge \widetilde{T}^{-1}\|v\|_{L^2}.$$

Now from (1.10) it is possible to remove the assumption of support in (1.4) by using a classical pseudo-differential (homogeneous) partition of unity. Eventually the theorem 1.2.3 can be extended to the following result.

THEOREM 1.3.1. *Assume* $n = 2$.

*Let* $q(t,x,\xi) \in C^\infty(\mathbf{R}_t \times \mathbf{R}_x^{n-1} \times \mathbf{R}_\xi^{n-1})$ *real valued such that (1.2) and (1.7) are fulfilled and*

$$\max_{\substack{t\in\mathbf{R}, x\in\mathbf{R}^{n-1} \\ \xi\in\mathbf{R}^{n-1}, |\xi|\le 2}} |(D_x^\alpha D_\xi^\beta q)(t,x,\xi)| = C_{\alpha\beta} < +\infty$$

*for each* $(\alpha,\beta)$. *Then there exist some positive constants* $C_0, T_0$, *depending on a finite fixed numbers of semi-norms of* $q$ *(the* $C_{\alpha\beta}$*), such that, for every* $u \in \mathcal{S}(\mathbb{R}^n)$ *such that* $u = 0$ *for* $|t| \geq T$, *with* $0 < T \leq T_0$,

$$\|D_t u + iq(t,x,D_x)u\|_{L^2(\mathbb{R}^n)} \geq C_0^{-1} T^{-1} \|u\|_{L^2(\mathbb{R}^n)} \quad .$$

c. These theorems have been proved by R. Beals and C. Fefferman [2] (in any dimension) replacing condition $\overline{(\psi)}$ by condition $(P)$ (i.e. (1.7) by $q(t,x,\xi)q(s,x,\xi) \geq 0$).

d. The specificity of the two-dimensional case comes from the fact that $q(t,x,\xi)$ satisfies (1.2) where $\xi$ is *one* real variable.

## §2. Some Hilbertian Lemmas

The lemmas stated and proved in this section are a natural sequel of the Nirenberg-Treves estimate ([8], Th. 26.8.1 in [5]). They are of course unrelated to the assumption $n = 2$.

**2.1 Notations.** Let $H$ be a complex Hilbert space and $\mathcal{L}(H)$ the Banach algebra of bounded linear operators on $H$. An operator $L \in \mathcal{L}(H)$ is selfadjoint (or real) if $L = L^*$ and antiselfadjoint (or purely imaginary) if $L^* = -L$. For $L \in \mathcal{L}(H)$, we set

$$(2.1) \qquad ReL = \frac{1}{2}(L + L^*), ImL = \frac{1}{2i}(L - L^*).$$

If $J$ is real and $K$ purely imaginary in $\mathcal{L}(H)$,

$$(2.2) \qquad [J, K] = JK - KJ = 2\mathrm{Re}(JK) \qquad \text{(thus real).}$$

Finally, if $B \in \mathcal{L}(H)$ is real, using its spectral decomposition, we get

$$(2.3) \qquad \begin{cases} B_\pm = \frac{1}{2}(|B| \pm B) \quad , & |B| = B_+ + B_- \\ B = B_+ - B_- \quad , & B_\pm \geq 0, \\ B_+B_- = B_-B_+ = 0 \quad . & \end{cases}$$

We can define the sign $S$ of $B, S = S_+ - S_-$, and we have

$$(2.4) \qquad S\pm \geq 0 \quad , \quad S_+S_- = S_-S_+ = 0, Id = S_+ + S_-,$$

$$(2.5) \qquad S_\pm B_\mp = 0 = B_\pm S_\mp \quad , \quad S_\pm B_\pm = B_\pm S_\pm = B_\pm,$$

and thus

$$(2.6) \qquad \begin{cases} S_+B = BS_+ = B_+ \quad , & S_-B = BS_- = -B_-, \\ SB_+ = B_+ = B_+S, & SB_- = -B_- = B_-S. \end{cases}$$

We get also

$$SB_{\pm}^{\frac{1}{2}} = \pm S_{\pm} B_{\pm}^{\frac{1}{2}} = \pm B_{\pm}^{\frac{1}{2}} = B_{\pm}^{\frac{1}{2}} S. \tag{2.7}$$

DEFINITION 2.1.1. *Let $B$ real $\in \mathcal{L}(H)$. An operator $S = S_{+} - S_{-}$ is a pseudo-sign of $B$ if the properties (2.4), (2.5) and (2.7) (and thus (2.6)) are fulfilled. Note that, from (2.4), we obtain that a pseudo-sign is unitary and selfadjoint.*

### 2.2 The Nirenberg-Treves Estimate.

We shall not recall here the basic estimate proved by Nirenberg and Treves in [8], used also by R. Beals and C. Fefferman [2]. The reader can consult the theorem 26.8.1 in [5]. Nevertheless we'll recall the basic lemma leading to this estimate, namely the lemma 26.8.2 in [5].

LEMMA 2.2.1. *(Nirenberg-Treves [8], lemma 28.6.2 in [5].) Let $A$ and $B \in \mathcal{L}(H), B$ real. Then, with operator norms,*

$$\| \left[B_{\pm}^{\frac{1}{2}}, \left[B_{\pm}^{\frac{1}{2}}, A\right]\right] \| \leq \frac{10}{3} \|A\|^{\frac{1}{4}} \|[B,A]\|^{\frac{1}{2}} \|[B,[B,A]]\|^{\frac{1}{4}}. \tag{2.8}$$

*Remark 2.2.2*: Let $g$ be a fixed positive definite quadratic form on $\mathbf{R}^{2n}$ and set, for $T \in \mathbf{R}^{2n}$,

$$g^{\sigma}(T) = \sup_{g(U)=1} [T,U]^2,$$

where $[,]$ is the symplectic form on $\mathbf{R}^{2n}$. We define $\lambda = (\sup \frac{g(T)}{g^{\sigma}(T)})^{-\frac{1}{2}}$ and assume $\lambda \geq 1$. A function $\theta \in C^{\infty}(\mathbf{R}^{2n})$ belongs to $S(\lambda^{\mu}, g)$ if the following estimates hold:

$$|\theta^{(k)}(X)T^k| \leq \lambda^{\mu} C_k(\theta) g(T)^{\frac{k}{2}}.$$

Assume that $A, B$ are operators on $L^2(\mathbf{R}^n)$ with Weyl symbols $a, b$ such that $a \in S(\lambda, g), b \in S(1, g), b$ real-valued, $a, b$ supported in a $g$-ball of radius $\leq 1$. (See section 18.5 in [5]). The lemma 2.2.1, in spite of the fact that $(b^w)_+^{\frac{1}{2}}$ is not a pseudo-differential operator, shows that the $\mathcal{L}(L^2)$ norm of $\|[B_+^{\frac{1}{2}},[B_+^{\frac{1}{2}},A]]\|$ is bounded by $\omega_n C_{p(n)}(a)C_{p(n)}(b)\lambda^{\frac{1}{4}}.1.\lambda^{-\frac{1}{4}} = \omega_n C_{p(n)}(a)C_{p(n)}(b)$, where $\omega_n, p(n)$ depend only on the dimension. The important fact here is of course that the right-hand side of (2.8) is estimated by semi-norms of the symbols, independently of $\lambda$.

### 2.3 A New Energy Estimate.

We are interested in this section in an ordinary differential equation in a Hilbert space $H$. Let $Q(t) \in \mathcal{L}(H)$ function of $t \in \mathsf{R}$. We'll study the operator

$$\frac{d}{dt} - Q(t) \tag{2.9}$$

acting on $u : \mathsf{R} \longrightarrow H$, continuously differentiable.

We shall assume

$$Q(t) = \mathsf{R}e(B(t)A(t)), \tag{2.10}$$

where $A(t), B(t)$ are in $\mathcal{L}(H)$, real, uniformly continuous as functions of $t \in \mathsf{R}$, with

$$A(t) \geq 0. \tag{2.11}$$

Our main assumption is that there exists a $t$-weakly measurable pseudo-sign of $B(t)$ (definition 2.1.1), $M(t)$, which is non-decreasing, i.e.

$$(M(t_2) - M(t_1))(t_2 - t_1) \geq 0, t_1, t_2 \text{ real }. \tag{2.12}$$

Note that if $B$ is time-independent $M = sign(B)$ satisfies obviously (2.12).

LEMMA 2.3.1. *Let $u \in C_0^1(\mathsf{R}, H)$ (continuously differentiable functions from $\mathsf{R}$ to $H$ with compact support). Then, for a t-weakly measurable real operator $M(t) \in \mathcal{L}(H)$ satisfying (2.12) and*

$$\sup_t \|M(t)\| < +\infty, \tag{2.13}$$

we have

$$\text{(2.14)} \qquad \mathrm{Re} \int < M(t)\dot{u}(t), u(t) >_H dt \leq 0,$$

($\dot{u} = \frac{du}{dt}$, $< \bullet, \bullet >_H$ *inner product in* $H$).

Before proving this lemma, let us remark that (2.13) means, after a formal integration by parts, that the operator $\dot{M}$ is non-negative (it should be a natural consequence of (2.12)). It would have been possible to use the theory of distributions valued in $\mathcal{L}(H)$[9] to prove this fact. Here we've preferred a simple elementary argument dealing with the very weak assumption of regularity on $M(t)$ (boundedness and measurability). Note that (2.13) is automatically fulfilled if $M(t)$ is a pseudo-sign of $B(t)(M(t)$ unitary is a consequence of (2.4)). Let us now prove the lemma.

By the Lebesgue's dominated convergence theorem, we have, setting

$$\equiv (u) = \mathrm{Re} \int < M(t)\dot{u}(t), u(t) > dt,$$

$$\equiv (u) = \lim_{h \to 0} (\equiv_h (u) = \mathrm{Re} \int < M(t)h^{-1}(u(t+h) - u(t)), u(t) > dt).$$

But, we have the identity

$$\begin{aligned} \equiv_h (u) = \mathrm{Re} &\int < h^{-1}(M(t-h) - M(t))u(t), u(t) > dt \\ + \mathrm{Re} &\int < M(t)u(t+h), h^{-1}(u(t) - u(t+h)) > dt. \end{aligned}$$

Using (2.12), we get

$$\equiv_h (u) \leq \mathrm{Re} \int < M(t)u(t+h), h^{-1}(u(t) - u(t+h)) > dt,$$

and applying the Lebesgue's dominated convergence theorem, we obtain

$$\equiv (u) \leq - \equiv (u),$$

which completes the proof of the lemma.

LEMMA 2.3.2. *Let $A, B$ be real operators in $\mathcal{L}(H), A \geq 0$ and $M$ a pseudo-sign of $B$ (cf. def. 2.1.1). Then*

$$\mathrm{Re}(M\mathrm{Re}(AB)) \geq -\frac{10}{3}\|A\|^{\frac{1}{4}}\|[A,B]\|^{\frac{1}{2}}\|[B,[B,A]]\|^{\frac{1}{4}}.$$

*Proof*: Set up

$$\begin{aligned} L = M\mathrm{Re}AB &= \frac{1}{2}MAB_+^{\frac{1}{2}}B_+^{\frac{1}{2}} - \frac{1}{2}MAB_-^{\frac{1}{2}}B_-^{\frac{1}{2}} \\ &+ \frac{1}{2}MB_+^{\frac{1}{2}}B_+^{\frac{1}{2}}A - \frac{1}{2}MB_-^{\frac{1}{2}}B_-^{\frac{1}{2}}A. \end{aligned}$$

So using (2.7) for the pseudo-sign $M$ we get

$$\begin{aligned} L = &\frac{1}{2}M[A, B_+^{\frac{1}{2}}]B_+^{\frac{1}{2}} + \frac{1}{2}B_+^{\frac{1}{2}}AB_+^{\frac{1}{2}} \\ &- \frac{1}{2}M[A, B_-^{\frac{1}{2}}]B_-^{\frac{1}{2}} + \frac{1}{2}B_-^{\frac{1}{2}}AB_-^{\frac{1}{2}} \\ &+ \frac{1}{2}B_+^{\frac{1}{2}}[B_+^{\frac{1}{2}}, A] + \frac{1}{2}B_+^{\frac{1}{2}}AB_+^{\frac{1}{2}} \\ &+ \frac{1}{2}B_-^{\frac{1}{2}}[B_-^{\frac{1}{2}}, A] + \frac{1}{2}B_-^{\frac{1}{2}}AB_-^{\frac{1}{2}}. \end{aligned}$$

Now the assumption $A \geq 0$ yields

$$\begin{aligned} 2\mathrm{Re}L \geq \mathrm{Re}\Big\{&M\left[[A, B_+^{\frac{1}{2}}], B_+^{\frac{1}{2}}\right] + MB_+^{\frac{1}{2}}[A, B_+^{\frac{1}{2}}] \\ &- M\left[[A, B_-^{\frac{1}{2}}], B_-^{\frac{1}{2}}\right] - MB_-^{\frac{1}{2}}[A, B_-^{\frac{1}{2}}] \\ &+ B_+^{\frac{1}{2}}[B_+^{\frac{1}{2}}, A] + B_-^{\frac{1}{2}}[B_-^{\frac{1}{2}}, A]\Big\}. \end{aligned}$$

Now remarking that $B_\pm^{\frac{1}{2}}$ is real, $[B_\pm^{\frac{1}{2}}, A]$ purely imaginary (by (2.2)) we get, by using (2.2) and (2.7),

$$\begin{aligned} \mathrm{Re}L \geq \mathrm{Re}&\left\{\frac{1}{2}M\left[\left[A, B_+^{\frac{1}{2}}\right], B_+^{\frac{1}{2}}\right] - \frac{1}{2}M\left[\left[A, B_-^{\frac{1}{2}}\right], B_-^{\frac{1}{2}}\right]\right\} \\ &+ \frac{1}{2}\left[B_+^{\frac{1}{2}}, \left[A, B_+^{\frac{1}{2}}\right]\right] + \frac{1}{2}\left[B_-^{\frac{1}{2}}, \left[A, B_-^{\frac{1}{2}}\right]\right] \\ &+ \frac{1}{2}\left[B_+^{\frac{1}{2}}, \left[B_+^{\frac{1}{2}}, A\right]\right] + \frac{1}{2}\left[B_-^{\frac{1}{2}}, \left[B_-^{\frac{1}{2}}, A\right]\right], \end{aligned}$$

which gives the result (the sum of the last four terms is 0 and $\|M\| \leq 1$), as a consequence of (2.8) in lemma 2.2.1.

LEMMA 2.3.3. *Let $A, B$, real operators in $\mathcal{L}(H), M$ a pseudo-sign of $B$. Then*

$$\|[M,[M,\mathrm{Re}(BA)]]\| \le \frac{80}{3}\|A\|^{\frac{1}{4}}\|[B,A]\|^{\frac{1}{2}}\|[B,[B,A]]\|^{\frac{1}{4}} \tag{2.16}$$

*Proof*: Notice that $[M,\mathrm{Re}(BA)]$ is purely imaginary (2.2) so $[M,[M,\mathrm{Re}(BA)]]$ is real (2.2).

Let us set

$L = 2[M,[M,\mathrm{Re}BA]]$,

$L = M(M(BA+AB)-(BA+AB)M)-(M(BA+AB)-(BA+AB)M)M.$

So, using $M^2 = I$ (cf definition 2.1.1), we get

$$\frac{1}{2}L = BA + AB - MBAM - MABM = 2\mathrm{Re}(BA - MBAM).$$

Moreover, we have, using (2.6),

$$MBAM = B_+^{\frac{1}{2}}B_+^{\frac{1}{2}}AM + B_-^{\frac{1}{2}}B_-^{\frac{1}{2}}AM,$$

and

$$\begin{aligned} MBAM = {} & \left[B_+^{\frac{1}{2}},\left[B_+^{\frac{1}{2}},A\right]\right]M + B_+^{\frac{1}{2}}AB_+^{\frac{1}{2}}M + \left[B_+^{\frac{1}{2}},A\right]B_+^{\frac{1}{2}}M \\ & + \left[B_-^{\frac{1}{2}},\left[B_-^{\frac{1}{2}},A\right]\right]M + B_-^{\frac{1}{2}}AB_-^{\frac{1}{2}}M + \left[B_-^{\frac{1}{2}},A\right]B_-^{\frac{1}{2}}M. \end{aligned}$$

Now, using (2.7), we obtain

$$\begin{aligned} MBAM = {} & \left[B_+^{\frac{1}{2}},\left[B_+^{\frac{1}{2}},A\right]\right]M + \left[B_-^{\frac{1}{2}},\left[B_-^{\frac{1}{2}},A\right]\right]M \\ & + B_+^{\frac{1}{2}}AB_+^{\frac{1}{2}} + \left[B_+^{\frac{1}{2}},A\right]B_+^{\frac{1}{2}} \\ & - B_-^{\frac{1}{2}}AB_-^{\frac{1}{2}} - \left[B_-^{\frac{1}{2}},A\right]B_-^{\frac{1}{2}}. \end{aligned}$$

Thus, we get

$$\begin{aligned} -BA + MBAM = {} & \left[B_+^{\frac{1}{2}},\left[B_+^{\frac{1}{2}},A\right]\right]M + \left[B_-^{\frac{1}{2}},\left[B_-^{\frac{1}{2}},A\right]\right]M \\ & + 2\left[B_+^{\frac{1}{2}},A\right]B_+^{\frac{1}{2}} + AB_+ \\ & - 2\left[B_-^{\frac{1}{2}},A\right]B_-^{\frac{1}{2}} - AB_- \\ & - BA. \end{aligned}$$

So, using (2.2), we have

$$\frac{1}{4}L = \mathrm{Re}(BA - M\dot{B}AM) =$$
$$= - \mathrm{Re}\left\{\left[B_+^{\frac{1}{2}}, \left[B_+^{\frac{1}{2}}, A\right]\right] M + \left[B_-^{\frac{1}{2}}, \left[B_-^{\frac{1}{2}}, A\right]\right] M\right\}$$
$$- \left[\left[B_+^{\frac{1}{2}}, A\right], B_+^{\frac{1}{2}}\right] + \left[\left[B_-^{\frac{1}{2}}, A\right], B_-^{\frac{1}{2}}\right]$$
$$+ \mathrm{Re}([B, A]) \qquad \text{(Note that this last term is 0).}$$

The lemma 2.2.1 and the previous identity give (2.16).

LEMMA 2.3.4. *Let $u(t), M(t)$ as in lemma 2.3.1 with $M(t)^2 = Id$. Let $\omega(t)$ be a smooth real valued function. Then*

$$-2\mathrm{Re}\int < (E(t)M(t)E(t))\dot{u}(t), u(t) >_H dt \tag{2.17}$$
$$\geq \int (2\dot{\omega}(t) - 2|\omega(t)\dot{\omega}(t)|)\|u(t)\|_H^2,$$

*with $E(t) = Id + \omega(t)M(t)$.*

We can note here that this lemma is a minoration of $\frac{d}{dt}(EME) = 2\mathrm{Re}(\dot{E}ME) + E\dot{M}E$ (cf remark after the lemma 2.3.1). Let us prove the lemma:

We have $\theta(u) = -2\mathrm{Re}\int < (E(t)M(t)E(t))\dot{u}(t), u(t) >_H dt$, and using $M(t)^2 = I$, we get

$$\theta(u) = -2\mathrm{Re}\int < \left(1 + \omega(t)^2\right) M(t)\dot{u}(t), u(t) >_H dt$$
$$- 2\mathrm{Re}\int 2w(t)\langle \dot{u}(t), u(t)\rangle_H dt.$$

So, by the lemma 2.3.1

$$\theta(u) \geq -2\mathrm{Re}\int < M(t)\frac{d}{dt}(\omega(t)u(t)), \omega(t)u(t) >_H dt$$
$$+ 2\mathrm{Re}\int \langle M(t)\omega(t)\dot{\omega}(t)u(t), u(t) >_H$$
$$+ \int 2\dot{\omega}(t)\|u(t)\|_H^2 dt.$$

Using again the lemma 2.3.1 we get

$$\theta(u) \geq \int (2\dot{\omega}(t) - 2|\omega(t)\dot{\omega}(t)|)\|u(t)\|_H^2 dt,$$

which completes the proof.

Let us introduce some notations. We shall note by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{R}, H)$ with inner product

$$(u,v)_{\mathcal{H}} = \int_{\mathbb{R}} \langle u(t), v(t)\rangle_H dt. \tag{2.18}$$

If $A(t), B(t)$ are operators in $\mathcal{L}(H)$, functions of $t \in \mathbb{R}$ we shall note

$$\nu_0(A,B) = \max_{t\in\mathbb{R}} \frac{10}{3} \|A(t)\|^{\frac{1}{4}} \|[A(t), B(t)]\|^{\frac{1}{2}} \|[B(t), [B(t), A(t)]]\|^{\frac{1}{4}} \tag{2.19}$$

The main result of this section is the following:

LEMMA 2.3.5. *Let $H$ be an Hilbert space and $A(t), B(t)$ real operators in $\mathcal{L}(H)$, uniformly continuous as functions of $t \in \mathbb{R}$, with $A(t) \geq 0$. We assume that there exists a $t$-weakly measurable pseudo-sign of $B(t)$ (definition 2.1.1), $M(t)$, satisfying (2.12). We set $Q(t) = \mathrm{Re}(A(t)B(t)), P = \frac{1}{i}\frac{d}{dt} + iQ(t)$. Then if $0 < \delta \leq 2^{-5}\nu_0(A,B)^{-1}$ (cf. (2.19)), $u \in C_0^1(\mathbb{R}, H)$ with diameter (supp $u$) $\leq \delta$*

$$\|Pu\|_{\mathcal{H}} \geq 2^{-4}\delta^{-1}\|u\|_{\mathcal{H}}, \qquad \textit{(cf. (2.18))}. \tag{2.20}$$

*Proof*: Let us set $N(t) = E(t)M(t)E(t)$, where $M(t)$ is a $t$-weakly measurable pseudo-sign of $B(t)$ satisfying (2.12),

$$E(t) = Id + \omega(t)M(t), \tag{2.21}$$

with $\omega(t) = t\delta^{-1}\chi(t\delta^{-1})$ where $\chi \in C_0^\infty(\mathbb{R}, [0,1]), \chi(s) \equiv 1$ on $|s| \leq \frac{1}{2}, \chi(s) \equiv 0$ on $|s| \geq 1, \delta$ a positive parameter.

Let us compute, for $u \in C_0^1(\mathbf{R}, H), u = 0$ if $|t| \geq \frac{\delta}{2}$,

$$
(2.22) \qquad
\begin{aligned}
&2\mathrm{Re}(Pu, iEMEu)_{\mathcal{H}} = A(u),\\
&A(u) = \int 2\mathrm{Re}\langle \frac{1}{i}\dot{u}(t) + i(\mathrm{Re}(A(t)B(t)))u(t), iE(t)M(t)E(t)u(t)\rangle_H dt,\\
&A(u) = -2\mathrm{Re}\int \langle \dot{u}(t), E(t)M(t)E(t)u(t)\rangle_H dt\\
&\qquad + 2\mathrm{Re}\int \langle E(t)M(t)E(t)Q(t)u(t), u(t)\rangle_H dt.
\end{aligned}
$$

We obtain easily that, for $|t| \leq \frac{\delta}{2}$,

$$
2\dot{\omega}(t) - 2|\omega(t)|\ |\dot{\omega}(t)| \geq \delta^{-1},
$$

and thus, using the lemma 2.3.4 we get

$$
(2.23) \qquad
\begin{aligned}
A(u) \geq \delta^{-1}\|u\|_{\mathcal{H}}^2 &+ 2\mathrm{Re}\int \langle E(t)M(t)[E(t), Q(t)]u(t), u(t)\rangle_H dt\\
&+ \int \langle 2\mathrm{Re}\{M(t)\mathrm{Re}(A(t)B(t))\}(E(t)u(t)), (E(t)u(t))\rangle_H dt
\end{aligned}
$$

Now using the lemma 2.3.2, to handle the last term in the right-hand side of (2.23), and $\|E(t)\|_{\mathcal{L}(H)} \leq 2$, we get, as $M(t)^2 = Id$, using (2.21), (2.2) for the first integral in (2.23)

$$
\begin{aligned}
A(u) \geq \delta^{-1}\|u\|_{\mathcal{H}}^2 &- 8\nu_0\|u\|_{\mathcal{H}}^2\\
&+ \int \langle [M(t), [M(t), \mathrm{Re}(A(t)B(t))]]u(t), u(t)\rangle_H \omega(t) dt.
\end{aligned}
$$

The lemma 2.3.3 gives (as $0 \leq \omega(t) \leq 1) A(u) \geq (\delta^{-1} - 8\nu_0 - 8\nu_0)\|u\|_{\mathcal{H}}^2$.

So the Cauchy-Schwarz inequality applied to (2.22) eventually gives

$$
\|Pu\|_{\mathcal{H}} \geq 2^{-4}\delta^{-1}\|u\|_{\mathcal{H}},
$$

if $\delta \leq 2^{-5}\nu_0^{-1}$, that is the result.

## 3. Pseudo-Differential Operators

### 3.1 Factorization.

We have to prove the estimate (1.8) for a normalized $(\overline{\psi})_M$ function $q$ (definition 1.2.2). Let us set

$$\omega \in C_0^\infty\left(\mathrm{R}^1_\xi, [0,1]\right), = \begin{cases} 1 \text{ for} & \xi \geq 2 \\ 0 \text{ for} & \xi \leq 1. \end{cases} \tag{3.1}$$

We can note that, using (1.2) (recall $n = 2$), we have

$$q(t,x,\xi) = q(t,x,1)\xi\omega(\xi) + (1-\omega(\xi))q(t,x,\xi). \tag{3.2}$$

But the assumption (1.4) on the support of $q$ and (3.1) imply that the symbol $(1 - \omega(\xi))q(t,x,\xi) = 0$ if $\xi \geq 2$ or $\xi \leq -1$. It is thus clearly sufficient to prove (1.8) for the symbol $q(t,x,1)\xi\omega(\xi)$, because the $\mathcal{L}(L^2(\mathrm{R}^{n-1} = \mathrm{R}^1))$ norm of the operator with symbol $(1-\omega(\xi))q(t,x,\xi)$ is estimated (uniformly in $t$) by a finite number (depending only on the dimension) of $C_{0\alpha\beta}$ (cf. (1.5),(1.6)). But using the classical quantization of symbols, we have

$$\begin{aligned} Op(q(t,x,1)\xi\omega(\xi)) &= Op(q(t,x,1))Op(\xi\omega(\xi)) \\ &= \frac{1}{2}(Op(q(t,x,1))Op(\xi\omega(\xi)) + Op(\xi\omega(\xi))Op(q(t,x,1))) \\ &+ \frac{1}{2}[Op(q(t,x,1)), Op(\xi\omega(\xi))]. \end{aligned} \tag{3.3}$$

But the last term (the bracket) is a pseudo-differential operator of order $O$ whose semi-norms can be estimated by those of $q$. As above we can neglect this term and prove (1.8) replacing $q(t,x,D_x)$ by

$$\begin{aligned} Q &= \frac{1}{2}(Op(q(t,x,1))Op(\xi w(\xi))) \\ &+ \frac{1}{2}(Op(\xi w(\xi))Op(q(t,x,1))) \end{aligned} \tag{3.4}$$

### 3.2 Non-Homogeneous Reduction.

Let us begin by a

CLAIM 3.2.1. *It is enough to prove (1.8)( with $Q$ (cf. (3.4)) replaced by*

$$Q_\nu = Re(Op(q(t,x,1))Op(\xi w(\xi)\theta_\nu(\xi))) \tag{3.5}$$

*$\nu$ integer $\geq 1, \theta_\nu \in C_0^\infty(\mathsf{R},[0,1])$, such that*

$$\forall\alpha, \exists C_\alpha, \forall\nu, \forall\xi, |\theta_\nu^{(\alpha)}(\xi)| \leq C_\alpha 2^{-\nu|\alpha|}, \tag{3.6}$$

*(The important fact is that the $C_\alpha$ are independent of $\nu$), and*

$$2^{\nu-1} \leq |\xi| \leq 2^{\nu+1} \text{ when } \xi \in \text{ supp } \theta_\nu. \tag{3.7}$$

*Under the preceding assumptions the only point to be checked is that the constants $C_0, T_0, M_0$ are independent of $\nu$.*

This claim is a very particular case of the following lemma.

LEMMA 3.2.2. *Let us assume that $q(t,x,\xi)$ is real valued $\in S(\lambda(x,\xi), g_{x,\xi})$ (see definition 18.4.2 in [5]) uniformly in $t \in \mathsf{R}$, with $\lambda(x,\xi) = \sup_T \left(\frac{g_{x,\xi}(T)}{g^\sigma_{x,\xi}(T)}\right)^{-1/2} \geq 1$, and $g$ $\sigma$-temperate (definition 18.5.1 in [5]). In order to prove (1.8) with $q(t,x,D_x)$ replaced by $q(t,x,\xi)^w$ (see (18.5.3) in [5]), it is enough to prove it for $q$ supported in a $g$-ball of radius $\leq 1$ and $q \in S(\lambda, g)$ with constants $C_0, T_0, M_0$ independent of $\lambda$.*

We shall not prove this lemma, which is a straightforward application of sections 18.5 and 18.6 in [5]. Note that the proof involves some symbols with values in $\ell^2$ (as in the proof of lemma 18.6.10 in [5]) and can be obtained also by using the notion of confined symbols introduced in [1].

### 3.3 Summarizing the Reductions.

In order to prove the theorem 1.2.3 (and thus the theorem 1.2.1) it is enough to prove the following lemma.

LEMMA 3.3.1. *Let $b(t,x)$ real valued $\in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$ such that $(b(t,x) = q(t,x,1))$*

$$\sup_{t,x} |(D_x^\alpha b)(t,x)| \leq C_\alpha \quad , \text{ with } \max_{|\alpha| \leq M} C_\alpha \leq 1, \tag{3.8}$$

$$b(t,x) > 0 \text{ implies } b(s,x) \geq 0 \text{ if } s \geq t. \tag{3.9}$$

*Let $a_\lambda(t,x,\xi)$ non-negative $\in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^{n-1} \times \mathbb{R}_\xi^{n-1})$, such that*

(3.10)

$$\sup_{t,x,\xi} |(D_x^\alpha D_\xi^\beta a_\lambda)(t,x,\xi)| \lambda^{|\beta|-1} \leq C_{\alpha\beta}, \text{ with } \max_{|\alpha|+|\beta| \leq M} C_{\alpha\beta} \leq 1, \lambda \text{ a parameter } \geq 1.$$

*Assume moreover that $b$ and $a_\lambda$ are zero on $|t| \geq 1, b(t,x) = 0$ for $|x| \geq 1$,*

$$a_\lambda(t,x,\xi) = 0 \text{ for } |x|^2 + \lambda^{-2}|\xi|^2 \geq 1. \tag{3.11}$$

*Then, there exist $C_0, T_0, M_0$, such that, if $M \geq M_0$, the conclusion of the theorem 1.2.3 holds, with $q(t,x,D_x)$ replaced by $\mathrm{Re}\, b^w a_\lambda^w$.*

## §4. Proof's End

### 4.1 The $\overline{\psi}$ Condition.

Let $b(t,X) \in C^\infty(\mathbf{R}_t \times \mathbf{R}^d_X)$ supported in $|t| \le 1$, such that

$$b(t,X) > 0 \text{ implies } b(s,X) \ge 0 \text{ if } s \ge t. \tag{4.1}$$

Then we define

$$t_+(X) = \inf\{t, t \in [-1,+1], b(t,X) > 0\}, \tag{4.2}$$

$$t_-(X) = sup\ \{t, t \in [-1,+1], b(t,X) < 0\} \tag{4.3}$$

If $b(t,X) \le 0$ (resp. $\ge 0$) for all $t$ we'll set $t_+(X) = +1$ (resp. $t_-(X) = -1$).

We have the following obvious

CLAIM 4.1.1. *For each $X \in \mathbf{R}^d$*

$$-1 \le t_-(X) \le t_+(X) \le +1, \tag{4.4}$$

$$b(t,X) \le 0 \qquad \text{for } t \le t_-(X), \tag{4.5}$$

$$b(t,X) = 0 \qquad \text{for } t_-(X) \le t \le t_+(X), \tag{4.6}$$

$$b(t,X) \ge 0 \qquad \text{for } t \ge t_+(X). \tag{4.7}$$

*An important consequence of this claim is that*

$$\begin{gathered} b(t,X) sgn(t - t_+(X)) = |b(t,X)|, \\ \text{with } s(t,X) = sgn(t - t_+(X)) = \begin{cases} +1 & \text{if} \quad t > t_+(X) \\ -1 & \text{if} \quad t \le t_+(X) \end{cases} \end{gathered} \tag{4.8}$$

Note also that $s$ is a *bounded measurable* function, and

$$\frac{\partial s}{\partial t} \geq 0 \text{ as a measure.} \tag{4.9}$$

**4.2 Specificity of the Factorization.**

Let us set, for $b$ as in lemma 3.3.1,

$$B(t) = Op(b(t,x))(= b^w) \tag{4.10}$$

as a bounded operator in the Hilbert space $H = L^2(\mathbb{R}^{n-1})$ (multiplication by the function $b(t,x)$). We define, with $s(t,x)$ defined in (4.8),

$$M(t) = Op(s(t,x)), \tag{4.11}$$

$M(t)$ is the bounded operator in $H = L^2(\mathbb{R}^{n-1})$ defined by the multiplication by the $L^\infty$ function $s(t,x)(t \in \mathbb{R}, x \in \mathbb{R}^{n-1})$. The operator $M(t)$ is obviously a $t$-weakly measurable pseudo-sign of $B(t)$ (def. 2.1.1) and satisfies (2.12) (it is obvious because only the $x \in \mathbb{R}^{n-1}$ variable is involved; it would not have been the case if $b$ was depending on $(x,\xi)$ and so $s$: the quantization of $s$ by (4.11) would have been possible – a $\mathcal{S}'$ distribution can be quantized – but the $L^2$ boundedness of $M$ and overall the non-negativity of $MB$ would have been false).

**4.3 Final $L^2$ estimate.**

Let us set

$$A(t) = a_\lambda^w(t) + C_1, \tag{4.12}$$

where $C_1$ is a constant such that

$$a_\lambda^w(t) + C_1 \geq 0 \qquad \text{on } L^2(\mathbb{R}^{n-1})$$

(which is a consequence of the Gårding inequality for the metric $dx^2 + \frac{d\xi^2}{\lambda^2}$, cf. Th. 18.6.7 in [5]; $C_1$ depends only on a finite number of $C_{\alpha\beta}$ in (3.10)). The calculus of pseudo-differential operators in the metric $dx^2 + \frac{d\xi^2}{\lambda^2}$ (cf. Th. 18.5.4 in [5]) and the $L^2$ boundedness of symbols with weight 1 (cf. Th. 18.6.3 in [5]) allows us to compute $\nu_0(A, B)$ given by (2.19).

We have

$$\nu_0(A,B) \leq \frac{10}{3}\left(C_2\lambda^1\right)^{\frac{1}{4}}\left(C_3\lambda^0\right)^{\frac{1}{2}}\left(C_4\lambda^{-1}\right)^{\frac{1}{4}} = C_5,$$

where $C_2, C_3, C_4, C_5$ depends only on a finite number of $C_\alpha$ in (3.8) and $C_{\alpha\beta}$ in (3.10). Then, using the lemma 2.3.5 (for $H = L^2(\mathrm{R}^{n-1})$, $A(t)$ given in (4.12), $B(t)$ given by (4.10), $M(t)$ by (4.11)) we obtain the lemma for $\mathrm{Re}(b^w(a_\lambda^w + C_1))$ and we can neglect the term $C_1$ by using the large constant $\delta^{-1}$ in (2.20). The proof is complete.

## REFERENCES


[1] BONY, LERNER, *Quantification asymptotique et microlocalisations d'ordre supérieur*, Exposés II, III (1986-87), Ecole Polytechnique. Seminaire EDP, and article to appear.

[2] R. BEALS, C. FEFFERMAN, *On local solvability of linear partial differential equations*, Ann. of Math. **97** (1973), 482-498.

[3] EGOROV, *The canonical transformations of pseudo-differential operators*, Uspehi Mat. Nauk. **24:5** (1969), 235-236.

[4] HÖRMANDER, *Propagation of singularities and semi-global existence theorems for (pseudo-)differential operators of principal type*, Ann. of Math. **108** (1978), 569-609.

[5] HÖRMANDER, The analysis of linear partial differential operators (1985), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo. 4 volumes.

[6] MALGRANGE, Ideals of differentiable functions (1966), Tata Institute, Bombay, and Oxford University Press.

[7] MOYER, *Local solvability in two dimensions: necessary conditions for the principle-type case*, University of Kansas. Mimeographed manuscript.

[8] NIRENBERG, TREVES, *On local solvability of linear partial differential equations I. Necessary conditions. II. Sufficient conditions. Correction.* Comm. Pure Appl. Math., . **23**, 1-38 and 459-509 (1970); **24**, 279-288 (1971).

[9] L. SCHWARTZ, *Theorie des distributions à valeurs vectorielles*, I. Ann. Inst. Fourier **7** (1957), 1-141, Grenoble.