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Estimates for a number of negative eigenvalues of the Schrödinger operator with intensive magnetic field


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1. In this lecture I give estimates from above and from below for a maximal dimension \( N \leq \infty \) of a linear subspace \( \mathcal{L} \subset C_0^\infty(X) \) on which a quadratic form

\[
Q(u) = \int \left[ \sum_{j,k} g^{jk}(D_j - V_j)u \cdot (D_k - V_k)u + V|u|^2 \right] dx
\]

(1)

is negative definite; here \( X \) is a domain in \( \mathbb{R}^d, d = 3, g^{jk} = g^{kj}, V_j, V \) are real-valued and \( g^{jk}, g^{jk} V_j, g^{jk} V_k + V \in L^1_{\text{loc}}(X) \); we use Einstein' summation rule. Then \( Q \) is correctly defined on \( C_0^\infty(X) \). Let us assume that

(\( H_1 \)) \( c^{-1} \leq |2|^{-2} g^{jk}(x) \sum_j \sum_k \leq c \quad \forall x \in X, \zeta \in \mathbb{R}^d \setminus 0 \)

and there are given functions \( \delta, \zeta, \xi, \xi_0, \xi_1, ... \) on \( X \) such that

(\( H_2 \)) \( \zeta_m(x) \geq 0, \xi(x) \geq 0, |\xi(x) - \xi(y)| \leq |x - y| \)

and for every point \( y \in X' = \frac{1}{2} X \), \( \xi(y) > 1 \) in \( X \cap B(y, \xi(y)) \) the following conditions are fulfilled:

(\( H_3 \)) \( c^{-1} \leq \zeta_m(x) / \zeta_m(y) \leq c, \quad \sum_j \zeta_m \leq c \zeta / \delta \), \( m = 0, 1, ..., \)

\( \xi_1 \geq \zeta / \delta \), \( \zeta \geq \zeta_2 \geq 1 / \delta \),

(\( H_4 \)) \( |D^B g^{jk}| \leq c \delta^{-|B|} \),

\( |D^B F_j| \leq c \zeta_1 \delta^{-|B|} \),

\( |D^B V| \leq c \zeta_2 \delta^{-|B|} \) \( \forall B: |B| \leq K < \infty \),

(\( H_5 \)) \( \delta X \cap B(y, \delta(y)) = \{ x_k = z(x_k) \} \cap B(y, \delta(y)) \) with

\( |D^B z| \leq c \delta^{1-|B|} \) \( \forall B: |B| \leq K \)

where \( B(y, \delta(y)) \) is an open ball with a center \( y \) and a radius \( \delta(y) \), \( x_k = (x_1, x_3) \) etc, \( k = k(y) = 1, 2, 3 \) in \( H_5 \), \( F_j = i \varepsilon^{jk} D_k V_1 \)
are components of the vector intensity of the magnetic field, $E_{jk}^{1}$ is absolutely skew-symmetric pseudo-tensor with $E_{123} = 1/\sqrt{g}$, $g = \det (g_{jk})$, $(g_{jk}) = (g_{jk})^{-1}$. Let $F = (g_{jk}F_{jF}^{k})^{1/2}$ be a scalar intensity of the magnetic field. Let us further assume that

(H₆) $\forall \gamma \in \{x', \varphi_{1} > c \frac{\varphi}{\gamma}\}$ in $X \cap B(y, \gamma(y))$

$$F \geq c^{-1} \varphi_{1},$$

(H₅) $\forall \gamma \in \{x', \varphi_{1} > c \varphi/\gamma, \gamma + F \leq \varepsilon \varphi_{2}\}$

$$B(y, \gamma(y)) \subseteq X$$

with $\varepsilon > 0$.

Moreover, let us assume that for every $\gamma \in X''' = \{x', \varphi_{1} > c \varphi_{2}^{2}, \varphi_{2} < c_{1}^{-1} \varphi_{1}, \gamma + F \leq \varepsilon \varphi_{2}^{2}\}$ in $B(y, \gamma(y))$ the following inequalities are fulfilled:

(H₇) $|D^{\beta}(\gamma + (2j+1)F)| \leq c \varphi_{2}^{2} \gamma - |\beta| \quad \forall \beta: |\beta| \leq K$

with $j = j(y) \in \mathbb{Z}^{+}, c_{1} > 8c$; then $X'''$ is a union of the disjoint domains $X_{j}''$.

Let

$$\gamma_{1} = \gamma^{2} |\nabla(\gamma/F)| \varphi_{1} / \varphi_{2}^{2} + \gamma \min_{j \in \mathbb{Z}^{+}} |\gamma + (2j+1)F|^{1/2} / \varphi_{2}$$

on $x'$ and $\gamma_{2} = \gamma_{1} \varphi_{1} / \varphi_{2}$ on $X'''$.

Finally, let us assume that

(H₈) $Q(u) \geq c^{-1} \int (|\nabla u|^{2} - W|u|^{2}) \, dx \quad \forall u \in C_{0}^{\infty}(x'')$

where $x'' = \{x, \varphi < 2 \}$, $W \in L^{1}_{1oc}(X), W > 0$.

Our the first principal result is

**Theorem 1.** Let conditions (H₄)-(H₈) be fulfilled. Then

$$\mathcal{N} - CR_{1} - C'R_{2} \leq N \leq \mathcal{N} + C(R_{1} + R_{3}) + C'R_{2}$$

where
\[ \mathcal{N} = \left( \frac{1}{2} \pi^{2} \right) \sum_{j=0}^{\infty} \int_{X'} (V + (2j+1)F)^{-1/2} F \sqrt{g} \, dx , \]

\[ z^\pm = \max (\pm \varepsilon, 0), \quad R_1 = \sum_{t=1}^{5} R_{1t}, \quad R_2 = \sum_{t=1}^{2} R_{2t}, \]

\[ R_{11} = \int_{X'} \int \xi_1^2 \gamma^{-1} \, dx , \]
\[ \{ x', x^{''''} \}, \quad V + F \leq \varepsilon \xi_2 \gamma \]

\[ R_{12} = \int_{X'} \int \xi_1 \gamma_1^{-1} \, dx , \]
\[ \{ x', x^{''''} \}, \quad V + F \leq \varepsilon \xi_2, \quad \gamma_1 \geq \xi^{-1/2+\sigma'} \gamma_1^{1/2+\sigma'} \]

\[ R_{13} = \int_{X'} \int \xi_1 \xi_2^{1/2+\sigma'} \gamma^{-1/2+\sigma'} \, dx , \]
\[ \{ x', x^{''''} \}, \quad V + F \leq \varepsilon \xi_2, \quad \gamma_1 \leq \xi^{-1/2+\sigma'} \gamma_1^{1/2+\sigma'} \]

\[ R_{14} = \sum_{j=0}^{\infty} \int_{X'} \int \xi_1^2 \gamma_2^{-1} \, dx , \]
\[ \{ x^{''''''}, V + (2j+1)F \leq \varepsilon \xi_2^2, \quad \gamma_2 \geq \xi_2^{-1/2+\sigma'} \gamma_1^{1/2+\sigma'} \}

\[ R_{15} = \sum_{j=0}^{\infty} \int_{X'} \int \xi_2 \xi_2^{1/2+\sigma'} \gamma^{-1/2+\sigma'} \, dx , \]
\[ \{ x^{''''''}, V + (2j+1)F \leq \varepsilon \xi_2^2, \quad \gamma_2 \leq \xi_2^{-1/2+\sigma'} \gamma_1^{1/2+\sigma'} \}

\[ R_{21} = \int_{X'} \int \xi_1^3 \xi_1^{-s} \gamma^{-2s} \, dx , \]

\[ R_{22} = \int_{X^{''''}} \int \xi_2^2 \xi_2^{-s} \gamma^{-1-s} \, dx , \]

\[ R_3 = \int_{X^{''''}} \int \xi_3^{3/2} \, dx , \]
here and in what follows \( \xi > 0, \sigma' > 0, s \) are arbitrary and \( C = C(c), C' = C'(c, c_2, \xi, \sigma', s) \), \( K = K(\sigma; s) \) in \( (H_4, H_2) \).

Remark 2. If conditions \((H_1)-(H_6)\) are fulfilled and if 
\( \mathcal{N} + R_1 + R_2 + R_3 < \infty \) then \( Q \) is semi-bounded from below on \( L^2(X) \) and hence it generates a self-adjoint Schrödinger operator 
\( A = (D_j - \mu V_j)g^{jk}(D_k - \mu V_k) + V \) on \( X \) with the Dirichlet boundary condition; then \( N \) is a dimension of its invariant negative subspace.

Theorem 1 is a more refined and general version of the principal theorems announced in \([1,2]\). Moreover, under a certain condition of a global nature concerning integral curves of the vector field \((F_1,F_2,F_3)\) one can derive a more precise estimates. If \( A \) depends on parameters then theorem 1 implies asymptotics of \( N \) with respect to these parameters (see e.g. \([1,2]\) ).

2. The following assertion is the crucial step in the proof of theorem 1:

**Theorem 3.** Let 
\[
A_{\mu,h} = (hD_j - \mu V_j)g^{jk}(hD_k - \mu V_k) + V
\]
with the Dirichlet boundary condition be a self-adjoint semi-bounded Schrödinger operator with the discrete spectrum and with the polynomial growth of the eigenvalue counting function \( N(\lambda) \) as \( \lambda \to \infty \); here \( h \in (0,1], \mu > 1 \); let \( e(x,y,\lambda,\mu,h) \) be a Schwartz kernel of its spectral projector. Let \( \gamma \subseteq X \) and in \( B(y,1) \subseteq X \) conditions \((H_1)-(H_6)\) be fulfilled with \( \gamma = S_\gamma = S_1 = 1 \); moreover, let \( \psi \in C_0^\infty(B(y,1/2)), 0 \leq \psi \leq 1, |D^\beta \psi| \leq c_2 \psi^{-|\beta|} \forall \beta : |\beta| \leq K. \)

Then

(i) The following estimate holds:
\[
\mathcal{R} = \left| \int (e(x,x,0,\mu,h) - S(x,\mu,h)h^{-d}) \psi^2(x) \, dx \right| \leq ...
\]
\[ Ch^{-2}(1 + \eta h) \int_{B(y,1)} \gamma_1^{-1} \, dx + \begin{cases} B(y,1), & \gamma_1 \geq h^{1/2-\sigma'} \end{cases} \]

\[ |\eta h^{1/2-\sigma'} \text{ mes } \begin{cases} B(y,1), & \gamma_1 \leq h^{1/2-\sigma'} \end{cases} | + C'h^{-1} \]

where

\[ S(x, \eta h) = (1/2 \pi)^{d-1} \sum_{j=0}^{\infty} (V + (2j+1)\eta h^F)^{(d-2)/2} \mu h^F \sqrt{g}. \]

(i) If \( V + \mu h^F \geq \mathcal{E} \) in \( X \cap B(y,1) \) (and not necessarily \( B(y,1) \subset X \) here) then

\[ e(x,x,0,\mu,h) \leq C'h^S \mu^{-S} \forall x \in X \cap B(y,1/2). \]

(iii) If \( V = -(2j+1)F + \zeta^2 V' \) with \( \zeta \in (h,1] \) and \( j \in \mathbb{Z}^+ \)

and if \( V' \) satisfies \((H_4)_3\) with \( \gamma = \zeta = 1 \) then for \( \mu = h^{-1} \)

\[ R \leq Ch^{-2}(1 + \int_{B(y,1)} \gamma_2^{-1} \, dx + \begin{cases} B(y,1), & \gamma_2 \geq (h/\zeta)^{1/2-\sigma'} \end{cases} \]

\[ (h/\zeta)^{-1/2-\sigma'} \text{ mes } \begin{cases} B(y,1), & \gamma_2 \leq (h/\zeta)^{1/2-\sigma'} \end{cases} + C'h^{-1} \zeta^{-1} ; \]

here \( \zeta \geq \zeta \) in the definition of \( \gamma_2 \).

(iv) Moreover, if \( V' \geq \mathcal{E} \) in \( B(y,1) \) then

\[ e(x,x,0,\mu = h^{-1},h) \leq C'h^{-2}(h/\zeta)^S \forall x \in B(y,1/2). \]

The proof of theorem 3 is complicated and it is based on the quasiclassical microlocal analysis of the non-stationary Schrödinger equation with parameters \( \mu, h \). Without conditions \((H_6)',(H_6)\) the similar assertion holds for \( \mu \leq c, d \geq 2 \) and it is the basis of the proof of the principal theorems of \([3]\). When all these asymptotics are established we generalize them first to arbitrary balls and then complete the proof of theorem 1 by means of an appropriate partition of unity and Rosenblum variational estimate for an eigen-
value counting function for operator generated by $Q$ in $L^2(X'', Jdx)$ with an admissible weight function $J$. One can find the similar procedure in [4].

3. Let us consider now the case $d = 2$. Certainly, now $F = F^3$, $F^3 = i(D_1V_2 - D_2V_1)/\sqrt{g}$. This case is not completely investigated yet. However I have proved the following

**Theorem 4.** Let $d = 2$ and all the conditions of theorem 3 be fulfilled. Then

(i) If $Y_1 \geq c^{-1}$ then

$$\mathcal{R} \leq c \mu^{-1} + c'h^{-1}.$$

(ii) Assertion (ii) of theorem 3 holds.

(iii) If $V = -(2j+1)\mu h F + \zeta^2 V'$ with $\zeta \in (h, 1]$ and $j \in \mathbb{Z}^+$ and if $V'$ satisfies (H4)3 with $\zeta = \varrho = 1$ and if $\delta_2 \geq c^{-1}$ (here $\delta_2 = \zeta$ in the definition of $Y_2$) then

$$\mathcal{R} \leq c \zeta^{-2} + c'h \zeta^{-1}.$$

(iv) On the other hand, if

$$|V + (2j+1)\mu h F| \geq \zeta^2 \geq c \mu^{-2} \quad \forall j \in \mathbb{Z}^+ \quad \forall x \in B(y,1)$$

then

$$|e(x,x, \lambda_2, \mu, h) - e(x,x, \lambda_1, \mu, h)| \leq c'(h/\zeta)^5$$

$$\forall x \in B(y,1/2) \quad \forall \lambda_1, \lambda_2 \in (-\zeta^2/c, \zeta^2/c)$$

(3).

(v) Moreover, if

$$|V + (2j+1)\mu h F + \mu^{-2} RV^2/8F| \geq \zeta^2 \geq c \mu^{-2}$$

$$\forall j \in \mathbb{Z}^+ \quad \forall x \in B(y,1)$$

where $R$ is a scalar curvature associated with the metrics $(g_{jk}/F)$ and if $\mu^{-1} + \mu h \leq \delta = \delta(c, \varepsilon)$ then inequality (3) holds.

Let us note that in the two-dimensional case the presence of the intensive magnetic field can improve the remainder estimate; on the
other hand, in this case the gaps in the quasiclassical limit of the spectrum can appear. The role of Landau levels \( E_j = V + (2j+1)\mu \hbar F \) is more important in the two-dimensional case than in three-dimensional one; there is a correction \( \Delta E_j = \mu^{-2} R(E_j - V)^2/8F \) to these levels.

4. Finally, it is well-known that if \( X = \mathbb{R}^d \), \( d = 2, 3 \), \( g^{jk}, F^j \) are constant and \( V = 0 \) then

\[
e(x,x,\lambda,\mu,\hbar) =
\frac{(1/2 \pi^{d-1})}{\sum_{j=0}^{\infty} (\lambda - (2j+1)\mu \hbar F)_+^{(d-1)/2}} \mu^{1-d} F \sqrt{g};
\]

in particular,

\[
\sigma(A) = \sigma_{ac}(A) = \left[\mu \hbar F, \infty\right) \quad \text{for } d = 3,
\]

\[
\sigma(A) = \sigma_{ess}(A) = \sigma_{pp}(A) = \left\{ (2j+1)\mu \hbar F, j \in \mathbb{Z}^+ \right\}
\]

for \( d = 2 \).

References

1. V.Ivrii, Proc. ICM-86, Berkeley (to appear)