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Linear and nonlinear field equations


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The aim of this lecture is to illustrate how some recent geometric techniques which were usual to derive global existence and long time existence results for non-linear wave equations ([1], [2], [3]) can be applied to tensorial field equations. We limitate ourselves here in describing the results which we have obtained in collaboration with D. Christodoulou, to the linear Maxwell and Spin - 2 equations in Minkowski space (see [4]). The latter are a linearised version of the Einstein equations in vacuum and their study important in our attempt to prove the global non-linear stability of the Minkowski metric.

Consider the Minkowski space $\mathbb{R}^{3+1}$ with canonical coordinates $(x^\alpha) \; \alpha = 0,1,2,3$ and metric

$$\begin{equation}
\text{d}s^2 = \eta_{\alpha\beta} \text{d}x^\alpha \text{d}x^\beta
\end{equation}$$

where $\eta$ is the diagonal matrix with entries $(-1,1,1,1)$. The coordinate $x^0$ is usually denoted by $t$. The following vector fields are conformal killing i.e. vector fields $X$ so that $\mathcal{L}_X \eta$ is proportional to $\eta$.

(2i) The 4 generators of the translation group

$$T_\mu = \frac{\partial}{\partial x_\mu} \quad \mu = 0,1,2,3$$

(2ii) The 6 generators of the Lorentz group

$$\Omega_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}$$

where $x_\mu = \eta_{\mu\nu} x^\nu$
(2iii) The scaling vector field

\[ S = x^\mu \frac{\partial}{\partial x^\mu} \]

(2iv) The 4 accelerations vector fields

\[ K_\mu = -2x_\mu S + \langle x, x \rangle \frac{\partial}{\partial x^\mu} \]

with \( \langle x, x \rangle = \eta_{\mu\nu} x^\mu x^\nu \).

The Lie algebra \( \Pi \) generated by \( T, \Omega, S \) plays a very important role in what follows. Given a tensor \( U \) in \( \mathbb{R}^{3+1} \), we define the norms

\[ \|U(t)\|^2 = \sum_{t=1}^{s} \left| \mathcal{L}_{x_1} \ldots \mathcal{L}_{x_k} U(t, x) \right|^2 \, dx \]

with the sum taken over all generator \( x_1 \ldots x_k \), \( 0 \leq k \leq s \), of \( \Pi \).

Here, \( \mathcal{L}_{x_1} \ldots \mathcal{L}_{x_k} U \) denotes the repeated Lie derivatives of \( U \) with respect to \( x_1 \ldots x_k \) and \( |.| \) denotes the euclidian norm in \( \mathbb{R}^{3+1} \).

Also, \( x \) refers to \( x^1, x^2, x^3 \), \( dx = dx^1 dx^2 dx^3 \).

The Maxwell equations in \( \mathbb{R}^{3+1} \) apply to antisymmetric 2-tensors \( F_{\alpha\beta} \) which are required to satisfy the following two pairs of equations

\[ (M_1) \quad F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \]

\[ (M_{i1}) \quad F_{\alpha\beta}^{;\beta} = 0 \]

where \( F_{\alpha\beta;\gamma} \) denotes the covariant differentiation of \( F \) relative to the flat Minkowski metric. The energy momentum tensor of \( (M_1) \) is given by the 2-tensor
\[ Q_{\alpha\beta} = F_{\alpha\gamma} F_{\beta}^\gamma + *F_{\alpha\gamma} *F_{\beta}^\gamma \]

where \( *F \) is the Hodge dual of \( F \) i.e. \( *F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta} \) and \( \epsilon_{\alpha\beta\gamma\delta} \) the components of the volume 1-form of \( \mathbb{R}^{3+1} \). We remark that \( Q \) has the following properties.

- symmetric in \( \alpha, \beta \)
- traceless, i.e. \( \eta^{\alpha\beta} Q_{\alpha\beta} = 0 \)
- satisfies the positive energy condition i.e. given any two time-like vector fields \( X, Y \), \( \langle X, X \rangle < 0 \), \( \langle Y, Y \rangle < 0 \), both future oriented, we have:
  \[ Q(X, Y) = Q_{\alpha\beta} X^\alpha Y^\beta > 0 \]
- \( Q_{\alpha\beta} = 0 \)

The energy momentum tensor allows one to derive energy estimates for (M). Let \( F \) be a solution of (M), and consider \( X \) a time-like vector field. Let \( P^\alpha = Q_{\alpha\beta} X^\beta \) be the \( X \)-momentum of \( F \). Then,

\[ P^\alpha = \frac{1}{2} Q^{\alpha\beta} (X_{\alpha;\beta} + X_{\beta;\alpha}) \]

The expression \( X_{\alpha;\beta} + X_{\beta;\alpha} \) is precisely \( \Box_X \eta_{\alpha\beta} \) and thus proportional to \( \eta \), if we choose \( X \) to be conformal killing. On the other hand, since \( Q \) is traceless, we conclude that any choice of a conformal killing vector field leads to a conservation law in (5) i.e.

\[ P^\alpha = 0 \]

Integrating (5') on slots \([0, t] \times \mathbb{R}^3\) we infer that,

\[ \int_{t=\text{const}} Q(T_o, X) dx = \int_{t=0} Q(T_o, X) dx \]

where \( T_o = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t} \).
According the positive energy condition, $Q(T, X)$ is everywhere positive if $X$ is time-like. The only two choices of conformal killing time-like vector fields are $X = T_o$ and $X = K_o$. In fact, let $\bar{K}_o = K_o + T_o$.

Then, according to (6), we have

$$\int_{t=\text{const}} Q(T_o, \bar{K}_o) \, dx = \int_{t=0} Q(T_o, \bar{K}_o) \, dx$$

According to (6'), we introduce the norm

$$\|F(t)\|^2 \Pi_s = \left( \int_{\mathbb{R}^3} Q(T_o, \bar{K}_o) \, dx \right)^{1/2}$$

where $Q$ is the energy momentum tensor of $F$. Also, we define

$$\|F(t)\|^2 \Pi_{s,n} = \sum \|\ell_{X_1} \ldots \ell_{X_k} F(t)\|^2$$

with the sum extended over all choices of vector fields $X_1, \ldots, X_k$, $0 < k < s$, among the generators of $\Pi$. Now, due to the conformal equivalence of the equations (M), we can easily check that if $F$ is a solution, the so is $\ell_X F$ for any conformal vector field $X$. As a consequence, we conclude that

$$\|F(t)\|^2 \Pi_{s,n} = \|F(0)\|^2 \Pi_{s,n}$$

and finite if the right hand side is finite. Since the right hand side depends only on initial conditions for $F$ at time $t=0$, we conclude that $\|F(t)\|^2 \Pi_{s,n}$ can be made globally finite, by requiring appropriate conditions at infinity, for the initial data. Finally, we concise this feet to derive uniform decay properties for $F$. To state our theorem, we need to introduce null frames in $\mathbb{R}^{3+1}$. Thus, let $e^+ = \partial_t + \partial_2$, $e^- = \partial_t - \partial_2$ and $e_1, e_r$ vector fields orthogonal to $e^+, e^-$, and to each other, and of length one. The vector field $\frac{\partial}{\partial r}$ is the radial vector field $\frac{x_i}{|x|} \partial_i$ with $|x|^2 = \sum_{i=1}^{3} (x^i)^2$. 

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We decompose $F$ relative to the null frame according to:

$$
\alpha_A = F_{A^+}, \quad \alpha_\alpha = F_{A^-}, \quad A = 1, 2
$$

(9)

$$
\rho = F_{+-}, \quad \sigma = F_{+-}
$$

Here, $\alpha$ and $\alpha_\alpha$ are vectors tangent to the spheres $|x| = \text{const}$ in $\mathbb{R}^3$ while $\rho$ and $\sigma$ are scalars. Clearly they determine the full tensor $F$.

The finiteness of the norm $\|F(t)\|_{\Pi, s}$ can be used to prove the following

**Theorem 1**: Let $F$ be a solution of the Maxwell equations (M) with initial conditions at $t=0$ for which the norm $I_s = \|F(0)\|_{\Pi, s}^s$, $s \geq 2$, is finite, then

(i) \[ |F(t,x)| \leq C(1+t)^{-5/2} I_2 \]

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \leq \frac{t}{2} + 1$

(ii) \[ |\alpha(t,x)| \leq C(1 + |t - |x| |)^{-3/2} (1 + t + |x|)^{-1} I_2 \]

\[ |(\rho, \sigma)(t,x)| \leq C (1 + |t - |x| |)^{-1/2} (1 + t + |x|)^{-2} I_2 \]

\[ |\alpha(t,x)| \leq C (1 + t + |x|)^{-5/2} I_2 \]

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| > \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of $F$ in the interior, $|x| < \frac{t}{2} + 1$, or for the derivatives $\alpha, \alpha_\alpha, \rho, \sigma$ relative to the null frame $e_+, e_-, e_1, e_2$ in the exterior $|x| > \frac{t}{2} + 1$ (see [4]).
In the second part of this lecture, I will indicate how similar results, based on the same ideas, can be used to derive decay estimates for the Spin-2 equations. These are equations satisfied by 4-tensors $W_{\alpha \beta \gamma \delta}$ which have all the symmetry properties of the Riemann curvature tensor of metric satisfying the Einstein vacuum equations. Namely,

(i) \[ W_{\alpha \beta \gamma \delta} = -W_{\beta \alpha \gamma \delta} = -W_{\alpha \beta \delta \gamma} \]
\[ W_{\alpha \beta \gamma \delta} = W_{\gamma \delta \alpha \beta} \]

(ii) \[ W_{\alpha \beta \gamma \delta} + W_{\alpha \gamma \delta \beta} + W_{\alpha \delta \beta \gamma} = 0 \]

(iii) \[ W_{\alpha \beta} = 0 \]

The Spin-2 equations are

\[(\text{Sp}) \quad W_{\alpha \beta \gamma \delta;\epsilon} + W_{\alpha \beta \delta \epsilon;\gamma} + W_{\alpha \beta \epsilon \gamma;\delta} = 0 \]

As the Maxwell equations, the Spin-2 equations are conformal invariant. In particular, for any solution $W$ and any conformal vector field $\mathcal{E}_X$, $\mathcal{E}_X W$ is also a solution. What corresponds to the energy momentum tensor for the Maxwell equations is now a 4-tensor $Q$ defined by

\[(10) \quad Q_{\alpha \beta \gamma \delta} = \epsilon_{\mu \nu} W_{\alpha \mu \gamma \nu} W_{\beta \delta}^\nu + *W_{\alpha \mu \gamma \nu} *W_{\beta \delta}^\nu \]

with $*W_{\alpha \beta \gamma \delta} = \epsilon_{\mu \nu} W_{\mu \nu \gamma \delta}$ the Hodge dual of $W$.

One can prove that $Q$ satisfied the following properties

- $Q$ is symmetric and traceless relative to all pair of indices
- $Q$ satisfied the positive energy condition i.e. given any $X, Y$ time like and future oriented:

\[ Q(X, X, Y, Y) = Q_{\alpha \beta \gamma \delta} X^\alpha X^\beta Y^\gamma Y^\delta > 0 \]
whenever $W$ is a solution of (Sp).

One can now proceed as in the derivation of the energy identities for the Maxwell equations to show that

$$
\int_{t=\text{cst}e} Q(T_o, T_o, \vec{k}_o, \vec{k}_o) dx = \int_{t=0} Q(T_o, T_o, \vec{k}_o, \vec{k}_o) dx.
$$

Or, introducing the norm

$$
\|W(t)\|^\# = \left( \int_{\mathbb{R}^3} Q(T_o, T_o, \vec{k}_o, \vec{k}_o) dx \right)^{1/2}
$$

With $Q$ the energy momentum tensor of $W$, and also,

$$
\|W(t)\|^\#_{\Pi,s} + \left( \sum_{i=1}^{n} \left\| \sum_{k=1}^{s} W(t) \right\|^2 \right)^{1/2}
$$

for any generators $X_i, \ldots, X_{i_k}$, $0 \leq k \leq s$ of $\Pi$,

$$
\|W(t)^\#\|_{\Pi,s} = \|W(0)^\#\|_{\Pi,s}
$$

We now decompose $W$ relative to the same null frame introduced above, and introduce

$$
\alpha_{AB} = W_{A+B+}, \quad \alpha_{AB} = W_{A-B-},
$$

$$
\beta_A = \frac{1}{2} W_{A+}; +, \quad \beta_A = \frac{1}{2} W_{A-}; -
$$

$$
\rho = \frac{1}{4} W_{+}; +, \quad \sigma = \frac{1}{4} * W_{+}; +
$$

Clearly, $\alpha, \alpha, \beta, \beta, \rho, \sigma$ completely determine $W$, and we can prove the following
Theorem 2: Let $W$ be a solution of (Sp) with initial conditions at $t = 0$ for which $I_s = \|W(0)\|^s < +\infty$ for some $s > 2$. Then

(i) $|W(t,x)| \leq C (1+t)^{-\frac{7}{2}} I_2$

for any $t > 0$, $x \in \mathbb{R}^3$, $|x| < \frac{t}{2} + 1$

(ii) $|g(t,x)| \leq C (1 + |t - |x||)^{-5/2} (1 + t + |x|)^{-1} I_2$

$|\mathbf{g}(t,x)| \leq C (1 + |t - |x||)^{-3/2} (1 + t + |x|)^{-2} I_2$

$|\rho(t,x)| \leq C (1 + |t - |x||)^{-1/2} (1 + t + |x|)^{-3} I_2$

$|\alpha(t,x)| \leq C (1 + t + |x|)^{-7/2} I_2$

for any $t > 0$, $x \in \mathbb{R}^3$, $|x| \geq \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of $W$ (see [4]).

The spirit of these linear estimates can be adjusted to treat the non linear Einstein equations. This, I hope, will be done in a series of papers together with D. Christodoulou.

REFERENCES


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