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ANALYTIC APPROXIMATION FOR HOMOGENEOUS
SOLUTIONS OF LINEAR PDE's.

by M.S. BAOUENDI

Let $P(x,D)$ be a differential operator with analytic coefficients in an open set of $\mathbb{R}^n$. Assume that the principal symbol of $P$ is nowhere identically zero. It is natural to ask the following question:

Is it true that any distribution solution of $P(x,D)u = 0$ is locally a limit of real analytic solutions of the same equation?

The answer to this question is not known. However an affirmative answer is given in Baouendi-Trèves [3] (see also [2] for first order overdetermined systems) when $P$ has simple (complex) characteristics. An affirmative answer is also given in Baouendi-Rothschild [1] when $P$ is a left invariant operator defined on a general Lie group. Detailed proofs could be found in [1] and [3].

First we state the result obtained in [3]. Denote by $t$ the variable in $\mathbb{R}$, by $x$ the one in $\mathbb{R}^n$. Let $\Omega$ be an open set in $\mathbb{R} \times \mathbb{R}^n$ containing the origin. We consider a first order linear differential operator of the form

$$L = I_{\mathbb{N}} - \sum_{j=1}^{n} A_j(t,x)D_{x_j} - A_0(t,x),$$

where $A_j$ are real-analytic in $\Omega$ valued in the space of complex $N \times N$ matrices, and $I_{\mathbb{N}}$ is the identity matrix. Set

$$a(t,x,\xi) = \sum_{j=1}^{n} A_j(t,x)\xi_j.$$ 

We assume that for every $(t,x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ the matrix $a(t,x,\xi)$ has $N$ distinct eigenvalues $\lambda_j(t,x,\xi)$, $j = 1, \ldots, N$.

**Theorem 1:**

Let $h \in \mathcal{W}^{1}(\Omega')$, $0 \in \Omega' \subset \Omega$, satisfying $Lh = 0$. There exist an
open neighborhood of $0$, $\Omega'' \subset \Omega'$, and a sequence of analytic functions $h_\nu$ in $\Omega''$ satisfying:

(i) $L h_\nu = 0$ in $\Omega''$
(ii) $\lim h_\nu = h$ in $C'(\Omega'')$.

Furthermore if $h$ is of class $C^k$, then the convergence in (ii) is in $C^k(\Omega'')$.

Now we state the result in [1].

**Theorem 2:**
Let $P$ be a left invariant differential operator defined on a Lie group $G$. For every open set $U \subset G$, neighborhood of the identity $e \in G$, there exists another open neighborhood of $e$, $W \subset G$, such that if $u$ is a distribution on $G$ satisfying $Lu = 0$ in $U$, then there exists a sequence $u_\nu$ of real analytic functions defined in $W$ and satisfying

(i) $L u_\nu = 0$ in $W$,
(ii) $\lim u_\nu = u$ in $C'(W)$.

Furthermore if $u$ is of class $C^k$, then the convergence in (ii) is in $C^k(W)$.

We sketch now the proof if theorem 1 in the case of a single complex vector field, i.e. $N = 1$. Set

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} a_j(t, x) \frac{\partial}{\partial x_j},$$

where $a_j$ are analytic functions in $\Omega = I \times U$, $0 \in I \subset \mathbb{R}$, $0 \in U \subset \mathbb{R}^n$. Let $h \in C^1(\Omega)$, $L h = 0$, and $g \in C^\infty_0(U)$, $g = 1$ near the origin in $\mathbb{R}^n$. Set

$$u(t, x) = g(x) \ h(t, x) \ , \ Lu = f.$$

Note that $f$ vanishes in a neighborhood of $x = 0$ for all $t \in I$.

For $j = 1, \ldots, n$, denote by $Z_j(t, x)$ the solution of the Cauchy problem
and set $Z = (Z_1, \ldots, Z_n)$.

For $\nu \in \mathbb{Z}_+$ define the operator $K_\nu$ by

\((K_\nu u)(t,x) = \frac{(\nu)^n}{(\nu!)} \int_{\mathbb{R}^n} e^{-\nu \|Z(t,x)-Z(t,y)\|^2} \det(Z'_y(t,y))u(t,y)dy.\)

The operator $K_\nu$ has the following properties :

(a) $K_\nu (Lu) = L(K_\nu u)$

(b) $\lim_{\nu \to \infty} K_\nu u = u$ uniformly in a fixed neighborhood of the origin in $\mathbb{R}^{n+1}$

(c) $K_\nu f$ extends holomorphically in $x$ to a fixed neighborhood of the origin in $\mathbb{C}^n$, and there converges to 0.

Assuming (a) (b) and (c), set

$h_\nu = K_\nu u - \nu_\nu$

where $\nu_\nu$ is the solution of

$L \nu_\nu = K \nu f \quad \nu_\nu \big|_{t=0} = 0.$

It follows from (c) that $\lim_{\nu \to \infty} \nu_\nu = 0$. Therefore (a) and (b) imply that we have (i) and (ii) of the conclusion of theorem 1.

Note that the operator $K_\nu$ defined by (1) can be written

\((K_\nu u)(t,x) = \frac{1}{(2\pi)^n} \int e^{i[Z(t,x)\xi-Z(t,y)\xi]-\epsilon \|\xi\|^2} \det(Z'_y(t,y))u(t,y)dyd\xi.\)

We limit ourselves to mention that the proof of theorem 1 in
the general case (i.e. $N > 1$) is done by reducing the system $L$ to a diagonal one, at least microlocally. Operators similar to (2) are introduced, where $Z(t,x)\xi$ is replaced by $\psi(t,x,\xi)$ satisfying

$$\partial_t \psi - \lambda(t,x,\partial_x \psi) = 0,$$

$$\psi|_{t=0} = x.\xi,$$

$\lambda$ stands for one of the eigenvalues of the matrix $a$. The exponential function in (2) is multiplied by analytic amplitudes determined by geometrical optics.

The proof of theorem 2 is based on the use of convolution with a suitable Gaussian defined near $e \in G$, and the use of the Campbell-Hansdorff formula in order to prove a result similar to (c) above.

REFERENCES

