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Remarks on the Navier-Stokes equations


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REMARKS ON THE NAVIER-STOKES EQUATIONS

by L. NIRENBERG

This talk is a report of work in progress of L. Caffarelli, R. Kohn and L. Nirenberg [1] extending the results of V. Scheffer [2-4]. It concerns weak solutions of the incompressible Navier-Stokes equations in 3 space dimensions of fluid flow. The velocity vector

$$u = (u_1, u_2, u_3)$$

satisfies (using summation convention and subscripts t, i to denote differentiation with respect to time or x_i)

$$\begin{align*}
    u_t^j - \Delta u^j + u^i u_i^j + p_j &= 0 \quad j = 1, 2, 3 \\
    \nabla \cdot u &= u_j^j = 0.
\end{align*}$$

Here $\Delta$ is the Laplace operator in the space variables, and $p$ represents the pressure; viscosity has been normalized to be one. For simplicity we assume here that the forcing term on the right of (1) is zero.

The initial value problem consists in prescribing

$$u(x, 0) = u_0(x).$$

We suppose $u_0$ has finite energy $E_0 = \int |u_0|^2 dx$.

If we consider a flow in a fixed domain $G$ rather than all of $\mathbb{R}^3$, we also prescribe some boundary conditions, for example the values of $u(x, t)$ for $x \in \partial G$, $t > 0$, say zero. Since the classical work of J. Leray and subsequently, E. Hopf, one knows the existence of weak solutions of (1), (2) for $t > 0$ with finite energy for any time :

$$\int |u(x, t)|^2 dx \leq C(T) \quad \text{for } 0 \leq t \leq T.$$ 

and

$$\int_0^T \int_G |Du|^2 dx \, dt \leq C(T)$$
where $|Du|^2 = \sum_{i,j} (u^i_j)^2$. Furthermore one has the energy inequality which we express in the following form: \( \forall T > 0 \), for \( \phi \in C_0^\infty (t \leq T) \), \( \phi(x,t) \geq 0 \), we have
\[
(5) \quad \int_{t=T} \phi |u|^2 dx + \int_0^T \int \phi |Du|^2 dxdt \leq \int_0^T \frac{1}{2} |u|^2 (\phi_t + \Delta \phi) + (|u|^2 + p) \phi_t \phi dx dt .
\]

Formally, one obtains this with equality, if one multiplies (1) by \( \phi u^j \), integrates in \((x,t)\) and sums over \(j\). However up to now one has only proved the existence of a weak solution of (1), (2) satisfying the inequality (5).

Since the early 50's (see the books by O. Ladyzhenskaya [5], J. L. Lions [6] and R. Temam [7]) there has been much work devoted to the following questions:

1. Is the weak solution, described above, of the initial (or initial boundary value) problem unique? If one can prove some further regularity of the solution then it is (see [5-7]).

2. Can the solution(s) develop singularities? If so can they eat energy in the sense that one may have strict inequality in (5)? For the initial value problem one knows that after some time \( t = T_0(E) \) the velocity \( |u| \) is finite. \( u \) is then \( C^\infty \) with respect to the space variables.

A weaker form of 2 is:

3. Can the solution develop singularities in case \( u_0(x) \) is a nice function?

Let \( S \) be the complement of the largest open set (in space-time) in which \( u \in L^\infty_{loc} \), i.e. \( S \) is the set where \( |u| \) becomes infinite. Treating the initial value problem in all of \( \mathbb{R}^3 \) in [1], and the initial boundary value problem in [3], Scheffer proved the following result.

**Theorem 1 (Scheffer):** The 5/3-Hausdorff measure of \( S \) is finite: \( H^{5/3}(S) < \infty \).

In [1] we localize his arguments and extend them to give the following improvement.

**Theorem 2:** If \( u \) is a weak solution in an open set in \( \mathbb{R}^3 \times \mathbb{R} \) satisfying (3) - (5) then \( H^1(S) = 0 \).

The proof is based on two propositions. The first is a local form of the key proposition of Scheffer in [2]. In the following, if \( P = (x_o, t_o) \) we denote by \( Q_r = Q_r(P) \) a circular cylinder in \((x,t)\) space given by
Proposition 1 : There is an absolute constant $\delta > 0$ such that if $u$ is a solution of $(1)$, $(2)$ in $Q_1(P)$, $P = (0,0,0,1)$ with

$$\int_{Q_1(P)} (|u|^3 + |u|p) \, dx \, dt + \int_0^1 (\int_0^1 \frac{|p|}{|x|^{1/2}} \, dx) \, dt \leq \delta,$$

then $|u|$ is finite in a neighborhood of $P$ in $Q_1(P)$.

A reformulation of this result is the following obtained by scaling - if $u$ is a solution, so is $\lambda u(\lambda x, \lambda^2 t) \quad \forall \lambda > 0$.

Proposition 1' : If $P \in S$ then

$$\lim_{r \to 0} \frac{1}{r^2} \int_{Q_r(P)} (|u|^3 + |u|p) \, dx \, dt + r^{-7/2} \int_{t_0}^{t} \int_{|x-x_0|<r} |p| \, dx \, dt \geq \delta'.

Making use of this, and various interpolation estimates, as well as the relationship

$$\Delta p = u^i_j u^j_i,$$

we prove

Proposition 2 : There is an absolute constant $\delta' > 0$ such that if $P \in S$ then

$$\lim_{r \to 0} \frac{1}{r^4} \int_{Q_r(P)} |Du|^2 \, dx \, dt \geq \delta'.$$

Proposition 1 is proved with the aid of special test functions $\varphi$ in (5) approximating the fundamental solution of the backward heat equation, with singularity at $P$.

REFERENCES


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