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On some localized estimates for pseudo-differential operators


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ON SOME LOCALIZED ESTIMATES FOR
PSEUDO-DIFFERENTIAL OPERATORS

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Let \( P = p^w(x,D) \) (Weyl convention) be a classical ps.d.op. in \( \mathbb{R}^n \) with principal symbol \( p^w_m \) positively homogeneous of degree \( m \).

Let \( p_o \), say \( p_o = (0,\xi) \), be a point in the cotangent space of \( \mathbb{R}^n \) and consider problems of the following types:

(A) Determine those \( \mu \) for which there exist a constant \( C \) and a ps.d.op. \( R \) of order strictly less than \( \mu \) near \( p_o \) such that
\[
\|u\|_{(\mu)}^2 \leq C\|Pu\|^2 + \|Ru\|^2 \quad ; \quad u \in C^\infty_0(\mathbb{R}^n),
\]
or in case \( p \) is real, \( p^w_m \geq 0 \).

(B) Determine those \( \mu \) for which there is a lower bound
\[
(Pu,u) \geq C\|u\|_{(\mu)}^2 + \|Ru\|^2.
\]

Sometimes when \( \mu \) is kept fixed we also look for the possible constants \( C \) that can occur.

In case \( p_o^w(p_o) \neq 0 \) the standard calculus for pseudo-differential operators gives us a simple answer. In the other cases one has to localize the estimates near the characteristic variety \( \Sigma \) and get corresponding problems for operators with polynomial coefficients obtained from the Taylor series of \( (x,\xi) = p(p_0 + \lambda(x,\xi)) \) when \( p \in \Sigma \), and \( \lambda \) is a small parameter. Thus for example to have (B) with \( \mu = (m-1)/2 \) (the sharp Gårding inequality) and a constant \( C \) implies lower bounds for the eigenvalues of the harmonic oscillator type operators which are obtained from a Taylor expansion up to the second order along \( \Sigma \). Sharper results in these direction are obtained by Hörmander [2].

In Egorov [1] it is shown that his theorem about the validity of (A) with \( \mu = m - \delta \) under the condition \( (\psi) \) when not all the commutators \( p_I \) of length \( |I| \) of \( \text{Re} p \) and \( \text{Im} p \) vanish when \( |I|(1-\delta) \leq 1 \), essentially relies upon estimates of the following form:

\[
(1) \quad M\|u\| + \|u_t^v\| \leq C_0\|u_t^v - Q(t)u\|.
\]

Here \( Q \) is either multiplication by a polynomial \( q(t) \) or an operator \( v(y) \rightarrow F(t,y) + G(t)y \) \( v(y) \) with polynomial coefficients acting on \( L^2(\mathbb{R}_y^2) \). The condition \( (\psi) \) implies that \( (Q(t)v,v) \) can only change sign
from - to + for fixed \( v \) and \( M \) is related to \( \sum_{|I| (1-5) \leq 1} |P_I|^{1/|I|} \). Thus for example \( M = \sum |q^{(j)}(0)|^{1/(j+1)} \) in the first case. In this case a simple proof for (1) is given if one observes that there is a constant \( C_A \) only depending on \( A \) so that :

\[
(2) \quad \|u_t\|^2 + \|u\|^2 \leq C_A \|u' - hu\|
\]

if \( h \) can only change sign from - to + and in addition satisfies the following :

\[
(3) \quad \text{measure } \{ t; |h(t)| < A^{-1} \} < 1 ,
\]

\[
(4) \quad \int_{\mathbb{R}} \max(0, -h'(t) - |h(t)|) dt \leq A.
\]

One then obtains (1) when \( Q(t) = q(t) \) by a symplectic dilation.

References


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