THE CRITICAL EXPONENT OF THE ARSHON WORDS

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Abstract. Generalizing the results of Thue (for \( n = 2 \)) [Norske Vid. Selsk. Skr. Mat. Nat. Kl. 1 (1912) 1–67] and of Klepinin and Sukhanov (for \( n = 3 \)) [Discrete Appl. Math. 114 (2001) 155–169], we prove that for all \( n \geq 2 \), the critical exponent of the Arshon word of order \( n \) is given by \( (3n - 2)/(2n - 2) \), and this exponent is attained at position 1.

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1. Introduction

In 1935, the Russian mathematician Solomon Efimovich Arshon\(^*\) [1,2] gave an algorithm to construct an infinite cube-free word over 2 letters, and an algorithm to construct an infinite square-free word over \( n \) letters for each \( n \geq 3 \). The binary word he constructed turns out to be exactly the celebrated Thue-Morse word, \( t = 01101001...[4,11] \); the square-free words are now known as the Arshon words. For \( n \geq 2 \), the Arshon word of order \( n \) is denoted by \( a_n = a_0a_1a_2... \)

A square or a 2-power is a word of the form \( xx \), where \( x \) is a nonempty word. Similarly, an \( n \)-power is a word of the form \( x^n = xx...x \) (\( n \) times). The notion of integral powers can be generalized to fractional powers. A non-empty finite word \( z \) over a finite alphabet \( \Sigma \) is a fractional power if it has the form \( z = x^n y \), where \( x \) is a non-empty word, \( n \) is a positive integer, and \( y \) is a prefix of \( x \), possibly empty. If \( |z| = p \) and \( |x| = q \), we say that \( z \) is a \((p/q)\)-power, or \( z = x^{p/q} \).

Keywords and phrases. Arshon words, critical exponent.

\(^*\)Vilenkin, in his 1991 article "Formulas on cardboard" [12], says that Arshon was arrested by the Soviet regime and died in prison, most likely in the late 1930’s or early 1940’s.
Let $\alpha > 1$ be a real number. A right-infinite word $w$ over $\Sigma$ is said to be $\alpha$-power-free (resp. $\alpha^+$-power-free), or to avoid $\alpha$-powers (resp. $\alpha^+$-powers), if no subword of it is an $r$-power for any rational $r \geq \alpha$ (resp. $r > \alpha$). Otherwise, $w$ contains an $\alpha$-power. The critical exponent of $w$, denoted by $E(w)$, is the supremum of the set of exponents $r \in \mathbb{Q}_{\geq 1}$, such that $w$ contains an $r$-power; it may or may not be attained.

Arshon constructed the words $a_n$, $n \geq 3$, as square-free infinite words. But actually, these words avoid smaller powers. In 2001, Klepinin and Sukhanov [8] proved that $E(a_3) = 7/4$, and the bound is attained; that is, $a_3$ avoids $(7/4)^+$-powers. In this paper we generalize the result of Klepinin and Sukhanov, and prove the following theorem:

**Theorem 1.1.** Let $n \geq 2$, and let $a_n = a_0a_1a_2\ldots$ be the Arshon word of order $n$. Then the critical exponent of $a_n$ is given by $E(a_n) = (3n - 2)/(2n - 2)$, and $E(a_n)$ is attained by a subword beginning at position 1.

## 2. Definitions and notation

Let $\Sigma_n = \{0, 1, \ldots, n-1\}$ be an alphabet of size $n$, $n \geq 3$. In what follows, we use the notation $a \pm 1$, where $a \in \Sigma_n$, to denote the next or previous letter in lexicographic order, and similarly we use the notation $a + b$, $a - b$, where $a, b \in \Sigma_n$; all sums of letters are taken modulo $n$.

Define two morphisms over $\Sigma_n$ as follows:

\[
\varphi_{e,n}(a) = a(a + 1)\ldots(n - 1)01\ldots(a - 2)(a - 1), \quad a = 0, 1, \ldots, n - 1;
\]

\[
\varphi_{o,n}(a) = (a - 1)(a - 2)\ldots0(n - 1)\ldots(a + 1)a, \quad a = 0, 1, \ldots, n - 1.
\]

The letters ‘$e$’ and ‘$o$’ stand for “even” and “odd”, respectively. Both $\varphi_{e,n}$ and $\varphi_{o,n}$ are $n$-uniform (that is, $|\varphi_{e,n}(a)| = n$ for all $a \in \Sigma_n$, and similarly for $\varphi_{o,n}$) and marked (that is, $\varphi_{e,n}(a)$ and $\varphi_{e,n}(b)$ have no common prefix or suffix for all $a \neq b \in \Sigma_n$, and similarly for $\varphi_{o,n}$). The Arshon word of order $n$ can be generated by alternately iterating $\varphi_{e,n}$ and $\varphi_{o,n}$: define an operator $\varphi_n : \Sigma_n^n \to \Sigma_n^n$ by

\[
\varphi_n(a_i) = \begin{cases} 
\varphi_{e,n}(a_i), & \text{if } i \text{ is even;} \\
\varphi_{o,n}(a_i), & \text{if } i \text{ is odd.}
\end{cases}
\]

That is, if $u = a_0a_1\ldots a_n \in \Sigma_n^n$, then $\varphi_n(u) = \varphi_{e,n}(a_0)\varphi_{o,n}(a_1)\varphi_{e,n}(a_2)\varphi_{o,n}(a_3)\ldots$ The Arshon word of order $n$ is given by

\[
a_n = \lim_{k \to \infty} \varphi_n^k(0).
\]

Note that $\varphi_n^k(0)$ is a prefix of $\varphi_n^{k+1}(0)$ for all $k \geq 0$, and the limit is well defined.
**Example 2.1.** For $n = 3$, the even and odd Arshon morphisms are given by

$$\varphi_{e,3} : \begin{cases} 
0 &\rightarrow 012 \\
1 &\rightarrow 120, \\
2 &\rightarrow 201
\end{cases}$$

$$\varphi_{o,3} : \begin{cases} 
0 &\rightarrow 210 \\
1 &\rightarrow 021, \\
2 &\rightarrow 102
\end{cases}$$

and the Arshon word of order 3 is given by

$$a_3 = \lim_{k \to \infty} \varphi_{3,k}(0) = \varphi_{e,3}(0) \varphi_{o,3}(1) \varphi_{e,3}(2) \varphi_{o,3}(0) \ldots$$

It is not difficult to see that when $n$ is even, the $i$'th letter of $a_n$ is even if and only if $i$ is an even position (for a formal proof, see Séébold [9,10]). Therefore, when $n$ is even, the map $\varphi_n$ becomes a morphism, denoted by $\alpha_n$:

$$\alpha_n(a) = \begin{cases} 
\varphi_{e,n}(a), & \text{if } a \text{ is even;} \\
\varphi_{o,n}(a), & \text{if } a \text{ is odd.}
\end{cases} \quad (2.3)$$

When $n$ is odd no such partition exists, and indeed, $a_n$ cannot be generated by iterating a morphism. This fact was proved for $a_3$ by Berstel [3] and Kitaev [6,7], and for any odd $n$ by Currie [5].

An occurrence of a subword within $a_n$ is a triple $(z,i,j)$, where $z$ is a subword of $a_n$, $0 \leq i \leq j$, and $a_i \ldots a_j = z$. In other words, $z$ occurs in $a_n$ at positions $i, \ldots, j$. We usually refer to an occurrence $(z,i,j)$ as $z = a_i \ldots a_j$. The set of all subwords of $a_n$ is denoted by $\text{Sub}(a_n)$. The set of all occurrences of subwords within $a_n$ is denoted by $\text{Occ}(a_n)$. An occurrence $(z,i,j)$ contains an occurrence $(z',i',j')$ if $i \leq i'$ and $j \geq j'$.

A subword $v$ of $a_n$ admits an interpretation by $\varphi_n$ if there exists a subword $v' = v_0v_1 \ldots v_kv_{k+1}$ of $a_n$, $v_i \in \Sigma_n$, such that $v = y_0\varphi_n(v_1 \ldots v_k)x_{k+1}$, where $y_0$ is a suffix of $\varphi_n(v_0)$ and $x_{k+1}$ is a prefix of $\varphi_n(v_{k+1})$. The word $v'$ is called an ancestor of $v$.

For an occurrence $z \in \text{Occ}(a_n)$, we denote by $\text{inv}(z)$ the inverse image of $z$ under $\varphi_n$. That is, $\text{inv}(z)$ is the shortest occurrence $z' \in \text{Occ}(a_n)$ such that $\varphi_n(z')$ contains $z$. Note that the word (rather than occurrence) $\text{inv}(z)$ is an ancestor of the word $z$, but not necessarily a unique one.

Following Currie [5], we refer to the decomposition of $a_n$ into images under $\varphi_n$ as the $\varphi$-decomposition, and to the images of the letters as $\varphi$-blocks. We denote the borderline between two consecutive $\varphi$-blocks by ‘|’; e.g., $i|j$ means that $i$ is the last letter of a block and $j$ is the first letter of the following block. If $z = a_i \ldots a_j \in \text{Occ}(a_n)$ begins at an even position we write $z = a_i a_{i+1}a_{i+2} \ldots$, and similarly for an occurrence that begins at an odd position.
3. General properties of the Arshon words

Lemma 3.1. For all \( n \geq 2 \), \( a_n \) contains a \((3n-2)/(2n-2)\)-power beginning at position 1.

Proof. For \( n = 2 \), \( a_2 = t = 0110\ldots \), which contains the 2-power 11 at position 1. For \( n \geq 3 \), \( a_n \) begins with

\[
\varphi_{e,n}(0)\varphi_{o,n}(1)\varphi_{e,n}(2) = 012\ldots(n-1)0(n-1)\ldots 21\ldots(n-1)01 = 0(12\ldots(n-1)0(n-1)\ldots 2)^{(3n-2)/(2n-2)}.
\]

□

Example 3.2.

\[
\begin{align*}
a_3 &= \varphi_{e,n}(0)\varphi_{e,n}(1)\varphi_{o,n}(2) = 012|021|021\ldots = 0(1202)^{7/4}1\ldots, \\
a_4 &= \varphi_{e,n}(0)\varphi_{e,n}(1)\varphi_{o,n}(2) = 0123|0321|2301\ldots = 0(123032)^{10/6}1\ldots, \\
a_5 &= \varphi_{e,n}(0)\varphi_{e,n}(1)\varphi_{o,n}(2) = 01234|04321|23401\ldots = 0(1234032)^{13/8}1\ldots.
\end{align*}
\]

Corollary 3.3. The critical exponent of \( a_n \) satisfies \((3n-2)/(2n-2) \leq E(a_n) \leq 2\) for all \( n \geq 2 \).

Proof. For \( n = 2 \), it is well known that \( E(a_n) = E(t) = 2 \) [4,11]. For \( n \geq 3 \), we know by Arshon [1,2] that \( a_n \) is square-free, and so \( E(a_n) \leq 2 \). The lower bound follows from Lemma 3.1. □

Lemma 3.4. Let \( n \geq 3 \), and let \( i, j \in \Sigma_n \).

1. If \( ij \in \text{Occ}(a_n) \), then \( j = i \pm 1 \).
2. The borderline between two consecutive \( \varphi \)-blocks has the form \( i|ji \) or \( ij|i \).

Moreover, a word of the form \( iji \) can occur only at a borderline.

Proof. If \( ij \) occurs within a \( \varphi \)-block, then \( j = i \pm 1 \) by definition of \( \varphi_n \). Suppose \( i \) is the last letter of a \( \varphi \)-block and \( j \) is the first letter of the next \( \varphi \)-block, and let \( kl = \text{inv}(ij) \). Assume \( j \neq i \pm 1 \), and suppose further that \( ij \) is the first pair that satisfies this inequality. Then \( l = k \pm 1 \), and so there are four cases:

\[
\begin{align*}
\varphi_n(kl) &= \varphi_{e,n}(k)\varphi_{o,n}(k+1), \quad \varphi_n(kl) = \varphi_{o,n}(k)\varphi_{e,n}(k+1), \\
\varphi_n(kl) &= \varphi_{e,n}(k)\varphi_{o,n}(k-1), \quad \varphi_n(kl) = \varphi_{o,n}(k)\varphi_{e,n}(k-1).
\end{align*}
\]

But it is easy to check that for all the cases above, \( j = i \pm 1 \), a contradiction.

For the second assertion, observe that by definition of \( \varphi_n \), a \( \varphi \)-block is either strictly increasing or strictly decreasing, and two consecutive blocks have alternating directions. By the above, a change of direction can have only the form \( iji \) or \( ij|i \). □

Definition 3.5 (Currie [5]). A mordent is a word of the form \( iji \), where \( i, j \in \Sigma_n \) and \( j = i \pm 1 \). Two consecutive mordents occurring in \( a_n \) are either near mordents,
far mordents, or neutral mordents, according to the position of the borderlines:

\[
\begin{align*}
\text{i|ji u kl|k} & \quad \text{near mordents, } |u| = n - 4; \\
\text{ij|ji u kl|k} & \quad \text{far mordents, } |u| = n - 2; \\
\text{ij|ji u kl|k} & \quad \text{neutral mordents, } |u| = n - 3; \\
\text{ij|ji u kl|k} & \quad \text{neutral mordents, } |u| = n - 3.
\end{align*}
\]

Note that for \( n = 3 \), near mordents are overlapping: \( a_3 = 012|021|021| \ldots \)

Since \( a_n \) is square-free, a \( p/q \)-power occurring in \( a_n \) has the form \( x y z \), where \( q = |xy|, p = |x y z|, \) and both \( x, y \) are nonempty.

**Definition 3.6.** Let \( z = a_i \ldots a_j \in \text{Occ}(a_n) \) be a \( p/q \)-power. We say that \( z \) is left-stretchable (resp. right-stretchable) if the \( q \)-period of \( z \) can be stretched left (resp. right), i.e., if \( a_{i-1} = a_{i+q-1} \) (resp. \( a_{j+1} = a_{j-q+1} \)). If the \( q \)-period of \( z \) can be stretched neither left nor right, we say that \( z \) is an unstretchable \( p/q \)-power.

Since the critical exponent is a supremum, it is enough to consider unstretchable powers when computing it.

**Lemma 3.7.** Let \( n \geq 3 \). Let \( z = x y z = (x y)^{p/q} \in \text{Occ}(a_n) \) be an unstretchable power such that \( |x| \leq n \) and \( x \) contains no mordents. Then \( p/q \leq (3n-2)/(2n-2) \).

**Proof.** Since \( |x| \leq n \), it is enough to consider \( y \) such that \( |y| \leq n - 2 \), for otherwise we would get that \( p/q < (3n - 2)/(2n - 2) \). Therefore, \( |xy| = q \leq 2n - 2 \) and \( |z| \leq 3n - 2 \). We get that \( x y \) is contained in at most \( 3 \) consecutive \( \varphi \)-blocks and \( z \) is contained in at most \( 4 \) consecutive \( \varphi \)-blocks. Suppose \( z \) is not contained in \( 3 \) consecutive \( \varphi \)-blocks. Let \( B_0 B_1 B_2 B_3 \) be the blocks containing \( z \), and assume that \( B_0 \) is even (the other case is similar). Since \( |x| \leq n \), necessarily \( x y \) begins in \( B_0 \) and ends in \( B_2 \). Since \( x \) contains no mordents, \( x \) has to start at the last letter of \( B_0 \); otherwise, we would get that \( x \) cannot extend beyond the first letter of \( B_1 \), and since \( |y| \leq n - 2 \), we would get that \( z \) is contained in \( 3 \) \( \varphi \)-blocks. Therefore, the letters of \( x \) are decreasing. Now, since \( |xy| \leq 2n - 2 \), the second occurrence of \( x \) begins at least \( 3 \) letters from the end of \( B_2 \). Since \( B_2 \) is an even block, we get a contradiction if \( |x| > 1 \). But if \( |x| = 1 \) then \( z \) is contained in \( B_0 B_1 B_2 \). We can assume therefore that \( z \) is contained in \( 3 \) consecutive \( \varphi \)-blocks, \( B_0 B_1 B_2 \). We assume that \( B_0 \) is even (the other case is symmetric).

If \( x y \) is contained in one block then, because \( B_0, B_2 \) are even and \( B_1 \) is odd, necessarily \( |x| = 1 \), and so \( p/q \leq 3/2 < (3n - 2)/(2n - 2) \). If \( x y \) begins in \( B_0 \) and ends in \( B_2 \), then, since \( |y| \leq n - 2 \), the first \( x \) occurrence has to end at the third letter of \( B_1 \) or later. Since \( x \) contains no mordents, this implies that \( x y \) begins at the last letter of \( B_0 \) and the letters of \( x \) are decreasing. Since \( B_2 \) is even, again \( |x| = 1 \).

Assume \( x y \) begins in \( B_0 \) and ends in \( B_1 \). Again, because \( B_0 \) is even and \( B_1 \) is odd, in order for \( x \) to contain more than one letter the second \( x \) occurrence has to start either at the last letter of \( B_1 \), or at the first letter of \( B_2 \).
Let $B_0 = \varphi_{e,n}(i)$. Then there are four cases for $B_1, B_2$:

1. $B_1 = \varphi_{o,n}(i+1)$, $B_2 = \varphi_{e,n}(i)$;
2. $B_1 = \varphi_{o,n}(i-1)$, $B_2 = \varphi_{e,n}(i)$;
3. $B_1 = \varphi_{o,n}(i+1)$, $B_2 = \varphi_{e,n}(i+2)$;
4. $B_1 = \varphi_{o,n}(i-1)$, $B_2 = \varphi_{e,n}(i-2)$.

We now check what the maximal possible exponent is in each of these cases. Without loss of generality, we can assume $i = 0$. We use the notation $z = xyx'$, where $x'$ is the second occurrence of $x$ in $z$.

**Case 1:** $B_0B_1B_2 = [01\ldots(n-1)][0(n-1)\ldots1][01\ldots(n-1)]$.

If $x'$ starts at the last letter of $B_1$ then $|x| = 1$, since 10 does not occur anywhere before. If $x'$ starts at the first letter of $B_2$, the only possible power is the $3n/2n$-power $B_0B_1B_0$, which contradicts the hypothesis $|y| \leq n - 2$.

**Case 2:** $B_0B_1B_2 = [01\ldots(n-1)](n-2)(n-3)\ldots0(n-1)[01\ldots(n-1)]$.

By the same argument, either $|x| = 1$ or $z$ is a $3n/2n$-power.

**Case 3:** $B_0B_1B_2 = [01\ldots(n-1)][0(n-1)\ldots1][23\ldots(n-1)][01]$.

If $x'$ starts at the last letter of $B_1$, we get the $(3n-2)/(2n-2)$-power described in Lemma 3.1. If $x'$ starts at the first letter of $B_2$, then $x$ has to start at the $2$ in $B_0$. But then the power is left-stretchable, to the $(3n-2)/(2n-2)$-power described above.

**Case 4:** $B_0B_1B_2 = [01\ldots(n-1)](n-2)(n-3)\ldots0(n-1)(n-2)(n-1)0\ldots(n-3)]$.

If $x'$ starts at the last letter of $B_1$, then $x$ has to start at the last letter of $B_0$. But then $|x| = 2$, since $(n-1) \neq (n-3)$. We get that $z$ is an $(n+2)/n$-power, and $(n+2)/n < (3n-2)/(2n-2)$ for all $n \geq 3$. If $x'$ starts at the first letter of $B_2$, then $x$ has to start at the second last letter of $B_0$. Again, $|x| = 2$, and $z$ is an $(n+2)/(n+2)$-power, where $(n+2)/(n+2) < (3n-2)/(2n-2)$ for all $n \geq 2$. □

In what follows, we will show that in order to compute $E(a_n)$, it is enough to consider powers $xyx$ such that $|x| \leq n$ and $x$ contains no mordents.

**Definition 3.8.** Let $z$ be a subword of $a_n$. We say that $(z_1, z_2)$ is a **synchronization point** of $z$ under $\varphi_n$ if $z = z_1z_2$, and whenever $\varphi_n(u) = v_1zv_2$ for some $u, v_1, v_2 \in \text{Sub}(a_n)$, we have $u = u_1u_2$, $\varphi_n(u_1) = v_1z_1$, and $\varphi_n(u_2) = z_2v_2$. That is, $z_1z_2$ is always a borderline in the $\varphi$-decomposition of $z$, regardless of the position in $a_n$ where $z$ occurs. We say that a subword $z \in \text{Sub}(a_n)$ is **synchronized** if it can be decomposed unambiguously under $\varphi_n$, in which case it has a unique ancestor.

**Lemma 3.9.** If $z \in \text{Sub}(a_n)$ has a synchronization point then $z$ is synchronized.

**Proof.** Suppose $z$ has a synchronization point, $z = u|v$. If $|u| = |v| = 1$ then $z$ cannot have a synchronization point at $u|v$, since $uv$ occurs either in $\varphi_{e,n}(u)$ or in $\varphi_{o,n}(u + 1)$. Therefore, at least one of $u, v$ has length $> 1$. Suppose $|u| > 1$. If the last two characters of $u$ are increasing, we know that an even $\varphi$-block ends at $u$ and an odd $\varphi$-block starts at $v$, and vice versa if the last two characters of $u$ are
Lemma 4.1. Let \( n \geq 4 \), and let \( x = xyx = (xy)^{p/q} \in \text{Occ}(a_n) \) be an unstretchable \( p/q \)-power, such that \( x \) has a synchronization point. Then there exists an \( r/s \)-power \( z' \in \text{Occ}(a_n) \), such that \( p = nr, q = ns, \) and \( z = \varphi_n(z') \).

Proof. Since \( x \) has a synchronization point, it has a unique decomposition under \( \varphi_n \). Suppose \( x \) does not begin at a borderline of \( \varphi \)-blocks. Then \( x = t|w, \) where \( t \) is a nonempty suffix of a \( \varphi \)-block, and \( z = t|wyt|w \). But since the interpretation is unique, both occurrences of \( t \) must be preceded by a word \( s \), such that \( st \) is a \( \varphi \)-block. Thus \( z \) can be stretched by \( s \) to the left, a contradiction. Therefore, \( x \) begins at a borderline, and so \( y \) ends at a borderline. For the same reason, \( x \) must end at a borderline, and so \( y \) must begin at a borderline. We get that both \( x \) and \( y \) have an exact decomposition into \( \varphi \)-blocks, and this decomposition is unique. In particular, both occurrences of \( \varphi \) have the same inverse image under \( \varphi_n \). Let \( k, l \) be the number of \( \varphi \)-blocks composing \( x, y \), respectively. Then \( p = n(2k + l), q = n(k + l), \) and \( \varphi_n^{-1}(z) = \varphi_n^{-1}(x)\varphi_n^{-1}(y)\varphi_n^{-1}(x) \) is a \((2k + l)/(k + l)\)-power. \( \square \)

Corollary 3.11. To compute \( E(a_n) \), it is enough to consider powers \( z = xyx \) such that \( x \) has no synchronization points.

4. Arshon Words of Even Order

To illustrate the power structure in Arshon words of even order, consider \( a_4 \):

\[
\begin{array}{ccc}
\varphi_{e,4} & \varphi_{o,4} & \alpha_4 \\
0 \to & 0123 & 3210 & 0123 \\
1 \to & 1230 & 0321 & 0321 \\
2 \to & 2301 & 1032 & 2301 \\
3 \to & 3012 & 2103 & 2103 \\
\end{array}
\]

\( a_4 = 0123|0321|2301|2103|0123|2103|2301|0321|2301|2103|0123|0321|2301|0321|\ldots \)

Lemma 4.1. Let \( n \geq 4 \) be even, and let \( x \in \text{Sub}(a_n) \) be a subword that has no synchronization point. Then \( |x| \leq n \) and \( x \) contains no mordents.

Proof. In general, a mordent \( iji \) can admit two possible borderlines: \( ij|i \) or \( iji \). However, if \( n \) is even, all images under \( \alpha_n \) begin with an even letter and end with an odd letter; images of odd letters under \( \varphi_{e,n} \) and images of even letters under \( \varphi_{o,n} \) are never manifested. Therefore, every mordent admits exactly one interpretation: if \( i \) is even and \( j \) is odd the interpretation has to be \( iji \), and vice versa for odd \( i \). Thus, if \( x \) has no synchronization point it contains no mordents.

Suppose \( x \) contains no mordents. Then \( |x| \leq n + 2 \), and the letters of \( x \) are either increasing or decreasing. Assume they are increasing. If \( |x| = n + 2 \) then \( x \) has exactly one interpretation, \( x = i|(i+1)\ldots(i-1)i|(i+1) \), or else we would get decreasing. Since both \( \varphi_{e,n} \) and \( \varphi_{o,n} \) are uniform marked morphisms, and since we know \( \varphi \)-blocks alternate between even and odd, we can infer \( \text{inv}(z) \) unambiguously from \( u|v \).
that \( a_n \) contains two consecutive even blocks. If \( |x| = n + 1 \) then \emph{a priori} \( x \) has two possible interpretations: \( x = i(i+1) \ldots (i-1)i \) or \( x = i(i+1) \ldots (i-1)i \).

However, the first case is possible if and only if \( i \) is odd, since for an even \( n \) no \( \varphi \)-block ends with an even letter. Similarly, the second case is possible if and only if \( i \) is even.

Lemma 4.1, together with Corollary 3.11 and Lemma 3.7, completes the proof of Theorem 1.1 for all even \( n \geq 4 \).

5. Arshon words of odd order

To illustrate the power structure in Arshon words of odd order, consider \( a_5 \):

\[
\varphi_{e,5} \quad \varphi_{o,5}
\begin{array}{ccc}
0 & \rightarrow & 01234
\end{array}
\begin{array}{ccc}
43210
\end{array}
\begin{array}{ccc}
1 & \rightarrow & 12340
\end{array}
\begin{array}{ccc}
04321
\end{array}
\begin{array}{ccc}
2 & \rightarrow & 23401
\end{array}
\begin{array}{ccc}
10432
\end{array}
\begin{array}{ccc}
3 & \rightarrow & 34012
\end{array}
\begin{array}{ccc}
21043
\end{array}
\begin{array}{ccc}
4 & \rightarrow & 40123
\end{array}
\begin{array}{ccc}
32104
\end{array}
\]

\[
a_5 =
01234|04321|23401|21043|40123|43210|40123|23401|04321|23401|21043|\ldots
\]

\textbf{Lemma 5.1.} Let \( n \geq 3 \) be odd. Then every subword \( z \in \text{Sub}(a_n) \) with \( |z| \geq 3n \) has a unique interpretation under \( \varphi_n \).

\textbf{Proof.} Consider a subword that contains a pair of consecutive mordents, \( z = iji u klk \). If \( |u| = n - 4 \) (that is, these are near mordents), then \( z \) contains two synchronization points, \( z = i[ji u kl|k] \); otherwise, we get a \( \varphi \)-block that contains a repeated letter, a contradiction. Similarly, if \( |u| = n - 2 \) (a pair of far mordents), \( z \) contains the synchronization points \( z = i[ji u k|lk] \). To illustrate, consider \( a_5 \): let \( z = a_4 \ldots a_{10} = 404 3 212 \). A borderline 40\( i4 \) implies that 43212 is a \( \varphi \)-block, a contradiction; a borderline 2|12 implies that 40432 is a \( \varphi \)-block, again a contradiction. Now let \( z = a_8 \ldots a_{16} = 212 340 121 \). A borderline 2|12 implies that 121 is a prefix of a \( \varphi \)-block, while a borderline 12|1 implies that 212 is a suffix of a \( \varphi \)-block. Again, we get a contradiction.

If \( |u| = n - 3 \) (neutral mordents), then \( z \) has two possible interpretations, either \( z = i[ji u k|lk] \) or \( z = i[ji u k|lk] \). However, by Currie [5], \( a_n \) does not contain two consecutive pairs of neutral mordents: out of three consecutive mordents, at least one of the pairs is either near or far. (It is also easy to see that this is the case by a simple inverse image analysis: an occurrence of the form \( ij|i u k|lk v r|sr \) or \( i|ji u k|lk v r|sr \) implies that \( a_n \) contains a square of the form \( abab, a, b \in \Sigma_n \), a contradiction: by Arshon, \( a_n \) is square-free.)
Let \( z \in \text{Occ}(a_n) \) satisfy \( |z| = 3n \). If \( z \) contains a pair of near or far mordents, then \( z \) has a unique ancestor. Otherwise, \( z \) contains a pair of neutral mordents, \( iji u k|k \), where \( |u| = n - 3 \), and there are two possible interpretations: \( iji u k|k \) or \( iji u k|k \). Let \( i'j'i' \) be the mordent on the left of \( iji \), and let \( k'l'k' \) be the mordent on the right of \( k|k \). Since no two consecutive neutral mordents occur, \( i'j'i' \) and \( k'l'k' \) must form near or far mordents with \( iji \) and \( k|k \).

If the interpretation is \( iji u k|k \), then \( k'l'k' \) forms a near pair with \( k|k \), while \( i'j'i' \) forms a far pair with \( iji \). By assumption, \( z \) does not contain a near pair or a far pair, therefore \( z \) can contain at most \( n - 2 \) letters to the right of \( k|k \), and at most \( n \) letters to the left of \( iji \). Since \( |z| = 3n \), this means that either \( z = j'|i' \ x \ iji u k|k \ v \ k' \) or \( z = |i' \ x \ iji u k|k \ v \ k' \) where \( |x| = n - 2 \) and \( |v| = n - 4 \). Similarly if the interpretation is \( ijji u k|k \), then either \( z = j'|i' \ v \ ijji u kl|k \ x \ k' \) or \( z = i' \ v \ ijji u k|k \ x \ k' \) where \( |x| = n - 2 \) and \( |v| = n - 4 \). In any case, \( z \) contains enough letters to determine if the far mordent is on the left or on the right, and the interpretation is unique.

Example 5.2. For \( n = 5 \), the occurrence \( z = a_{21} \ldots a_{34} = 01234321040123 \), of length \( 3n - 1 = 14 \), has two possible interpretations under \( \varphi_5 \), as illustrated in Figure 1. However, if either of the left or right question marks is known, the ambiguity is solved: the top interpretation is valid if and only if the left question mark equals 4 (so as to complete the \( \varphi \)-block) and the right question mark equals 2 (so as to complete the near mordent). The bottom interpretation is valid if and only if the left question mark equals 1 (so as to complete the near mordent) and the right question mark equals 4 (so as to complete the \( \varphi \)-block).

Note. Lemma 5.1 is an improvement of a similar lemma of Currie [5], who proved that every occurrence of length \( 3n + 3 \) or more has a unique interpretation.

Corollary 5.3. The critical exponent of an odd Arshon word is the largest exponent of powers of the form \( z = xyx \), such that \( |x| < 3n \).

To compute \( E(a_n) \) we need to consider subwords of the form \( xyx \), with \( x \) unsynchronized. Moreover, the two occurrences of \( x \) should have different interpretations, or else we could take an inverse image under \( \varphi_n \). For a fixed \( n \), it would suffice to run a computer check on a finite number of subwords of \( a_n \); this is exactly the technique Klepinin and Sukhanov employed in [8]. For a general \( n \), we need a more careful analysis.
Lemma 5.4. Let \( n \geq 3 \), \( n \) odd. For all mordents \( \imath \) in \( a_n \),

1. \( \imath(\imath + 1) \) or \( \imath(\imath + 1)^{(c)} \)
2. \( \imath(\imath - 1) \) or \( \imath(\imath - 1)^{(c)} \)

Proof. A mordent \( \imath \) can admit two possible borderlines: \( \imath \iota \) or \( \imath \iota \). Consider the mordent \( \imath(\imath + 1) \). If the borderline is \( \imath(\imath + 1) \), then \( \imath(\imath + 1) \) is a suffix of an increasing \( \varphi \)-block, and so the block must be an image under \( \varphi_{n,n} \). By definition of \( \varphi_{n,n} \), \( \imath(\imath + 1) \) is the suffix of \( \varphi_{n,n}(\imath + 2) \). Since even and odd blocks alternate, the next block must be an image under \( \varphi_{n,n} \), and by definition, \( \imath \) is the prefix of \( \varphi_{n,n}(\imath + 1) \).

If the borderline is \( \imath(\imath + 1) \), then \( \imath(\imath + 1) \) is the prefix of a decreasing \( \varphi \)-block, and by similar considerations this block is \( \varphi_{n,n}(\imath + 2) \), while the previous block is \( \varphi_{n,n}(\imath + 1) \). The assertion for \( \imath(\imath - 1) \) is proved similarly.

Lemma 5.5. Let \( n \geq 3 \), \( n \) odd, and let \( z \in \text{Occ}(a_n) \).

1. \( \imath(\imath + 1) \) or \( \imath(\imath + 1)^{(c)} \) for some \( \imath \in \Sigma_n \), then \( |u| \geq n - 1 \);
2. \( \imath(\imath + 1) \) or \( \imath(\imath + 1)^{(c)} \) for some \( \imath \in \Sigma_n \), then \( |u| \geq n - 2 \).

Proof. Let \( z = \imath(\imath + 1) \), and suppose \( \imath(\imath) \) does not occur in \( u \) (otherwise, if \( u = \imath(\imath + 1)u' \), then \( z = \imath(\imath + 1)u'' \)). If \( |u| < n - 1 \) then \( z \) must contain a mordent in order for \( z \) to be repeated. But then the two occurrences of \( \imath \) have the same parity, a contradiction. The rest of the cases are proved similarly.

Lemma 5.6. Let \( n \geq 3 \) be odd, and let \( x = (xy)p/q \in \text{Occ}(a_n) \) be an unstretchable power, such that \( x \) is unsynchronized and contains a mordent. Then \( p/q < E(a_n) \).

Proof. Suppose \( x \) contains the mordent \( \imath(\imath + 1) \) (the case of \( \imath(\imath - 1) \) is symmetric). Then the two occurrences of the mordent have different interpretations, else we could take an inverse image under \( \varphi_n \) and get a power with the same exponent. By Lemma 5.4, there are two different cases, according to which interpretation comes first:

By Lemma 5.5, in both cases there must be at least \( n - 1 \) additional \( \varphi \)-blocks between the blocks containing the two \( \imath(\imath + 1) \) occurrences. Thus, in both cases \( q \geq n^2 + n - 1 \) (note that \( q \) is the length of the period, and can be measured from the beginning of \( \imath(\imath + 1) \) in the first \( x \) to just before \( \imath(\imath + 1) \) in the second \( x \)). Now, \( x \) is unsynchronized, and so by Lemma 5.1 \( |x| < 3n \). Thus, \( |x|/q < (3n - 1)/(n^2 + n - 1) < n/(2n - 2) \) for all \( n \geq 3 \), and so \( p/q = (|x| + q)/q < (3n - 2)/(2n - 2) \leq E(a_n) \). □
By Lemma 5.6, in order to compute $E(a_n)$ it is enough to consider powers $xyx$ such that $x$ is unsynchronized and contains no mordents. The longest subword that contains no mordents is of length $n + 2$, but such subword implies a far pair, and has a unique ancestor. Therefore, we can assume $|x| \leq n + 1$.

**Lemma 5.7.** Let $n \geq 3$ be odd, and let $z = xyx = (xy)^{p/q} \in \text{Occ}(a_n)$ be an unstretchable power, such that $x$ is unsynchronized, $x$ contains no mordents, and $|x| = n + 1$. Then $p/q < E(a_n)$.

**Proof.** Since $|x| = n + 1$ and $x$ contains no mordents, necessarily $x = ivi$, where $i \in \Sigma_n$ and either

$$v = (i + 1) \ldots (n - 1)01 \ldots (i - 2)(i - 1),$$

or

$$v = (i - 1) \ldots 10(n - 1) \ldots (i + 2)(i + 1).$$

Suppose the letters of $v$ are increasing, and assume without loss of generality that $i = 0$. Then $x$ admits two possible interpretations: $x = 01 \ldots (n - 1)0$ or $x = 01 \ldots (n - 1)0$. The ancestors of the first and second case are $\text{inv}(x) = 0^{(e)1^{(e)}}$ and $\text{inv}(x) = 0^{(e)1^{(e)}}$, respectively. Any other interpretation is impossible, since it implies $a_n$ contains two consecutive even $\varphi$-blocks.

As in the previous lemma, we can assume that the two $x$ occurrences of $z$ have different inverse images. There are two possible cases:

1. $\varphi_{\varphi(x)}(0) \varphi_{\varphi(x)}(1) \varphi_{\varphi(x)}(0) \ldots (n - 1)0 \ldots | 0 \ldots | \ldots | 0 \ldots | 1 \ldots (n - 1)0$.
2. $\varphi_{\varphi(x)}(0) \varphi_{\varphi(x)}(1) \varphi_{\varphi(x)}(0) \ldots (n - 1)0 \ldots | 0 \ldots | \ldots | 0 \ldots | 1 \ldots (n - 1)0.$

By Lemma 5.5, in both cases $y$ contains at least $n - 2$ additional $\varphi$-blocks. Therefore, $q \geq n^2 - n + 1$, and so $|x|/q \leq (n + 1)/(n^2 - n + 1) < n/(2n - 2)$ for all $n \geq 3$. Again, $p/q < (3n - 2)/(2n - 2) \leq E(a_n)$. \hfill $\Box$

By Lemma 5.7, to compute $E(a_n)$ for an odd $n \geq 3$ it is enough to consider powers of the form $z = xyx$ such that $|x| \leq n$ and $x$ contains no mordent. By Lemma 3.7, such powers have exponent at most $(3n - 2)/(2n - 2)$. This completes the proof of Theorem 1.1.

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References


