SQUARES AND CUBES IN STURMIAN SEQUENCES

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Abstract. We prove that every Sturmian word $\omega$ has infinitely many prefixes of the form $U_nV_n^3$, where $|U_n| < 2.855|V_n|$ and $\lim_{n \to \infty} |V_n| = \infty$. In passing, we give a very simple proof of the known fact that every Sturmian word begins in arbitrarily long squares.

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1. Introduction

Let $\mathcal{A}$ be a finite alphabet of letters and let $\omega$ be an infinite sequence of elements from $\mathcal{A}$. Using the terminology of combinatorics on words, $\omega$ is called an infinite word over $\mathcal{A}$, any string of its consecutive letters is called its factor, and any factor of $\omega$ starting from the first letter of $\omega$ is called its prefix.

For every positive integer $n$, let $p(\omega,n)$ be the number of distinct factors of $\omega$ of length $n$. Obviously, $1 \leq p(\omega,n) \leq |\mathcal{A}|^n$ for each $n \geq 1$. By an old result of Morse and Hedlund [23], for any word $\omega$ over $\mathcal{A}$, the complexity function $p(\omega,n)$ is either bounded by an absolute constant independent of $n$ (iff the word $\omega$ is ultimately periodic) or $p(\omega,n) \geq n + 1$ for each $n \geq 1$. The words $\omega$ for which $p(\omega,n) = n + 1$ for every $n \in \mathbb{N}$ exist and are called Sturmian words. Clearly, $p(\omega,1) = 2$ implies that a Sturmian word $\omega$ must be an infinite word over an alphabet of two letters.

It is well-known that the Fibonacci word

$$f = 010010100101001010010010100101001001\ldots,$$

which is the limit $f = \lim_{n \to \infty} f_n$ of the sequence of words $f_{-1} = 1, f_0 = 0$ and $f_{n+1} = f_n f_{n-1}$ for $n \geq 0$, is Sturmian. See a survey [9] for some extremal properties of the Fibonacci word. Sturmian sequences (also known as Beatty sequences) appear in symbolic dynamics, ergodic theory, number theory, computer graphics,
pattern recognition, crystallography, etc. See, for instance, [7,10,12,16,17,25,26]. For a more systematic exposition one can consult Chapter 2 in [20], Chapter 10 in [5] and also a collective book under pseudonym of Pytheas Fogg [24].

Given an infinite word \( \omega \) and a finite factor \( w \) of \( \omega \), it is often important to know the highest power of \( w \) which appears as a factor of \( \omega \). Let \( |w| \) be the length of the word \( w \). Then, for any fixed real number \( \tau > 0 \), the \( \tau \)th power of a finite word \( w \) is the word of length \( \lfloor |w| \tau \rfloor \) given by \( w^\tau = w^{\lfloor \tau \rfloor}u \), where \( u \) is the prefix of \( w \) of length \( (\lfloor \tau \rfloor - \lfloor \tau \rfloor)|w| \). For example, \( 01001^{2.1} = 01001010010 \). Let \( \tau_n \) be the supremum taken over \( \tau > 1 \) such that \( w^\tau \) is a factor of \( \omega \) for at least one factor \( w \) of \( \omega \) satisfying \( |w| = n \). (It is possible that \( \tau_n = \infty \) for some fixed \( n \in \mathbb{N} \).) Then the quantity \( \limsup_{n \to \infty} \tau_n \) is called the index of \( \omega \). It is known that the index of every Sturmian word is at least 3 (see [6,22,26] or Chap. 2 in [20]). On the other hand, by Theorem 1.2 of [8], there exist Sturmian words with index equal to 3. The index of \( \omega \) is often called a critical exponent of \( \alpha \) and sometimes is defined as \( \sup_{n \geq 1} \tau_n \). In the sense of this definition, it was shown recently that each number \( \alpha > 1 \) is a critical exponent of some infinite word [19] and that each number \( \alpha > 2 \) is a critical exponent of some infinite word over an alphabet of two letters [11].

For some applications, it is important not only to know whether a word \( \omega \) has a finite or infinite index and how large this index (or critical exponent) is, but one also needs to determine how far from the beginning of the word \( \omega \) a non-trivial power \( w^\tau \) with \( \tau > 1 \) occurs. For example, the fact that a non-trivial power of a longer and longer word occurs not far from the beginning of an infinite word is crucial in [1]. It is proved there that if \( \alpha \) is a Pisot number or a Salem number and \( \omega = (d_k)_{k \geq 1} \) is a bounded sequence of integers, which is stammering (see the definition below), then the number \( \sum_{k=1}^{\infty} d_k \tau^{-k} \) either belongs to the field \( \mathbb{Q}(\alpha) \) or is transcendental. (See also [15] for earlier work and [3] for subsequent work related to this old problem of digit distribution of an irrational algebraic number in base \( b \geq 2 \).) It is remarked in [1] that if \( \alpha \) is an arbitrary algebraic number then for the same conclusion a somewhat stronger condition on the word \( \omega \) is required. The paper [14] related to an unsolved Mahler’s problem [21] about the powers of \( 3/2 \) modulo 1 is another example where this kind of information is necessary for Sturmian words \( \omega \). More precisely, in [14] one needs to estimate the smallest value of the supremum \( \sup_{\tau \geq 0, \tau \geq 2} \frac{\tau^{2+\sigma}}{1+\tau} \) taken over all Sturmian words \( \omega \), where \( \omega \) has infinitely many prefixes of the form \( uv^\tau \), with \( |u| \leq \sigma |v| \).

Let \( \sigma \) and \( \tau \) be two real numbers satisfying \( 0 \leq \sigma < \infty \) and \( \tau > 1 \). Motivated by [1] (see also [3]), we say that an infinite word (sequence) \( \omega \) over an alphabet \( \mathcal{A} \) is a \((\sigma, \tau)\)-stammering word (or a \((\sigma, \tau)\)-stammering sequence) if there exist two sequences of finite words \( (U_n)_{n \geq 1} \) and \( (V_n)_{n \geq 1} \) over \( \mathcal{A} \) such that

(i) for any \( n \geq 1 \) the word \( U_nV_n^\tau \) is a prefix of \( \omega \);
(ii) \( |U_n| \leq \sigma |V_n| \) for every \( n \geq 1 \);
(iii) \( |V_n| \to \infty \) as \( n \to \infty \).

By the definition given in [1], a word \( \omega \) is called a stammering word if it is a \((\sigma, \tau)\)-stammering word for some fixed pair \((\sigma, \tau)\), where \( 0 \leq \sigma < \infty \) and \( \tau > 1 \). We remark that in terms of our definition it is proved in [1] that if for a word \( \omega \)
there is an integer \( t \geq 2 \) such that \( p(\omega, n) \leq tn \) for infinitely many \( n \in \mathbb{N} \) then \( \omega \) is a \((4t, 1 + 1/t)\)-stammering word.

**Theorem 1.** Every Sturmian word is a \((0, 2)\)-stammering word.

Theorem 1 is known. See, e.g., [4] or [13] for two different proofs. In general, the constant 2 cannot be replaced by \( 2 + \varepsilon \) with \( \varepsilon > 0 \) (see Thm. 1.1 in [8]). We give the proof of Theorem 1 in just few lines (after some preliminaries in Sect. 2).

The main result of this paper is the following:

**Theorem 2.** Every Sturmian word is a \((2.855, 3)\)-stammering word.

In the proof of Theorem 2 we do not use the concepts of the slope \( \alpha \), where \( \alpha \) in an irrational number satisfying \( 0 < \alpha < 1 \), and the intercept \( \varrho \) of the Sturmian word \( \omega \), whose \( n \)th symbol over the alphabet \( \{0, 1\} \) is given as the difference \( \lfloor \alpha(n+1) + \varrho \rfloor - \lfloor \alpha n + \varrho \rfloor \) (see [23] or Chap. 2 in [20]). Since we need some information on the prefix of a Sturmian word \( \omega \) before a factor that is a cube occurs, the problem cannot be reduced to the study of characteristic Sturmian word (i.e., \( \varrho = 0 \)) with the same slope and then observing that the word \( \omega \) has the same factors as the corresponding characteristic word (as is usually done).

The proof of Theorem 2 is completely self-contained. The only simple fact we use in the preliminary Section 2 is that the word \( \omega \) over an alphabet \( \{a, b\} \) is Sturmian if and only if \( \omega \) is aperiodic and for every finite (possibly empty) factor \( w \) of \( \omega \), at most one of the words \( awa \) and \( bwb \) is the factor of \( \omega \) (see, e.g., Prop. 2.1.3 and Thm. 2.1.5 in [20]).

2. Sturmian words

**Lemma 3.** Let \( \omega \) be a Sturmian word over \( \{a, b\} \) that starts with the letter \( a \). Then there is a unique integer \( k \geq 0 \) such that \( \omega \) is composed of the blocks \( A = ab^{k+1} \) and \( B = ab^k \) only. The word \( \omega' \) obtained from \( \omega \) by replacing \( ab^{k+1} \) with \( A \) and \( ab^k \) with \( B \) is a Sturmian word over \( \{A, B\} \).

**Proof.** The word \( \omega \) can be expressed in the form \( ab^{k_1}ab^{k_2}ab^{k_3}... \) with some integer \( k_1, k_2, k_3, ... \geq 0 \). Let \( k = \min\{k_1, k_2, k_3, ...\} \). Note that \( b^{k+2} \) cannot be a factor of \( \omega \), because then both \( ab^k a \) and \( b^{k+2} \) would be factors of \( \omega \), a contradiction. So \( \omega \) is composed of the blocks \( B = ab^k \) and \( A = ab^{k+1} \) only.

Consider the word \( \omega' \) over \( \{A, B\} \) obtained from \( \omega \). Clearly, \( \omega' \) is aperiodic. If it is not Sturmian then there exists a word \( X \) over \( \{A, B\} \) such that \( AXA \) and \( BXB \) are factors of \( \omega' \). Thus either \( BXBB \) or \( BXBA \) is a factor of \( \omega' \). In both cases, for some word \( Y \) over \( \{a, b\} \) obtained from \( X \) by replacing \( A \) by \( ab^{k+1} \) and \( B \) by \( ab^k \), the words \( b^{k+1}Yab^{k+1} = bb^kYab^k \) and \( ab^kYab^k \) are factors of \( \omega \), a contradiction. \( \square \)
We say that $\omega'$ is the block-word of the Sturmian word $\omega$. Lemma 3 also follows from a more general result of Justin and Vuillon [18] (see also [27]).

**Theorem 4.** Let $\omega = \omega_0$ be a Sturmian word over $\{A_0, B_0\}$ and let $(\omega_k)_{k \geq 1}$ be a sequence of words such that each $\omega_k$ is the block-word of $\omega_{k-1}$. Then there is a unique sequence of integers $s_1, s_2, s_3, \ldots \geq 0$ such that $\omega_k$ is a Sturmian word over the alphabet $\{A_k, B_k\}$, where

$$A_k = U_{k-1}V_{k-1}^{s_k+1}, \quad B_k = U_{k-1}V_{k-1}^{s_k} \quad \text{with} \quad \{U_{k-1}, V_{k-1}\} = \{A_{k-1}, B_{k-1}\}$$

for each $k \geq 1$. In particular, $B_k$ is a prefix of $A_k$ for every $k \geq 1$, so $|A_k| > |B_k|$, where $|A_k|$ and $|B_k|$ denote the lengths of the words $A_k, B_k$ in the alphabet $\{A_0, B_0\}$. Moreover, for infinitely many $k \in \mathbb{N}$, we have $|A_k| < 2|B_k|$. Finally, $|A_k|, |B_k| \to \infty$ as $k \to \infty$.

**Proof.** The sequence of Sturmian block-words $(\omega_k)_{k \geq 1}$ exists, by Lemma 3. If the first letter of $\omega_{k-1}$ is $A_{k-1}$ then, by Lemma 3, $A_k = A_{k-1}B_{k-1}^{s_k+1}$, $B_k = A_{k-1}B_{k-1}^{s_k}$, where $s_k \geq 0$. Therefore,

$$\frac{|A_k|}{|B_k|} = \frac{|A_{k-1}| + (s_k + 1)|B_{k-1}|}{|A_{k-1}| + s_k|B_{k-1}|} < 2.$$

Suppose the first letter of $\omega_{k-1}$ is $B_{k-1}$ for all sufficiently large $k$. Then $A_k = B_{k-1}A_{k-1}^{s_k+1}$, $B_k = B_{k-1}A_{k-1}^{s_k}$. If $s_k \geq 1$ for infinitely many $k \in \mathbb{N}$ then, for those $k$, we have

$$\frac{|A_k|}{|B_k|} = \frac{|B_{k-1}| + (s_k + 1)|A_{k-1}|}{|B_{k-1}| + s_k|A_{k-1}|} < 2.$$

Hence, in both cases, $|A_k| < 2|B_k|$ for infinitely many $k \in \mathbb{N}$.

Alternatively, there exists a positive integer $t$ such that, firstly, the first letter of $\omega_{k-1}$ is $B_{k-1}$ and, secondly, $A_k = B_k-1A_{k-1}, B_k = B_{k-1}$ for every $k \geq t$. We will show that this is impossible. Indeed, let $t \geq 1$ be an integer such that $\omega_{t-1}$ has a prefix $B_{t-1}A_{t-1}$. Then the words $\omega_k$, where $k = t - 1, \ldots, t + l - 2$, begin with $B_k$ (all equal to $B_{t-1}$). The word $\omega_{t+l-2}$ begins with $B_{t+l-2}A_{t+l-2}$. By our assumption, $\omega_{t+l-1}$ begins with $B_{t+l-1}$, hence $B_{t+l-1} = B_{t+l-1}A_{t+l-2}$ and $A_{t+l-1} = B_{t+l-1}A_{t+l-2}$.

Finally, it is clear that $|A_k| \to \infty$ as $k \to \infty$. Furthermore, $|B_k| \to \infty$ as $k \to \infty$, because the sequence $(|B_k|)_{k \geq 0}$ is non-decreasing and, as we just proved, $|B_k| > |A_k|/2$ for infinitely many $k \in \mathbb{N}$.

\[ \square \]

3. **Proofs of Theorems 1 and 2**

**Proof of Theorem 1:** Let $k$ be a sufficiently large integer. If the word $\omega_k$ begins with the letter $B_k$ then $B_k^2$ is a prefix of $\omega_k$, because $B_k$ is a prefix of $A_k$. Suppose that $\omega_k$ begins with $A_k$. Then, by Lemma 3, the word $\omega_k$ consists of the blocks $A_kB_k^{s_k+1}$ and $A_kB_k^{s_k}$ only, where $s = s_{k+1} \geq 0$. Clearly, $(A_kB_k^s)^2$ is a prefix of $\omega_k$, unless $\omega_k$ begins with the block $A_kB_k^{s_k+1}$. However, if it begins with $A_kB_k^{s_k+1}$
then, independent on whether the second block is $A_k B_{k+1}^s$ or $A_k B_{k+1}^s$, the word $\omega_k$ begins with $(A_k B_{k+1}^s)^2$, because $B_k$ is a prefix of $A_k$. Since, by Theorem 4, $|A_k|, |B_k| \to \infty$ as $k \to \infty$, this proves that every Sturmian word $\omega$ begins in arbitrarily long squares.

Proof of Theorem 2: Let $k$ be any of those (infinitely many) $k$’s for which $|B_k| \prec |A_k| < 2 |B_k|$. For brevity, let us write $A$ and $B$ for $A_k$ and $B_k$, respectively, so that $|B| < |A| < 2 |B|$. Below, without further notice, we shall use the fact that $B$ is a prefix of $A$.

By Lemma 3, the word $\omega = \omega_k$ consists either of the blocks $AB^s A$ and $AB^s$ only or of the blocks $BA^s A$ and $BA^s$ only, where $s = s_{k+1} \geq 0$. Suppose first that we have the blocks $AB^s A$ and $AB^s$, where $s \geq 2$. Then $AB^3$ is a prefix of this word, because $B$ is a prefix of $A$. Also, $|A| < 2 |B|$. So if there are infinitely many such cases then, by Theorem 4 claiming that $|B_k| \to \infty$ as $k \to \infty$, $\omega$ is a $(2, 3)$-stammering word, which is more than required. Another simple case is when we have the blocks $BA^s A$ and $BA^s$, where $s \geq 3$, only. Then $BA^4$ is a prefix of this word and $|B| < |A|$. So if there are infinitely many such cases then $\omega$ is a $(1, 3)$-stammering word, which is more than required.

We claim that in the remaining cases, listed in the table below, we have either a cube occurring as a prefix of $\omega$ (in which case $\omega$ is a $(0, 3)$-stammering word) or $\omega$ has one of the prefixes listed in the third column of the table. Note that each prefix there has the form $UV^3$, where $U$ and $V$ are some words over the alphabet $\{A, B\}$. The maximal value of the quotient $|U|/|V|$ is given in the last column of the table. For each $UV^3$, the upper bound for the constant $|U|/|V|$ is calculated using the inequality $|B| < |A| < 2 |B|$.

<table>
<thead>
<tr>
<th></th>
<th>$AB^2 A$</th>
<th>$AB^s A(BAB),^s$, $(AB)^s(BA)^s$, $(AB)^s BA(BAB)^s$</th>
<th>$AB^s$</th>
<th>$A(AB)^3$, $A^3 BA(AB)^3$, $ABA^3$, $ABA(AB)^3$, or case 1</th>
<th>$BA^2 A$</th>
<th>$A^2 BA^3$</th>
<th>$BA^3$, $BA^2 A^3$, $BA^4 BBA(A^2 B)^3$</th>
<th>$BA^2 B$, $BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$</th>
<th>$BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$</th>
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<td>$AB^2 A$</td>
<td>$AB^s A(BAB),^s$, $(AB)^s(BA)^s$, $(AB)^s BA(BAB)^s$</td>
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<td>$BA^2 A$</td>
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<td>2</td>
<td>$AB^2 A$</td>
<td>$AB^s A(BAB),^s$, $(AB)^s(BA)^s$, $(AB)^s BA(BAB)^s$</td>
<td>$AB^s$</td>
<td>$A(AB)^3$, $A^3 BA(AB)^3$, $ABA^3$, $ABA(AB)^3$, or case 1</td>
<td>$BA^2 A$</td>
<td>$A^2 BA^3$</td>
<td>$BA^3$, $BA^2 A^3$, $BA^4 BBA(A^2 B)^3$</td>
<td>$BA^2 B$, $BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$</td>
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<td>$BA^3$, $BA^2 A^3$, $BA^4 BBA(A^2 B)^3$, $BA^4 BABA(A^2 B)^3$</td>
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We begin with case 1, when $\omega$ consists of the blocks $AB^2 A$ and $AB$. If the first block is $AB^2$ then $AB^3$ is a prefix of $\omega$. It is one of the values listed in the third column of the first row. Suppose that $AB$ is the first block. If the next block is $AB$ again then $\omega$ begins with $(AB)^3$, which is a cube. Alternatively, the next block is $AB^2$, so $\omega$ has one of the two prefixes $ABAB^2 A$ or $ABAB^2 A$. In the latter case, independent of the third block, $A(BAB)^3$ a prefix of $\omega$ (which is in the table). Suppose that the prefix is $ABAB^2 A$. If the next block is $AB$ then $(AB)^2(BA)^3$ is a prefix of $\omega$. Let $AB^2$ be the next block. Then two possibilities are $ABAB^2 A$ and $ABAB^2 A$ and $ABAB^2 A$. The first possibility gives the prefix $(ABAB^2)^3$ which is a cube, whereas the second possibility gives $(AB)^2 BA(BAB)^3$. 
From $|B| < |A| < 2|B|$, we find that the quotients

$$\frac{|A|}{|B|}, \frac{|A|}{|A| + 2|B|}, \frac{2|A| + 2|B|}{|A| + |B|}, \frac{3|A| + 3|B|}{|A| + 2|B|},$$

are all smaller than $9/4$.

Consider case 2 when $\omega$ consists of the blocks $AB$ and $A$. If the word $\omega$ begins with $AA$ then it begins with a cube $A^3$. Similarly, if $\omega$ begins with $AD$ then a cube $(AB)^3$ is a prefix of $\omega$. So there are two possibilities $AAB$ or $ABA$. As above it is easy to see that $AABAB$ gives $A(AB)^3$, which is one of the prefixes in the corresponding row. Otherwise, $ABA$ splits into $AABA$ (which gives the prefix $A^2BA^3$) and $AABAAB$. Here, the next block $A$ leads to $(A^2B)^3$. Assume that the next block is $AB$. Then the prefix $AABAABAB$ leads to the prefix $A^2BA(AB)^3$.

The second possibility $ABA$ gives $ABA^3$ if the next block is $A$. Otherwise, we have the following two cases $ABAABA$ (which leads to $ABA(AB)^3$) and $ABAABA$. In the latter case, the next $AB$ leads to the cube $(ABA)^3$, whereas the next $A$ gives $ABAABA$. Although this leads to $ABAABA$, we do not stop here, because the prefix $ABAAB$ before the cube $A^3$ occurs is too large. Instead, since $\omega_k$ begins with $(AB)A(AB)A^2 = (A_kB_k)A_k(A_kB_k)A_k^2$, we observe that, by Theorem 4, the word $\omega_{k+1}$ consists of the blocks $A_{k+1} = AB$ and $B_{k+1} = A$ only. Its prefix in the alphabet $\{A_{k+1}, B_{k+1}\}$ is $A_{k+1}B_{k+1}A_{k+1}B_{k+1}^2$. Here,

$$|A_{k+1}|/|B_{k+1}| = (|A| + |B|)/|A| \in (3/2, 2) \subset (1, 2),$$

so $|B_{k+1}| < |A_{k+1}| < 2|B_{k+1}|$ and we are back to the case 1 for the word $\omega_{k+1}$ instead of $\omega_k$. Now, since $|B| < |A| < 2|B|$, the quotients

$$\frac{|A|}{|A| + |B|}, \frac{3|A| + |B|}{|A| + |B|}, \frac{|A| + |B|}{|A| + |B|}, \frac{2|A| + |B|}{|A| + |B|},$$

(calculated for $A(AB)^3$, $A^2BA(AB)^3$, $ABA^3$, $ABA(AB)^3$, respectively) are all smaller than $7/3$. For the prefix $A^2BA^3$ the quotient $(2|A| + |B|)/|A|$ is at most 3. This is greater than 2.855, so we split case 2 into two subcases 2a and 2b. The subcase 2b will be analyzed later.

Consider case 3 when $\omega$ consists of the blocks $BA^3$ and $BA^2$. The first block $BA^3$ is the first prefix of the third row. If $BA^2$ is followed by $BA^2$ then $\omega$ starts with $(BA^2)^3$. So the first two blocks are $BA^2$ and $BA^3$, giving $BA^2BA^3$. If the next block is $BA^3$ then $\omega$ begins with $B(A^2BA)^3$. The alternative case leads to $BA^2BA^3BA^2$. Independent of the fourth block, this leads to the prefix $BA^2BA(A^2B)^3$. This time,

$$\max \left( \frac{|B|}{|A|}, \frac{|B|}{3|A| + |B|}, \frac{3|A| + 2|B|}{|A| + 2|B|} \right) < \frac{5}{3}.$$

In case 4 we have the blocks $BA^2$ and $BA$. If the first two blocks are $BA$ and $BA$ then $\omega$ begins with a cube $(BA)^3$. Suppose $\omega$ begins with $BABA^2$. The next block
\( BA^2 \) leads to the prefix \( B(ABA)^3 \), whereas the next block \( BA \) gives \( BABABA^2BA \), which leads to \( (BA)^2(AB)^3 \). Next, let us consider the beginning \( BA^2BA \). Independent of the next block, this leads to the prefix \( (BA)(AB)^3 \). The remaining case is \( BA^2BA^2 \). If \( \omega \) does not begin with a cube, the next block must be \( BA \). The beginning \( BA^2BA^2BA \) leads to the prefix \( BA^2BA(AB)^3 \). We have

\[
\max \left( \frac{|B|}{2|A| + |B|}, \frac{2|A| + 2|B|}{|A| + |B|}, \frac{|A| + |B|}{|A| + |B|}, \frac{3|A| + 2|B|}{|A| + |B|} \right) < \frac{8}{5}
\]

Finally, in case 5, we have the blocks \( BA \) and \( B \). Both \( BB \) and \( BBA \) lead to the prefix \( B^3 \). The beginning \( BAB \) leads to the prefix \( BAB^3 \). Let the first two blocks be \( BA \) and \( BA \). If \( \omega_k \) does not start in a cube the next block must be \( B \). Since, by Theorem 4, the sequence \( \omega_{k+1} \) is Sturmian, the fourth block must be \( BA \), i.e., we have \( BABABA \). By the same argument, if the next block is \( B \) it must be followed by \( BA \). The prefix \( BABABBABBA \) leads to \( BA(AB)^3 \). Now suppose that the fifth block is \( BA \), i.e., we have \( (BA)^2B(AB)^2 \). In case the sixth block is \( BA \), we obtain \( (BA)^2B(AB)^3 \). Otherwise, if the sixth block is \( B \), we get \( (BA)^2B(BA^2B) \). Seventh block must be \( BA \) again. If the eight block is \( BA \) then \( \omega \) begins in a cube, so suppose that the eighth block is \( B \). Then by the above argument it must be followed by \( BA \), giving \( (BA)^2B(BA^2BBABBBA) \). This leads to the prefix \( (BA)^2BBABA(BAB)^3 \). Now, from \( |B| < |A| < |A| < |A| \), we obtain

\[
\max \left( \frac{|A| + |B|}{|A| + |B|}, \frac{2|A| + 3|B|}{|A| + |B|}, \frac{3|A| + 4|B|}{|A| + |B|} \right) < \frac{5}{2}
\]

For the prefix \( BAB^3 \) the quotient \( (|A| + |B|)/|A| \) is at most 3. Since this is greater than 2.855, we split case 5 into two subcases 5a and 5b.

This would finish the proof of the theorem with even better constant 8/3, unless for each sufficiently large \( k \) in the word \( \omega_k \) with \( |B_k| < |A_k| < 2|B_k| \) we have either case 2b or case 5b. Indeed, then the cases 1, 2a, 3, 4, 5a show that the word \( \omega \) has infinitely many prefixes of the form \( U_nV_n^3 \) with \( |U_n| < 8|V_n|/3 \) and \( \lim_{n \to \infty} |V_n| = \infty \).

To complete the proof assume that there is a \( k_0 \) such that for each \( k \geq k_0 \) satisfying \( 1 < q_k := |A_k|/|B_k| < 2 \) the word \( A_k^2B_kA_k^2 \) is a prefix of the word \( \omega_k \) consisting of the blocks \( A_k^2B_k \) and \( A_k \) (case 2b) or \( B_kA_kB_k^2 \) is a prefix of \( \omega_k \) consisting of the blocks \( B_kA_k \) and \( B_k \) (case 5b).

Let \( \delta = (3\sqrt{2} - 5)/10 = 0.17082 \ldots \) be the root of

\[
\delta^2 + \delta = 1/5.
\]

If there are infinitely many \( k \)'s for which we have case 2b and \( q_k \geq 1 + \delta \), then the proof is completed, because \( A_k^2B_kA_k^2 \) is a prefix of \( \omega_k \) and

\[
(2|A_k| + |B_k|)/|A_k| = 2 + 1/q_k \leq 2 + 1/(1 + \delta) = 2 + 5\delta < 2.855
\]

for each such \( k \). Similarly, if there are infinitely many \( k \)'s for which we have case 5b and \( q_k \leq 1 + 5\delta < 1.855 \), then the proof is also completed, because \( B_kA_kB_k^2 \) is
a prefix of $\omega_k$ and

$$
(|A| + |B_k|)/|B_k| = 1 + q_k \leq 2 + 5\delta < 2.855
$$

for each such $k$. So we can assume that $q_k < 1 + \delta$ in case 2b and $q_k > 1 + 5\delta$ in case 5b. In particular, no $k \geq k_0$ exists for which

$$
1 + \delta \leq q_k \leq 1 + 5\delta.
$$

Clearly, in case 2b the word $\omega_k$ is composed of the blocks $A_{k+1} = A_k B_k$ and $B_{k+1} = A_k$, so for the next word $\omega_{k+1}$ using $1 < q_k < 1 + \delta$ we obtain

$$
q_{k+1} = |A_{k+1}|/|B_{k+1}| = 1 + |B_k|/|A_k| = 1 + 1/q_k \in (1 + 5\delta, 2).
$$

Consequently, the word $\omega_{k+1}$ satisfies the condition 5b, namely, $\omega_{k+1}$ consists of the blocks $A_{k+2} = B_{k+1} A_{k+1}$ and $B_{k+2} = B_{k+1}$ and one of its prefixes must be $B_{k+1} A_{k+1} B_{k+1}^2$. By Lemma 3, the next block-word consists of the blocks

$$
A_{k+3} = B_{k+1} A_{k+1} B_{k+1}^2
$$

and

$$
B_{k+3} = B_{k+1} A_{k+1} B_{k+1}^2
$$

for some integer $s \geq 2$. If $s \geq 4$, then $B_{k+1} A_{k+1} (B_{k+1}^2)^3$ is a prefix of $\omega$. So the bound

$$
\frac{|A_{k+1}| + |B_{k+1}|}{2|B_{k+1}|} = \frac{1}{2} + \frac{q_{k+1}}{2} < \frac{3}{2} = 1.5 < 2.855
$$

gives the required estimate. Otherwise, let $2 \leq s \leq 3$. Then using $q_{k+1} = 1 + 1/q_k > 1 + 1/(1 + \delta) = 1 + 5\delta$ we obtain

$$
q_{k+3} = \frac{|A_{k+1}| + (s + 2)|B_{k+1}|}{|A_{k+1}| + (s + 1)|B_{k+1}|} = \frac{q_{k+1} + s + 2}{q_{k+1} + s + 1} \geq \frac{q_{k+1} + 5}{q_{k+1} + 4} \geq \frac{6 + 5\delta}{5 + 5\delta} = 1 + \delta
$$

and

$$
q_{k+3} = \frac{q_{k+1} + s + 2}{q_{k+1} + s + 1} = 1 + \frac{1}{q_{k+1} + s + 1} < 1.25.
$$

It follows that for some $k \geq k_0$ we have $q_k \in [1 + \delta, 1.25] \subset [1 + \delta, 1 + 5\delta]$, a contradiction. This completes the proof of the theorem. □

In fact, we proved Theorem 2 with the constant

$$
2 + 5\delta = \frac{3\sqrt{5} - 1}{2} = 2.8541\ldots
$$

which is slightly smaller than 2.855.
4. Concluding remarks

We already observed in Section 1 that the constant 3 of Theorem 2 is optimal. More precisely, for every $\varepsilon > 0$, there exists a Sturmian word which is not a $(\sigma, 3+\varepsilon)$-stammering word for every $\sigma \geq 0$. The constant 2.855 in Theorem 2 is not optimal! By some further analysis of different prefixes that can occur as prefixes of a Sturmian word $\omega$ before a cube this constant can be reduced. We do not know the best possible constant. However, one can show that the Fibonacci word is a $((\sqrt{5}+1)/2, 3)$-stammering word but is not a $((\sqrt{5}+1)/2-\varepsilon, 3)$-stammering word for every positive number $\varepsilon$.

Given any $\tau \leq 3$, let $\sigma(\tau)$ be the infimum over all $\sigma \geq 0$ such that every Sturmian word is a $(\sigma, \tau)$-stammering word. By Theorem 1, $\sigma(\tau) = 0$ for $\tau \leq 2$. Theorem 2 combined with the above observation implies that $1.618 < \sigma(3) < 2.855$.

Problem 1. Evaluate $\sigma(\tau)$ for each $\tau \in (2, 3]$.

One can also consider a similar problem if $\tau$ is not fixed. Following [2], we say that an infinite word (sequence) $\omega$ over an alphabet $A$ satisfies Condition $(\ast)_{g}$ if there exist two sequences of finite words $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ over $A$ and a sequence of positive real numbers $(\tau_n)_{n \geq 1}$ such that

(i) for any $n \geq 1$ the word $U_nV_n^{\tau_n}$ is a prefix of $\omega$;
(ii) $|U_nV_n^{\tau_n}| \geq g|U_nV_n|$ for every $n \geq 1$;
(iii) $|V_n^{\tau_n}| \to \infty$ as $n \to \infty$.

Then the Diophantine exponent of $\omega$, $\text{Dio}(\omega)$, is defined as the supremum of the real numbers $g$ for which $\omega$ satisfies Condition $(\ast)_{g}$.

Problem 2. Evaluate $D(S) := \inf_{\omega \in \text{Sturmian}} \text{Dio}(\omega)$.

Obviously, if some word is a $(\sigma, \tau)$-stammering word for a fixed pair $(\sigma, \tau)$ then it satisfies Condition $(\ast)_{g}$ for $g = (\sigma + \tau)/(\sigma + 1)$. Hence

$$D(S) \geq \sup_{\tau \in [2, 3]} \frac{\sigma(\tau) + \tau}{\sigma(\tau) + 1}.$$ 

Selecting $\tau = 2$ we obtain $D(S) \geq 2$. We do not know whether $D(S) = 2$ or $D(S) > 2$. The inequality $D(S) > 2$ (if proved) has some applications to Mahler’s problem: one can use the same method as in [14].

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