FROM BI-IDEALS TO PERIODICITY

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Abstract. The necessary and sufficient conditions are extracted for periodicity of bi-ideals. They cover infinitely and finitely generated bi-ideals.

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1. INTRODUCTION

The periodicities are fundamental objects, due to their primary importance in word combinatorics [8,9] as well as in various applications. The study of periodicities is motivated by the needs of molecular biology [6] and computer science. Particularly, we mention here such fields as string matching algorithms [4], text compression [13] and cryptography [11].

In different areas of mathematics, people consider a lot of hierarchies which are typically used to classify some objects according to their complexity. Here we deal with the hierarchy

\[ \mathcal{B} \supseteq \mathcal{P}, \]

where

\( \mathcal{B} \) is the class of bi-ideals,
\( \mathcal{P} \) is the class of periodic words.

This hierarchy comes from combinatorics on words, where these classes are being investigated intensively (cf. [2,8–10]). Bi-ideal sequences have been considered, with different names, by several authors in algebra and combinatorics [1,3,7,12,14].

Every bi-ideal \( x \) is the limit of some bi-ideal sequence \( (v_i) \). This bi-ideal sequence can be represented uniquely by the sequence \( (u_i) \), where \( v_0 = u_0 \) and

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∀i ≥ 0 v_{i+1} = v_i u_{i+1} v_i. We characterize the periodic words through this representation. At first we give an exhaustive description (Th. 3.7) of periodicity for all classes of bi-ideals. Then for periodic bi-ideals we demonstrate if every \( u_i \) appears infinitely often then every \( u_i \) is a power of the certain word. This leads to the effective method for finitely generated bi-ideals to check whether the bi-ideals are periodic.

2. Preliminaries

In this section we present most of the notations and terminology used in this paper. Our terminology is more or less standard (cf. [10]) so that a specialist reader may wish to consult this section only if need arise.

Let \( A \) be a finite non-empty set and \( A^* \) the free monoid generated by \( A \). The set \( A \) is also called an alphabet, its elements letters and those of \( A^* \) finite words.

The role of the identity element is performed by the empty word which is denoted by \( \lambda \). We set \( A^+ = A^* \{ \lambda \} \).

A word \( w \in A^+ \) can be written uniquely as a sequence of letters as \( w = w_1 w_2 \ldots w_l \), with \( w_i \in A, 1 ≤ i ≤ l, l > 0 \). The integer \( l \) is called the length of \( w \) and denoted \( |w| \). The length of \( \lambda \) is 0. We set \( w_0 = \lambda \) and \( \forall i w_{i+1} = w_i w \); \( w^+ = \bigcup_{i=1}^{\infty} \{ w^i \} \), \( w^* = w^+ \cup \{ \lambda \} \).

A positive integer \( p \) is called a period of \( w = w_1 w_2 \ldots w_l \) if the following condition is satisfied:

\[
1 ≤ i ≤ l - p \Rightarrow w_i = w_{i+p}.
\]

We recall the important periodicity theorem due to Fine and Wilf [5]:

**Theorem 2.1.** Let \( w \) be a word having periods \( p \) and \( q \) and denote by \( \gcd(p, q) \) the greatest common divisor of \( p \) and \( q \). If \( |w| ≥ p + q - \gcd(p, q) \), then \( w \) has also the period \( \gcd(p, q) \).

The word \( w' \in A^* \) is called a factor (or subword) of \( w \in A^* \) if there exist \( u, v \in A^* \) such that \( w = uwv \). The word \( u \) (respectively \( v \)) is called a prefix (respectively a suffix) of \( w \). The ordered triple \((u, w', v)\) is called an occurrence of \( w' \) in \( w \). The factor \( w' \) is called a proper factor if \( w \neq w' \). We denote respectively by \( F(w) \), \( \text{Pref}(w) \) and \( \text{Suff}(w) \) the sets of \( w \) factors, prefixes and suffixes.

An (indexed) infinite word \( x \) on the alphabet \( A \) is any total map \( x : \mathbb{N} \rightarrow A \). We set for any \( i ≥ 0, x_i = x(i) \) and write

\[
x = (x_i) = x_0 x_1 \ldots x_n \ldots
\]

The set of all the infinite words over \( A \) is denoted by \( A^\omega \).

The word \( w' \in A^* \) is a factor of \( x \in A^\omega \) if there exist \( u \in A^*, y \in A^\omega \) such that \( x = uw'y \). The word \( u \) (respectively \( y \)) is called a prefix (respectively a suffix).
of $x$. We denote respectively by $F(x)$, $\text{Pref}(x)$ and $\text{Suff}(x)$ the sets of $x$ factors, prefixes and suffixes. For any $0 \leq m \leq n$, both $x[m, n]$ and $x[m, n + 1]$ denote a factor $x_m x_{m+1} \cdots x_n$. The indexed word $x[m, n]$ is called an occurrence of $w'$ in $x$ if $w' = x[m, n]$. The suffix $x_n x_{n+1} \cdots x_{n+i} \cdots$ is denoted by $x[n, \infty)$.

If $v \in A^+$ we denote by $v^\omega$ the infinite word $v v \cdots v \cdots$. This word $v^\omega$ is called a periodic word. The concatenation of $u = u_1 u_2 \cdots u_k \in A^*$ and $x \in A^\omega$ is the infinite word $ux = u_1 u_2 \cdots u_k x_0 x_1 \cdots x_n \cdots$.

A word $x$ is called ultimately periodic if there exist words $u \in A^*$, $v \in A^+$ such that $x = uv^\omega$. In this case, $|u|$ and $|v|$ are called, respectively, an anti-period and a period of $x$.

A sequence of words of $A^*$

$v_0, v_1, \ldots, v_n, \ldots$

is called a bi-ideal sequence if $\forall i \geq 0 \ (v_{i+1} \in v_i A^* v_i)$. The term “a bi-ideal sequence” is due to the fact that $\forall i \geq 0 \ (v_i A^* v_i) $ is a bi-ideal of $A^*$.

**Corollary 2.2.** Let $(v_n)$ be a bi-ideal sequence. Then

$v_m \in \text{Pref}(v_n) \cap \text{Suff}(v_n)$

for all $m \leq n$.

A bi-ideal sequence $v_0, v_1, \ldots, v_n, \ldots$ is called proper if $v_0 \neq \lambda$. In the following the term bi-ideal sequence will be referred only to proper bi-ideal sequences.

If $v_0, v_1, \ldots, v_n, \ldots$ is a bi-ideal sequence, then there exists a unique sequence of words

$u_0, u_1, \ldots, u_n, \ldots$

such that

$v_0 = u_0, \ \forall i \geq 0 \ (v_{i+1} = v_i u_{i+1} v_i)$. Let us consider $u, v \in A^\infty = A^* \cup A^\omega$. Then $d(u, v) = 0$ if $u = v$, otherwise

$d(u, v) = 2^{-n}$,

where

$n = \max \{ |w| \mid w \in \text{Pref}(u) \cap \text{Pref}(v) \}$.

It is called a prefix metric.

Let $v_0, v_1, \ldots, v_n \ldots$ be an infinite bi-ideal sequence, where $v_0 = u_0$ and $\forall i \geq 0 \ (v_{i+1} = v_i u_{i+1} v_i)$. Since for all $i \geq 0$ the word $v_i$ is a prefix of the next word $v_{i+1}$ the sequence $(v_i)$ converges, with respect to the prefix metric, to the infinite word $x \in A^\omega$

$x = v_0 (u_1 v_0) (u_2 v_1) \cdots (u_n v_{n-1}) \cdots$

This word $x$ is called a bi-ideal. We say the sequence $(u_i)$ generates the bi-ideal $x$. 
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Convention. Let \( x \) be a bi-ideal generated by \((u_i)\), then \( x = \lim_{i \to \infty} v_i \), where \( v_0 = u_0 \) and \( v_{i+1} = v_i u_{i+1} v_i \). We adopt this notational convention henceforth.

Let \( x \) be an infinite word. A factor \( u \) of \( x \) is called recurrent if it occurs infinitely often in \( x \). The word \( x \) is called recurrent when any of its factors is recurrent.

**Proposition 2.3.** (see, e.g., [10]) A word \( x \) is recurrent if and only if it is a bi-ideal.

**Lemma 2.4.** (see, e.g., [10]) Let \( x \in A^\omega \) be an ultimately periodic word. If \( x \) is recurrent, then \( x \) is periodic.

Due to this lemma we can restrict ourselves. Therefore we investigate only the periodicity of bi-ideals and say nothing about ultimate periodicity.

### 3. The periodicity of bi-ideals

The following three lemmas are very easy, but they turn out to be extremely useful:

**Lemma 3.1.** If \( x = w^\omega \) and \( T \) is the minimal period of the word \( x \), then \( T \mid |w| \), i.e. \( T \) divides \( |w| \).

**Proof.** Let \( n = T |w| \), then both \( T \) and \( |w| \) are periods of the word \( x(0, n) \). Hence (Th. 2.1) \( t = \gcd(T, |w|) \) is a period of \( x(0, n) \). Now we have

\[
\forall i \ x(i, n) = x[i+1, n+i]
\]

Therefore \( t \) is a period of \( x \). Since \( T \) is the minimal period of the word \( x \), then \( t \geq T \geq \gcd(T, |w|) = t \). Hence \( T = \gcd(T, |w|) \), thereby \( T \mid |w| \). \( \square \)

**Lemma 3.2.** If \( x = u^\omega v \) and \( |w| = |v| \), then \( vy = y = v^\omega \).

**Proof.** Let \( |w| = t \) and \( |u| = k + 1 \), then \( v = x_{k+1} x_{k+2} \ldots x_{k+t} \), since \( |v| = |w| \). We have \( \forall i \ x_{i+t} = x_i \), therefore

\[
\forall j \in \mathbb{N} \forall s \ x_{k+j} = x_{k+j+s}. \quad \square
\]

**Lemma 3.3.** If \( \exists u \in A^+ \) \( ux = x \in A^\omega \), then a word \( x \) is periodic with the minimal period \( T \mid |u| \).

**Proof.** Let \( u = a_1 a_2 \ldots a_{t-1} \), where \( \forall j \ a_j \in A \), and \( y = ux \), then \( \forall i \ x_i = y_{i+t} \). Let \( y = ux = x \).

Hence

\[
\forall i \ y_i = x_i = y_{i+t}.
\]

This means that \( y \) is periodic with a period \( t \). Since \( y = x \), then \( x \) is periodic with a period \( t \) too. Let \( T \) be the minimal period of \( x \), then by Lemma 3.1 \( T \mid |u| \), i.e. \( T \mid |u| \). \( \square \)
Corollary 3.4. Let $|v|$ be the minimal period of $x = v^ω$.

If $v = x[k, k + |v|]$ then $|v| | k$.

Proof. If, for any $k$, $v = x[k, k + |v|]$, then (see Lem. 3.2)

$$x = x[0, k) v^ω = x[0, k)x.$$ 

Hence by Lemma 3.3 $|v| | x[0, k]$.

□

Lemma 3.5. If exists $n$ such that $v_n u \in v^*$ and $\forall i \in \mathbb{Z}^+$ ($u_{n+i} \in uv^*$), then $\forall i \in \mathbb{Z}^+$ ($v_{n+i} \in v^*v_n$).

Proof. If $i = 0$ then $v_{n+i} = v_n = \lambda v_n \in v^*v_n$.

Further, we shall prove the lemma by induction on $i$, i.e., suppose that $v_{n+i} \in v^*v_n$, namely,

$$\exists k \in \mathbb{N} (v_{n+i} = v^k v_n).$$

By assumption, $v_n u \in v^*$ and $u_{n+i+1} \in uv^*$, i.e.

$$\exists l \in \mathbb{N} (v_n u = v^l) \land \exists m \in \mathbb{N} (u_{n+i+1} = uv^m).$$

Hence

$$v_{n+i+1} = v_{n+i+1} v_{n+i+1} v_{n+i+1} = (v^k v_n) (uv^m) (v^k v_n) = v^k (v_n u) v^m + v^k v_n = v^k v^m + v^k v_n \in v^*v_n.$$

We have completed the inductive step.

□

Lemma 3.6. If $t$ is the period of the bi-ideal $x$ and $|v_n| \geq t$, then

$$\forall i \in \mathbb{Z}^+ u_{n+i} = u_{n+i} x.$$ 

Proof. We have $v_{n+i} = v_{n+i-1} u_{n+i} v_{n+i-1}$. Hence, if $i \in \mathbb{Z}^+$ then (Cor. 2.2)

$$\forall i \in \mathbb{Z}^+ \exists v_i v_{n+i} = v_n v_i v_n.$$

Now, by definition of $x$

$$x = v_n u_{n+1} v_n \ldots$$

$$x = v_{n+i} u_{n+i+1} v_{n+i} \ldots = v_n v_i^t v_n u_{n+i+1} v_n \ldots$$

By assumption, $x$ is periodic, therefore

$$x = v^ω,$$  where $|v| = t.$
Since $v \in \text{Pref}(v_n)$ then by Lemma 3.2
\[
x = v_n u_{n+1} x,
\]
\[
x = v_n u_{n+i+1} x.
\]
Hence $\forall i \in \mathbb{Z}_+ \ x = v_n u_{n+i} x$. Thus $\forall i \in \mathbb{Z}_+ \ u_{n+i} x = u_{n+i} x$.

**Theorem 3.7.** A bi-ideal $x$ is periodic if and only if
\[
\exists n \in \mathbb{N} \ \exists u \exists v \ (v_n u \in v^* \land \forall i \in \mathbb{Z}_+ \ u_{n+i} \in uv^*).
\]

**Proof.** $\Rightarrow$ Let $T$ be the minimal period of the word $x$, then $\exists n \in \mathbb{N} \ |v_n| \geq T$. Thus by Lemma 3.6
\[
\forall i \in \mathbb{Z}_+ \ u_{n+i} x = u_{n+i} x.
\]
Let $u$ be the longest word of the set $\bigcap_{i=1}^{\infty} \text{Pref}(u_{n+i})$ then
\[
\forall i \in \mathbb{Z}_+ \exists u'_i (u_{n+i} = uu'_i).
\]
Particularly, $\exists k u_{n+k} = u$. This means that
\[
\forall i \in \mathbb{Z}_+ \ uu'_i x = u_{n+i} x = u_{n+k} x = ux.
\]
Thus
\[
\forall i \in \mathbb{Z}_+ \ u'_i x = x.
\]
Hence by Lemma 3.3
\[
\forall i \in \mathbb{Z}_+ \ T \backslash |u'_i|.
\]
Thereby
\[
\forall i \in \mathbb{Z}_+ \ u'_i \in v^*,
\]
where $v = x[0,T)$. Thus
\[
\forall i \in \mathbb{Z}_+ \ u_{n+i} = uu'_i \in uv^*.
\]
Note
\[
x = v_n u_{n+1} v_1 \ldots = v_n uu'_1 v_n \ldots
\]
Since $u'_1 \in v^*$ and $v \in \text{Pref}(v_n)$, then [Lemma 3.2] $x = v_n ux$. Hence [Lem. 3.3]
$v_n u \in v^*$.

$\Leftarrow$ By Lemma 3.5
\[
\forall i \in \mathbb{N} \exists k_i \in \mathbb{N} \ v_{n+i} = v^{k_i} v_n.
\]
Since $\lim_{k \to \infty} |v_k| = \infty$ then $\lim_{i \to \infty} k_i = \infty$. Thus
\[
x = \lim_{k \to \infty} v_k = \lim_{i \to \infty} v_{n+i} = \lim_{i \to \infty} v^{k_i} v_n = v^\omega.
\]
4. Powers

**Observation.** If all $u_i \in w^*$ for some word $w \neq \lambda$, then the bi-ideal generated by $(u_i)$ is periodic.

The following example demonstrates the converse is not true in general.

**Example 4.1.** Let $x$ be the bi-ideal generated by $(u_i)$, where

\[
\begin{align*}
u_0 & = 0, \\
u_1 & = 1, \\
\forall i > 1 & \quad u_i = 00100.
\end{align*}
\]

Then

\[
\begin{align*}
v_0 & = 0, \\
v_1 & = 010, \\
v_2 & = 0100010010, \\
v_3 & = 01000100010010010.
\end{align*}
\]

and $x = \lim_{i \to \infty} v_i = (0100)^\omega$. Thus $x$ is periodic.

Nevertheless, if every $u_j$ appears infinitely often in $(u_i)$, then the converse is valid.

**Theorem 4.2.** Let $(u_i)$ be a sequence of words, which contains every $u_j$ infinitely often. The bi-ideal $x$ generated by $(u_i)$ is periodic if and only if

\[\exists w \forall u_i u_i \in w^*.\]

**Proof.** $\Rightarrow$ Let $x$ be a periodic bi-ideal, then by Theorem 3.7

\[\exists n \in \mathbb{N} \exists u \exists v (v_n u \in v^* \land \forall i \in \mathbb{Z}_+ \ u_{n+i} \in uv^*).\]

Hence by Lemma 3.5 $|v|$ is the period of $x$. Therefore we can assume that $|v|$ is the minimal period of $x$ and $|u| < |v|$. Since the sequence $(u_i)$ contains every $u_j$ infinitely often then by Theorem 3.7 $\forall i \in \mathbb{N} (u_i \in uv^*)$.

Now suppose that $u_i = u$ for all $i < m$ but $u_m = uv^k$, where $k > 0$. Then there exist $\alpha \in \mathbb{Z}_+$ and $y$ such that

\[x = u^\alpha v^k y.\]

(i) If $u = \lambda$ then $\forall i u_i \in v^*$.

(ii) Otherwise $u \neq \lambda$. Then (Corollary 3.4) $|v| \backslash |u|$. Hence, there exists $\beta \in \mathbb{Z}_+$ such that $\alpha |u| = \beta |v|$. Thus $x = v^\beta u^\alpha$. Contradiction, since $|u| < |v|$ and $|v|$ is the minimal period of $x$.

$\Leftarrow$ See Observation. \(\square\)
Now we turn our attention to the problem of effectiveness.

**Definition 4.3.** Assume that \((u_i)\) generates a bi-ideal \(x\). The bi-ideal \(x\) is called **finitely generated** if

\[ \exists m \forall i \forall j \,(i \equiv j \pmod{m} \Rightarrow u_i = u_j). \]

In this situation, we say that the \(m\)-tuple \((u_0, u_1, \ldots, u_{m-1})\) generates the bi-ideal \(x\).

**Theorem 4.4.** A bi-ideal \(x\) generated by \((u_0, u_1, \ldots, u_{m-1})\) is periodic if and only if

\[ \exists w \forall i \in \overline{0, m-1} \; u_i \in w^*. \]

**Proof.** As a corollary from Definition 4.3 and Theorem 4.2. \(\square\)

This theorem gives a method to generate nonperiodic bi-ideals. Let

\((u_0, u_1, \ldots, u_{m-1})\)

be any \(m\)-tuple chosen at random. Let \(v\) be any shortest word from the set

\[ \{u_0, u_1, \ldots, u_{m-1}\} \]

and \(w\) be the shortest prefix of \(v\) such that \(v \in w^+\). If there exists \(u_i\) such that \(u_i \notin w^*\) then the bi-ideal generated by \((u_0, u_1, \ldots, u_{m-1})\) is not periodic. This can be easily checked by a deterministic algorithm.

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**References**


