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CALCULI OF NET STRUCTURES AND SETS ARE SIMILAR

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Abstract. Three basic operations on labelled net structures are proposed: synchronised union, synchronised intersection and synchronised difference. The first of them is a version of known parallel composition with synchronised actions identically labelled. The operations work analogously to the ordinary union, intersection and difference on sets. It is shown that the universe of net structures with these operations is a distributive lattice and – if infinite pre/post sets of transitions are allowed – even a Boolean algebra. As a consequence, some representation theorems of this algebra are stated. The primitive objects are atomic net structures containing one transition with at most one pre-place or post-place (but not both). A simple example of a production system constructed by making use of the operations (and its transformations) is given. Some remarks on behavioural properties of compound nets are stated, in particular, how some constructing strategies may help to infer liveness. The latter issue is limited to semantics of place/transition nets without weights on arrows and with unbounded capacity of places and is not extensively investigated, since the main objective is focused on a calculus of net structures.

Mathematics Subject Classification. 68Q85.

1. Introduction

The ascertainment expressed in the title of this paper is prompted by the following observation. Any Petri net-like structure is uniquely represented by its set of transitions, if by a transition a pair of sets (or multisets) is understood: a pre-set and post-set of places. If, in such representation, a place occurs in more than one transition, then it is treated as one in the net structure

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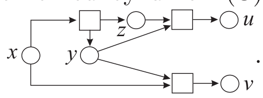
and, consequently, pictured as a place connected by arrows with all such transitions. Representing net structures as sets induces immediately natural way to combine them - just using set theoretical operations. For instance, net structures $\{(\{x\},\{y,z\}),(\{y,z\},\{u\})\}$ and $\{(\{x,y\},\{v\})\}$ when combined by union ($\cup$) yield $\{(\{x\},\{y,z\}),(\{y,z\},\{u\}),(\{x,y\},\{v\})\}$, pictorially 

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In this way, however, new branches (arrows) from/to places only are obtained. To obtain new branches from/to transitions, the latter must also be named (labelled), with the provision that no two distinct transitions may be labelled with the same label in a net structure. A combination of such net structures, called their *synchronised union*, is obtained by uniting pre-sets (post-sets) of transitions labelled identically in the constituents and leaving transitions labelled differently unchanged. Accordingly, other operations on sets are adapted to net structures. Various operations on nets have been treated since a long time in a number of works, for instance in [2,3,5,8,11–14,19–21]. In some of them, like [8,13,14,21], the category theoretic terms have been exploited to this end. However, the main objective of the latter works was a formal description of nets' behaviour, while the present paper aims rather at showing a close affinity of the proposed calculus of net structures and set calculus. Synchronised union, intersection and difference enjoy properties of respective operations on sets, hence a distributive lattice of the net structures and – if infinite pre/post sets of transitions are allowed – a Boolean algebra. Thus, some representation theorems hold for them (in [7] a complete lattice of branching processes evoked by a given Petri net has been constructed). Nonetheless, some remarks on how behaviour of a compound net depends on behaviour of its components are stated. Section 2 contains definitions of net structures, labelling of transitions, synchronised union, intersection and difference on net structures and their renaming operation. Some properties of these concepts are stated. In Section 3 the main property of the devised calculus is stated: it is a distributed lattice in any case and a Boolean algebra provided that cardinality of pre and post sets of transitions are not required to be finite. The calculus is illustrated by example of three factories showing how the calculus of net structures may be applied to combine small parts into a large system. Also some remarks on behavioural properties of nets are made. Section 4 contains three isomorphisms of the lattice of net structures, the last of them being an application of the Stone's representation theorem [18] to the calculus of net structures. In Section 5 some final remarks are made.

2. Labelled net structures and their compositions

2.1. Labelled net structure

Let $\mathbb{X}$ be a set – a universe of *net places* for all nets. An *unlabelled transition* over $\mathbb{X}$ is a pair $t = ({}^{\bullet}t,\, t^{\bullet})$ where ${}^{\bullet}t,\, t^{\bullet} \subseteq \mathbb{X}$. The set ${}^{\bullet}t$ is a *pre-set* and $t^{\bullet}$ a *post-set* of t. *An unlabelled net structure* over $\mathbb{X}$ is any set T of such transitions.

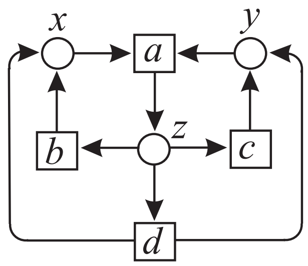
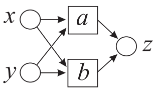


FIGURE 1.

Let members of T be labelled: $l_T : L_T \xrightarrow{onto} T$ is a labelling function, where L_T is a set of labels chosen for T so that $\mathbb{X} \cap L_T = \emptyset$. Say then "$a$ labels t in T" if $l_T(a) = t$. Thus, such a *labelled transition* is an indexed pair $(a,t)_T$ denoted a:t when it is clear from the context which T is involved. Any labelled transition of the form a:$(\emptyset,\emptyset)$ is called *isolated*. A triple $\langle L_T, T, l_T\rangle$ is a *labelled net structure*, which we often call for short just *net structure* and denote it also by T. This is, thus, a set of labelled transitions in which any label occurs once. The net structure $T = \{a : (\{x,y\},\{z\}),\ b : (\{z\},\{x\}),\ c : (\{z\},\{y\}),\ d : (\{z\},\{x,y\})\}$ is drawn in Figure 1. Here, $l_T(a) = (\{x,y\},\{z\})$, $l_T(b) = (\{z\},\{x\})$, $l_T(c) = (\{z\},\{y\})$, $l_T(d) = (\{z\},\{x,y\})$. Due to the labelling function and to the implication $a \neq b \Rightarrow a : t \neq b : t$, any ordinary Petri net structure may be represented as a set of labelled transitions. Also the so called non-simple nets, *i.e.* those where some distinct nodes (places or transitions) have identical pre and post sets, *e.g.*



is a set $\{a$:$(\{x,y\},\{z\})$,b:$(\{x,y\},\{z\})\}$. Since isolated transitions have no effect on nets' behaviour seen as evolution of state (a part of which, responsible for control, is marking), it is convenient to admit:

Assumption 2.1. Net structures T and T' are equal ($T = T'$) if they differ at most by isolated transitions, *i.e.* $\{a$:$(\emptyset,\emptyset)\} \cup T = T$, for any label a.

Remark. The manner of labelling admitted here follows that in programming languages, where a statement may be labelled by more than one label, while no two statements in different sites in a program may be labelled by the same label (*cf.* also TCSP – [9]). Petri net transition is seen as an operator on state, thus a counterpart of statement, and is a unique member of the set T. For our purpose such operator may be decorated by more than one label.

Introducing weights on arrows is straightforward: it requires taking multisets instead of sets.

2.2. Synchronised union ($\oplus$) of net structures

Since any net structure is a collection of labelled transitions, the operations on sets may be applied to them, but with a restriction: no two different transitions with the same label may appear in the resulting net structure (because labelling is a function!). For instance, the net structure T in Figure 1 is a union: $T = \{a : (\{x, y\}, \{z\}),\ b : (\{z\}, \{x\}),\ c : (\{z\}, \{y\})\} \cup \{d : (\{z\}, \{x, y\})\}$. Such partial union of net structures is capable of generating each net structure from "atoms" if by the latter we mean net structures containing one transition with arbitrary pre/post-set. However we need a total operation as well as more elementary "atoms", those of the form $\{a{:}(\{s\}, \emptyset)\}$ or $\{a{:}(\emptyset, \{s\})\}$ or $\{a{:}(\emptyset, \emptyset)\}$ with $s \in \mathbb{X}$ and a label a. To obtain arbitrary net structure from such atoms, we define operation called here a *synchronised union* denoted by $\oplus$. It is defined as follows: for the labelled net structures T_1, T_2: $\tau \in T_1 \oplus T_2$ iff **either** there exist labelled transitions $\tau_i = a{:}t_i \in T_i$ $(i = 1, 2)$ with $\tau = a{:}({}^\bullet t_1 \cup {}^\bullet t_2,\ t_1^\bullet \cup t_2^\bullet)$, **or** $\tau \in T_1 \cup T_2$ otherwise. Due to Assumption 2.1: $\{a{:}(\emptyset, \emptyset)\} \oplus T = T \oplus \{a{:}(\emptyset, \emptyset)\} = T$, thus any isolated transition plays a part of the neutral (zero) element for $\oplus$ and will be denoted by θ. Operation $\oplus$ is total, *i.e.* never two different transitions with the same label appear in the result. For instance, the net structure in Figure 1 is a synchronised union:

$$\begin{aligned} T =& \{a : (\{x\}, \emptyset)\} \oplus \{a : (\{y\}, \emptyset)\} \oplus \{a : (\emptyset, \{z\})\} \oplus \{b : (\{z\}, \emptyset)\} \oplus \{b : (\emptyset, \{x\})\} \oplus \\ & \{c : (\{z\}, \emptyset)\} \oplus \{c : (\emptyset, \{x\})\} \oplus \{d : (\{z\}, \emptyset)\} \oplus \{d : (\emptyset, \{x\})\} \oplus \{d : (\emptyset, \{y\})\}. \end{aligned}$$

In fact this representation of T is its $\oplus$-decomposition into atomic net structures (associativity of $\oplus$ allows for this notation). The above definitions imply:

Proposition 2.2.

(i) Synchronised union $\oplus$ of net structures is associative, commutative, idempotent and monotone with respect to relation $\sqsubseteq$ defined as $T_1 \sqsubseteq T_2 \Leftrightarrow T_2 = T_1 \oplus T_2$ being a partial order. The least net structure is θ.

(ii) $T_1 \oplus T_2 = lub(T_1, T_2)$ – the least upper bound (wrt $\sqsubseteq$) of the set $\{T_1, T_2\}$. The operation is total, *i.e.* $T_1 \oplus T_2$ is always a net structure.

(iii) If $L_{T_1} \cap L_{T_2} = \emptyset$ then $T_1 \oplus T_2 = T_1 \cup T_2$.

(iv) Each net structure is $\oplus$-decomposable into atomic net structures, *i.e.* containing a single transition with at most one place in its pre-set and then no place in its post-set or, symmetrically, the other way round.
Let $atoms[T]$ denote the set of atoms of T. Then $T = lub(atoms[T])$, $T_1 \oplus T_2 = lub(atoms[T_1] \cup atoms[T_2])$.

Proof. Evident. □

Synchronised union allows to build large nets from small components. It is a version of a parallel composition "$\|$" on nets with synchronised transitions studied *e.g.* in [12]. However "$\oplus$" fuses together not only transitions but also places, which is not the case with "$\|$". In this respect "$\oplus$" more resembles an operator on nets studied *e.g.* in [1].

Example 2.3. complete net structures. A net structure T is *complete* iff each place s has a *complement* in T *i.e.* a place $\overline{s}$ such that ${}^\bullet s = \overline{s}^\bullet$ and $s^\bullet = {}^\bullet \overline{s}$, where ${}^\bullet s = \{t \in T\colon\ s \in t^\bullet\}$ and $s^\bullet = \{t \in T\colon\ s \in {}^\bullet t\}$. A *complement* of T is a net structure $\overline{T}$ obtained from T by replacing pre-set (post-set) of each transition by its post-set (pre-set) with complemented places (do not mix this notion of complement with that in the lattice theory, where it complements to the greatest element of a lattice!). For any net-structure T the net-structure $T \oplus \overline{T}$, called a *complementation* of T, is complete. For instance, if T is the net-structure in Figure 1 then

$$\overline{T} = \{a : (\{\overline{z}\}, \{\overline{x}, \overline{y}\}),\ b : (\{\overline{x}\}, \{\overline{z}\}),\ c : (\{\overline{y}\}, \{\overline{z}\}),\ d : (\{\overline{x}, \overline{y}\}, \{\overline{z}\})\} \text{ and}$$

$$T \oplus \overline{T} = \{a : (\{x, y, \overline{z}\}, \{z, \overline{x}, \overline{y}\}),\ b : (\{z, \overline{x}\}, \{x, \overline{z}\}),\ c : (\{z, \overline{y}\}, \{y, \overline{z}\}),\ d : (\{z, \overline{x}, \overline{y}\}, \{x, y, \overline{z}\})\}$$

A more extensive use of the synchronised union is in Example 3.3.

2.3. Synchronised intersection ($\odot$) of net structures

Like in case of union, we strengthen ordinary intersection of sets to make it adequate operation on labelled net structures.

First, denote $SUB[T] = \{T' : T' \sqsubseteq T\}$, *i.e.* $SUB[T]$ is a set of all sub-net structures of a net structure T, and for a set A of net structures denote by $lub(A)$ the least upper bound (wrt $\sqsubseteq$) of A (if it exists). Obviously, $lub(SUB[T]) = T$. Note that for any sets A, B of net structures: $lub(A \cup B) = lub(A) \oplus lub(B)$ provided that the lubs involved exist. Indeed, since $lub(A), lub(B) \sqsubseteq lub(A \cup B)$ then by monotonicity of $\oplus$: $lub(A) \oplus lub(B) \sqsubseteq lub(A \cup B)$. Conversely, $T \in A \cup B \Leftrightarrow T \in A \vee T \in B \Rightarrow T \in SUB[lub(A)] \vee T \in SUB[lub(B)] \Leftrightarrow T \in SUB[lub(A)] \cup SUB[lub(B)] \Rightarrow T \in SUB[lub(A) \oplus lub(B)]$ (because $SUB[T_1] \cup SUB[T_2] \subseteq SUB[T_1 \oplus T_2]$ for any net structures T_1, T_2). Hence, $T \sqsubseteq lub(A) \oplus lub(B)$. By definition of the lub, $lub(A \cup B)$ is the least of all T' such that $T \sqsubseteq T'$ thus $lub(A \cup B) \sqsubseteq lub(A) \oplus lub(B)$. Therefore $lub(A \cup B) = lub(A) \oplus lub(B)$. From this equality, we get $lub(SUB[T_1] \cup SUB[T_2]) = lub(SUB[T_1]) \oplus lub(SUB[T_2]) = T_1 \oplus T_2$.
Second, by duality, let us define an operation $\odot$ on net structures as $T_1 \odot T_2 = lub(SUB[T_1] \cap SUB[T_2])$ called a *synchronised intersection* of T_1 and T_2. Assume $lub(\emptyset) = \theta$.

Proposition 2.4.

- (i) Synchronised intersection $\odot$ of net structures is associative, commutative, idempotent and monotone with respect to relation $\sqsubseteq$ defined as in Proposition 2.2 and $\theta \odot T = T \odot \theta = \theta$.
- (ii) $T_1 \odot T_2 = glb(T_1, T_2)$ – the greatest lower bound (wrt $\sqsubseteq$) of the set $\{T_1, T_2\}$. The operation is total, *i.e.* $T_1 \odot T_2$ is always a net structure.
- (iii) $T_1 \odot T_2 = T_1 \Leftrightarrow T_1 \oplus T_2 = T_2 \Leftrightarrow T_1 \sqsubseteq T_2$.
- (iv) $T \odot \overline{T} = \theta$, where $\overline{T}$ is the complement of T (see Ex. 2.3).
- (v) $T_1 \odot T_2 = lub(atoms[T_1] \cap atoms[T_2])$ (see Prop. 2.2(iv)).

Proof. Point (i) – evident, (ii) is demonstrated as follows. Let $A = SUB[T_1] \cap SUB[T_2]$. Since $A \subseteq SUB[T_1]$ and $A \subseteq SUB[T_2]$ then, obviously, $T_1 \odot T_2 = lub(A)$ exists in $SUB[T_1]$ and in $SUB[T_2]$, thus in $SUB[T_1] \cap SUB[T_2]$. Therefore $lub(A) \sqsubseteq T_1 \wedge lub(A) \sqsubseteq T_2$ and for each $T \in A$: $lub(A) \sqsubseteq T \Rightarrow lub(A) = T$ (because $lub(A)$, is a maximal element in A). Therefore $(T \sqsubseteq T_1 \wedge T \sqsubseteq T_2) \Rightarrow T \sqsubseteq lub(A)$, which means $lub(A) = glb(T_1, T_2)$. Points (iii), (iv), (v) – evident. □

Synchronised intersection allows to extract some fragments of net structures or to highlight synchronised part of a compound net structure. One may see some affinity of this operator to selection operator in relational data bases: it selects a set of transitions (*i.e.* a net structure) from one operand under requirements provided in the other.

Example 2.5. For T in Figure 1 and

$$U = \{a : (\{x, u\}, \{z\}),\ b : (\{z\}, \{x\}),\ e : (\{z\}, \{u\}),\ f : (\{z\}, \{x, u\})\},$$
$$T \odot U = \{a : (\{x\}, \{z\}),\ b : (\{z\}, \{x\})\}.$$

A more extensive use of the synchronised intersection is in Example 3.3.

2.4. Synchronised difference ($\ominus$) of net structures

By analogy to properties (iv) in Proposition 2.2 and (v) in Proposition 2.4 define a *synchronised difference* between T_1 and T_2 as $T_1 \ominus T_2 = lub(atoms[T_1] \setminus atoms[T_2])$. Let us limit ourselves to an analogue to the De Morgan laws in the calculus of sets:

Proposition 2.6. For any net structures T_1, T_2, T_3:
(i) $T_1 \ominus (T_2 \odot T_3) = (T_1 \ominus T_2) \oplus (T_1 \ominus T_3)$;
(ii) $T_1 \ominus (T_2 \oplus T_3) = (T_1 \ominus T_2) \odot (T_1 \ominus T_3)$.

Proof. Easy calculation. □

Remark. In general $T_1 \ominus T_2 \neq lub(SUB[T_1] \setminus SUB[T_2])$.

Synchronised difference allows to delete some fragments of net structures.

Example 2.7. For T in Figure 1 and U in Example 2.5

$$T \ominus U = \{a : (\{y\}, \emptyset), b : (\emptyset, \emptyset), c : (\{z\}, \{y\}), d : (\{z\}, \{x, y\})\} =$$
$$\{a : (\{y\}, \emptyset), c : (\{z\}, \{y\}), d : (\{z\}, \{x, y\})\} \qquad \text{(see Assumption 2.1).}$$

A more extensive use of the synchronised difference is in Example 3.3.

2.5. Renaming of place names and transition labels

So far we have defined three set theoretic-like operations on net structures. Now, the calculus will be supplied with the operation of renaming in the style of [15].

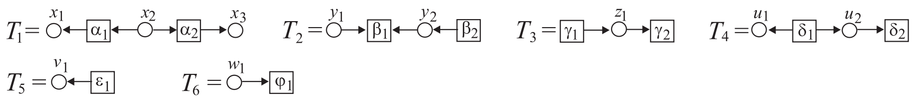
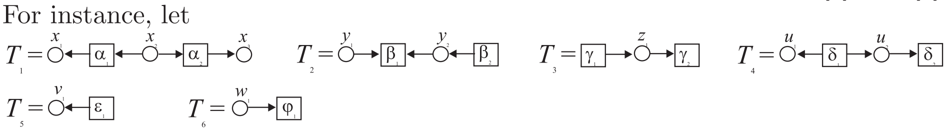


TABLE 1. Renaming functions.

It is a convenient tool for net design, since fragments of a system can be commited to different designers who work separately (a "supervisor" takes their results and makes suitable renaming in order to put them together using operations $\oplus, \odot, \ominus$). Let $X_T \subseteq \mathbb{X}$, $L_T \subseteq \mathbb{L}$ be sets of places (their names) and transition labels in a net structure T and let $f\colon X_T \cup L_T \to \mathbb{X} \cup \mathbb{L}$ be a mapping called a *renaming*, such that $x \neq y \Rightarrow f(x) \neq f(y)$ and $x \in X_T \Rightarrow f(x) \in \mathbb{X}$, $x \in L_T \Rightarrow f(x) \in \mathbb{L}$. Then, $T[f]$ denotes T with each $x \in X_T \cup L_T$ replaced by $f(x) \in X_{T[f]} \cup L_{T[f]}$. For instance, let



and let renaming be given in the Table 1.

Then the expression $(T_1[f_{T_1}] \oplus T_2[f_{T_2}] \oplus T_3[f_{T_3}] \oplus T_4[f_{T_4}]) \ominus (T_5[f_{T_5}] \oplus T_6[f_{T_6}])$ represents the net structure T in Figure 1.

3. LATTICE OF NET STRUCTURES

Let $\mathbb{T}$ be the set of all unlabelled transitions over the universe $\mathbb{X}$ of places and let $\mathbb{L}$ be a universe of labels. For $T \subseteq \mathbb{T}$ let $l_T\colon L_T \stackrel{onto}{\longrightarrow} T$ be a labelling function, where $L_T \subseteq \mathbb{L}$ is a set of labels in T. Additionally assume $L_\theta = \emptyset$. Formally, a labelled net structure, not being θ, is the function l_T, that is a set of all pairs $(a, l_T(a))$ denoted $a{:}l_T(a)$ with $l_T(a) \neq l_T(b) \Rightarrow a \neq b$. Let LNS denote the set of all labelled net structures along with θ, that is $LNS = \{l_T |\; T \subseteq \mathbb{T}\}$. For brevity and a part the labelled net structures play here, we retain notation T, possibly with indices, for them. It follows from Propositions 2.2 (*ii*) and 2.4 (*ii*) that $\langle LNS, \oplus, \odot\rangle$ is a lattice.
Moreover we have:

Theorem 3.1. *Algebra $\langle LNS, \oplus, \odot\rangle$ is a distributive lattice, i.e. $T_1 \odot (T_2 \oplus T_3) = (T_1 \odot T_2) \oplus (T_1 \odot T_3)$ and dually, $T_1 \oplus (T_2 \odot T_3) = (T_1 \oplus T_2) \odot (T_1 \oplus T_3)$ for any net structures T_1, T_2, T_3. Its least element is θ. If the universe $\mathbb{X}$ or $\mathbb{L}$ is infinite then assumption $|{}^\bullet t| < \infty$ and $|t^\bullet| < \infty$ for each transition $t \in \mathbb{T}$ deprives the lattice of the greatest element. Without this assumption, the greatest element is the labelled net structure $\mathbb{L} \times \{(\mathbb{X}, \mathbb{X})\}$ thus the lattice is a Boolean algebra. If $\mathbb{X}$ and $\mathbb{L}$ are finite then there are finitely many labelled net structures and the greatest element is their synchronised union (note that in this case $|{}^\bullet t| < \infty$ and $|t^\bullet| < \infty$ hold). For a given $T \in LNS$, the sublattice $\langle SUB[T], \oplus, \odot\rangle$ is a Boolean algebra with the greatest element T.*

Proof. **First**, note that $atoms(T_1 \odot (T_2 \oplus T_3)) = atoms((T_1 \odot T_2) \oplus (T_1 \odot T_3))$. Indeed, by easily checked equalities $atoms(T \overset{\oplus}{\underset{\odot}{}} T') = atoms(T) \overset{\cup}{\underset{\cap}{}} atoms(T')$ for any T, T': $atoms(T_1 \odot (T_2 \oplus T_3)) = atoms(T_1) \cap atoms(T_2 \oplus T_3) = atoms(T_1) \cap (atoms(T_2) \cup atoms(T_3)) = (atoms(T_1) \cap atoms(T_2)) \cup (atoms(T_1) \cap atoms(T_3)) = atoms(T_1 \odot T_2) \cup atoms(T_1 \odot T_3) = atoms((T_1 \odot T_2) \oplus (T_1 \odot T_3))$. Hence, $lub(atoms(T_1 \odot (T_2 \oplus T_3))) = lub(atoms((T_1 \odot T_2) \oplus (T_1 \odot T_3))) \Leftrightarrow T_1 \odot (T_2 \oplus T_3) = (T_1 \odot T_2) \oplus (T_1 \odot T_3)$ (see Proposition 2.1(iv)). The dual distribution law follows from a general property of distributed lattices.
Second, let $\mathbb{X} = \{x_i | \, i = 1, 2, 3, ...\}$, $\mathbb{L} = \{a\}$ and $T_i = \{a{:}(\{x_i\}, \emptyset)\}$. Then the least upper bound of the set $\{T_1, T_2, T_3, ...\}$ is the labelled net structure $T = \{a{:}(\{x_1, x_2, x_3, ...\}, \emptyset)\}$ containing one transition with infinite pre-set. The set LNS_{fin} of all labelled net structures over such $\mathbb{X}$ and $\mathbb{L}$ and containing transitions with finite pre/post sets only, contains $\{T_1, T_2, T_3, ...\}$ as its subset. Thus LNS_{fin} has no greatest element.
Third, in any case, $lub(LNS) = \mathbb{L} \times \{(\mathbb{X}, \mathbb{X})\}$. Indeed, $(\mathbb{X}, \mathbb{X})$ is the unlabelled transition with the universe $\mathbb{X}$ of places as its pre and post sets. The labelling $l_{\mathbb{T}}$: $\mathbb{L} \overset{onto}{\longrightarrow} \mathbb{T}$ defined as $l_{\mathbb{T}}(a) = (\mathbb{X}, \mathbb{X})$ is a constant function, thus the set $\{a{:}(\mathbb{X}, \mathbb{X}) | \, a \in \mathbb{L}\} = \mathbb{L} \times \{(\mathbb{X}, \mathbb{X})\} \in LNS$. Obviously $T \sqsubseteq \mathbb{L} \times \{(\mathbb{X}, \mathbb{X})\}$ and if $\mathbb{L} \times \{(\mathbb{X}, \mathbb{X})\} \sqsubseteq T'$ then $T' = \mathbb{L} \times \{(\mathbb{X}, \mathbb{X})\}$, for any $T, T' \in LNS$.
Fourth, evidently $lub(SUB[T]) = T$, thus T is the greatest element in $SUB[T]$.
□

Corollary 3.2. *Mappings* $h^{\odot}_{T_0}(T) = T_0 \odot T$ *for any* T_0 *are homomorphisms of* LNS *onto the sublattices of all net structures* T' *with* $T' \sqsubseteq T_0$ *and mappings* $h^{\oplus}_{T_0}(T) = T_0 \oplus T$ *for any* T_0 *are homomorphisms of* LNS *onto the sublattices of all net structures* T' *with* $T_0 \sqsubseteq T'$.

Although Theorem 3.1 directly follows from further Theorems 4.1 and 4.2, the latter, as simple versions of representation theorems, have been situated in Section 4 intended for that issue. A general consequence of distributivity of the net structures' lattice is Theorem 4.4. Although it follows from the general Stone's representation theorem [18], it will be proved, since its specificity (as a property of the universe of net structures) makes the proof quite simple.

Example 3.3. *Combining production units.* There are three production units: P_a, P_b, P_c making aircrafts, boats and cars respectively, and three agencies S_a, S_b, S_c responsible for delivery of the products to a trading company. They are represented by net structures – small loops in Figures 2 and can be combined and optimised in many ways. For instance, separate factories of aircrafts, boats and cars delivering their products at trader's premises (place s) are represented by net structures $A = P_a \oplus S_a$, $B = P_b \oplus S_b$, $C = P_c \oplus S_c$. All the three can be combined into one huge factory $ABC = A \oplus B \oplus C$ represented by a net depicted in Figure 3. Each factory may be optimised by subtracting net structures $T_\alpha = \{\alpha_1{:}(\{a_4\}, \emptyset), \alpha_2{:}(\emptyset, \{a_4\})\}$, $T_\beta = \{\beta_1{:}(\{b_4\}, \emptyset), \beta_2{:}(\emptyset, \{b_4\})\}$, $T_\gamma = \{\gamma_1{:}(\{c_4\}, \emptyset), \gamma_2{:}(\emptyset, \{c_4\})\}$, depicted in Figure 4. So, $A_{opt} = A \ominus T_\alpha$, $B_{opt} = B \ominus T_\beta$, $C_{opt} = C \ominus T_\gamma$. The optimised huge factory is $ABC_{opt} = A_{opt} \oplus B_{opt} \oplus C_{opt}$, which equals $ABC \ominus (T_\alpha \oplus T_\beta \oplus T_\gamma)$ and by the De Morgan law (Prop. 2.6) also $(ABC \ominus T_\alpha) \odot (ABC \ominus T_\beta) \odot (ABC \ominus T_\gamma)$. It is depicted in Figure 5. If the aircraft and boat factories made an agreement to share the car production then the following units come into being: $AC = A \oplus C$ and $BC = B \oplus C$. Their synchronised intersection is expected to be the car factory. Indeed, $AC \odot BC = (A \odot B) \oplus (A \odot C) \oplus (C \odot B) \oplus (C \odot C) = C$, since $C \odot C = C$ and every remaining summand equals θ. The intended initial marking of all the net structures occurring in this example is the following: one token in $a_2, a_4, b_2, b_4, c_2, c_4$ and no token in remaining places. Note however that the construction of this system and its transformations proceeded on net structures, that is unmarked nets.

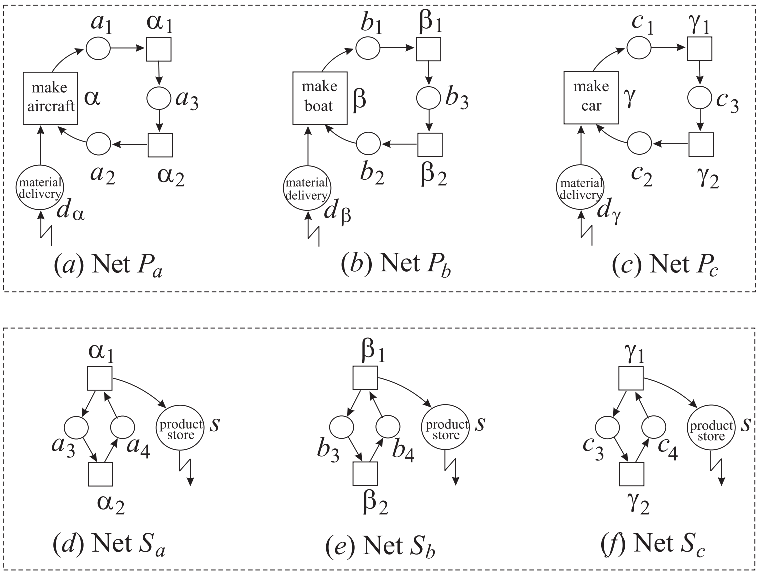


FIGURE 2. Production units and delivery agencies.

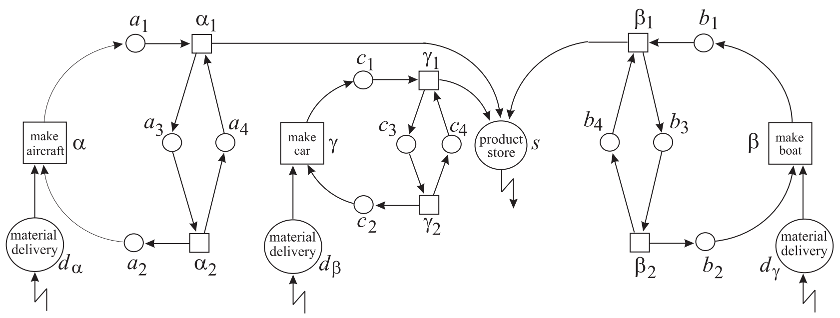


FIGURE 3. $ABC = (P_a \oplus S_a) \oplus (P_b \oplus S_b) \oplus (P_c \oplus S_c)$.

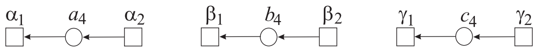


FIGURE 4. $T_\alpha, T_\beta, T_\gamma$ - substructures to be removed.

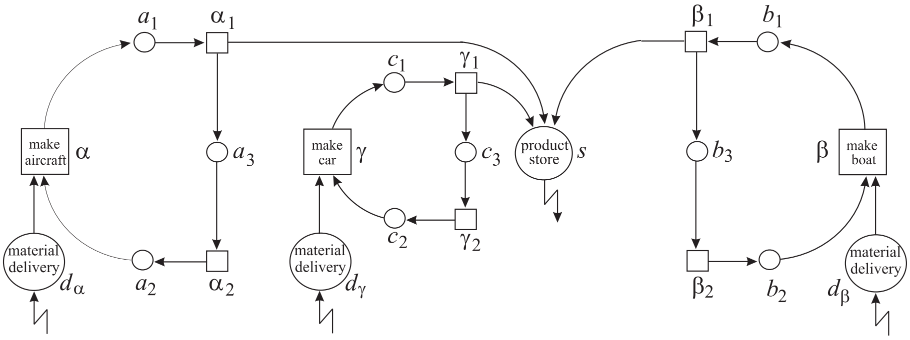
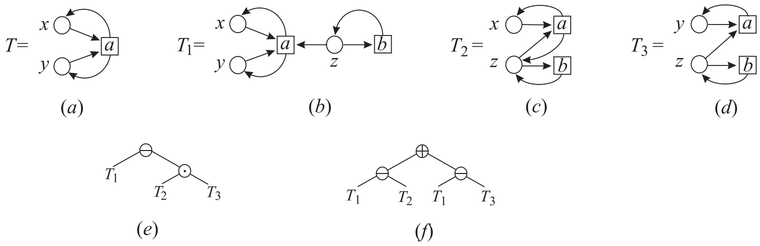


FIGURE 5. $ABC_{opt} = ABC \ominus (T_\alpha \oplus T_\beta \oplus T_\gamma)$.



(e) (f)

FIGURE 6. Parsing trees of net-structure T.

Remarks on behavioural properties of compound nets

Remark 3.4. Semantics of nets is a relation between markings. For simplicity, let us define it for Place/Transition nets without weights on arrows and with infinite capacity of places. As above, let $\mathbb{X}$ be a universe of places and $\mathbb{T}$ – the universe of all unlabelled transitions over $\mathbb{X}$. A *marking* is a function $M\colon\ \mathbb{X} \to \mathbb{N}$ where $\mathbb{N}$ is the set of all natural numbers, 0 including. Note that here the marking does not concern any particular net structure. Markings will be treated as multisets over $\mathbb{X}$ and usual operations $+, -$ and comparison $\leq$ applied component-wise to them. The set $\mathbb{N}^{\mathbb{X}}$ of all markings is denoted by $\mathbb{M}$. For a transition $t \in \mathbb{T}$ by $[[t]]$ is denoted a binary relation between markings, defined by $(M, M') \in [[t]]$ iff ${}^\bullet t \leq M \wedge M' = M - {}^\bullet t + t^\bullet$. Semantics of a (unlabelled) net structure $T \subseteq \mathbb{T}$ is a binary relation $[[T]] = \bigcup_{t \in T} [[t]]$. Additionally, $[[\emptyset]] = \emptyset$. The reflexive and transitive closure $[[T]]^*$ is a *reachability* relation in T. Note that an ordinary marking of a net T is obtained by restricting the marking defined above to the set of places in T. If T is labelled (that is $T \in LNS$) then its semantics is defined in the same way, since labels do not affect behaviour of a net: we assume $[[a{:}t]] = [[b{:}t]] = [[t]]$ for any $a, b \in L_T$. Owing to net-independence of marking, we have for any $T_1, T_2 \in LNS$: If $L_{T_1} \cap L_{T_2} = \emptyset$ then $[[T_1 \oplus T_2]] = [[T_1]] \cup [[T_2]]$. Indeed, by (iii) in Proposition 2.2 $[[T_1 \oplus T_2]] = [[T_1 \cup T_2]]$. Let $(M, M') \in [[T_1 \cup T_2]]$ thus $(M, M') \in [[a{:}t]]$ for a certain labelled transition $a{:}t \in T_1 \cup T_2$. By $L_{T_1} \cap L_{T_2} = \emptyset$: $a{:}t$ is either in T_1 or in T_2 but is not composed of transitions labelled by a in T_1 and in T_2. Thus, $(M, M') \in [[T_1]] \cup [[T_2]]$ hence $[[T_1 \oplus T_2]] \subseteq [[T_1]] \cup [[T_2]]$. Let $(M, M') \in [[T_1]] \cup [[T_2]]$ thus $(M, M') \in [[a{:}t_1]]$ for a certain $a{:}t_1 \in T_1$ or $(M, M') \in [[b{:}t_2]]$ for a certain $b{:}t_2 \in T_2$. Thus, $(M, M') \in [[T_1 \cup T_2]]$ hence $[[T_1]] \cup [[T_2]] \subseteq [[T_1 \oplus T_2]]$.

Remark 3.5. If different net expressions describe the same net structure then they describe different ways of building this net structure. The building process is exhibited by a parsing tree (there is evident grammar of the language of net expressions) of a net expression. Thus, with a given net structure a forest of parsing trees is associated. Some semantic properties of the net may sometimes be inferred from some parsing trees easier than from others. For instance, net structure in Figure 6(a) may be described either as $T_1 \ominus (T_2 \odot T_3)$ with the tree in Figure 6(e), or (by Prop. 2.6(i)) as $(T_1 \ominus T_2) \oplus (T_1 \ominus T_3)$ with the tree in Figure 6(f).
Suppose, the net structure T is marked by tokens in x and y. If property of liveness is concerned then the tree in Figure 6(f) allows to infer liveness of T immediately from liveness of $T_1 \ominus T_2 = \{a{:}(\{y\}, \{y\})\}$ and $T_1 \ominus T_3 = \{a{:}(\{x\}, \{x\})\}$. Indeed, it is easy to prove that if nets U, V with disjoint sets of places and with markings M_U, M_V are live then $U \oplus V$ is live in the marking $M_U \cup M_V$. Note that neither T_1 nor T_3 is live. □

4. Representation theorems for net structures

The lattice of net structures exhibits a close similarity to the elementary calculus of sets, more exactly, the lattice of subsets of a certain set. We state an isomorphism between the lattice of net structures and three set lattices.

Theorem 4.1. *There is one-to-one correspondence between each net structure T and $atoms[T]$ with: $atoms[T_1 \oplus T_2] = atoms[T_1] \cup atoms[T_2]$ and $atoms[T_1 \odot T_2] = atoms[T_1] \cap atoms[T_2]$.*

In other words, the lattices $\langle LNS, \oplus, \odot\rangle$ and $\langle\{atoms[T]: T \in LNS\}, \cup, \cap\rangle$ are isomorphic.

Proof. If $T_1 \neq T_2$ then obviously $atoms[T_1] \neq atoms[T_2]$. Let $atoms(T_1) = \{\alpha_{11}, ..., \alpha_{1n}\}$, $atoms(T_2) = \{\alpha_{21}, ..., \alpha_{2m}\}$. Then $atoms[T_1 \oplus T_2] = atoms[lub(atoms[T_1] \cup atoms[T_2])] = atoms[lub(\{\alpha_{11}, ..., \alpha_{1n}\} \cup \{\alpha_{21}, ..., \alpha_{2m}\})] = \{\alpha_{11}, ..., \alpha_{1n}\} \cup \{\alpha_{21}, ..., \alpha_{2m}\} = atoms[T_1] \cup atoms[T_2]$. For $\odot$ and $\cap$ – similarly. □

Theorem 4.2. *There is one-to one correspondence between each net structure* T *and* $SUB[T]$ *with:* $SUB[T_1 \oplus T_2] = SUB[T_1] \cup SUB[T_2]$ *and* $SUB[T_1 \odot T_2] = SUB[T_1] \cap SUB[T_2]$.
In other words, the lattices $\langle LNS, \oplus, \odot \rangle$ *and* $\langle \{SUB[T]: T \in LNS\}, \cup, \cap \rangle$ *are isomorphic.*

Proof. similar to that of Theorem 4.1 – replace *atoms* with SUB. □

An important fact in the general theory of lattices is the so called representation theorem originating from [18]. It states that for each distributive lattice there exists an isomorphic set lattice, *i.e.* a lattice whose members are sets and operations are ordinary union and intersection. Since the lattice of all net structures with synchronised union and intersection is distributive, it has a representation as a set lattice. Let us remind a few facts on ideals – in the context of our lattice – and take a look at such representation proposed in [18]. An *ideal* of $\langle LNS, \oplus, \odot \rangle$ is any non-empty subset $I \subseteq LNS$ satisfying (a) $(T_1 \in I \wedge T_2 \in I) \Rightarrow T_1 \oplus T_2 \in I$, (b) $(T_1 \sqsubseteq T_2 \wedge T_2 \in I) \Rightarrow T_1 \in I$ (note that (a) and $(b\ )$ are equivalent to $T_1 \in I \wedge T_2 \in I \Leftrightarrow T_1 \oplus T_2 \in I$). An ideal I is *prime* iff $I \neq LNS$ and for any T_1, T_2: $T_1 \odot T_2 \in I \Rightarrow (T_1 \in I \vee T_2 \in I)$. Obviously, if $\neg(T_1 \sqsubseteq T_2)$ then there exist ideals I such that $T_1 \in I \wedge T_2 \notin I$ (example: $\{T : T \sqsubseteq T_1\}$ is such ideal). An ideal I is *maximal* iff $I \neq LNS$ and for any ideal $I' \neq LNS$: $I \subseteq I' \Rightarrow I = I'$. In the proof of representation Theorem 4.4, the following fact will be used.

Proposition 4.3. Let $\neg(T_1 \sqsubseteq T_2)$ and denote by $\mathbb{Q}(T_1, T_2)$ the class of all ideals I of the lattice $\langle LNS, \oplus, \odot \rangle$, such that $T_1 \in I \wedge T_2 \notin I$. Then $\mathbb{Q}(T_1, T_2)$ has a maximal element which, due to distributivity of the lattice, is a prime ideal.

Different versions of this fact are proved (using axiom of choice or its equivalents) in most of expositions of lattice theory, *e.g.* [10,16].

Theorem 4.4. *For a net structure* $T \in LNS$ *let* $\mathbb{P}(T)$ *denote the class of all prime ideals* I *of the lattice* $\langle LNS, \oplus, \odot \rangle$*, such that* $T \notin I$*. Then:*

- (i) $\mathbb{P}(T_1 \oplus T_2) = \mathbb{P}(T_1) \cup \mathbb{P}(T_2)$ *and* $\mathbb{P}(T_1 \odot T_2) = \mathbb{P}(T_1) \cap \mathbb{P}(T_2)$,
- (ii) $\mathbb{P}$ *establishes one-to-one correspondence between each* T *and* $\mathbb{P}(T)$.

In other words, the lattices $\langle LNS, \oplus, \odot \rangle$ *and* $\langle \{\mathbb{P}(T): T \in LNS\}, \cup, \cap \rangle$ *are isomorphic.*

Proof. (i) $I \in \mathbb{P}(T_1 \oplus T_2) \Leftrightarrow T_1 \oplus T_2 \notin I \Leftrightarrow$ (because I is an ideal) $T_1 \notin I \vee T_2 \notin I \Leftrightarrow I \in \mathbb{P}(T_1) \vee I \in \mathbb{P}(T_2) \Leftrightarrow I \in \mathbb{P}(T_1) \cup \mathbb{P}(T_2)$. For $\odot$ and $\cap$ – the dual reasoning.
(ii) If $T_1 \neq T_2$ then $\neg(T_1 \sqsubseteq T_2 \wedge T_2 \sqsubseteq T_1)$ that is $\neg(T_1 \sqsubseteq T_2) \vee \neg(T_2 \sqsubseteq T_1)$. Hence, by Proposition 4.1, there exists a prime ideal $I \in \mathbb{Q}(T_1, T_2) \cup \mathbb{Q}(T_2, T_1)$ that is $T_1 \in I \wedge T_2 \notin I$ or $T_1 \notin I \wedge T_2 \in I$. Thus, $I \in \mathbb{P}(T_2) \wedge I \notin \mathbb{P}(T_1)$ or $I \in \mathbb{P}(T_1) \wedge I \notin \mathbb{P}(T_2)$, therefore $\mathbb{P}(T_1) \neq \mathbb{P}(T_2)$. □

Final remarks

Synchronised union of net structures is an operation appearing (in various guise) in many publications. It is usually called a "parallel combination with synchronisation" and concerns various models of concurrency, not only nets. Contrary to this, synchronised intersection and difference seem to be absent in the theory of parallel processing. This work is rather theoretically oriented and that is why all the operations have been devised to find and study a possibly close correspondence between calculus of net structures and sets. And, possibly, to take advantage of rich substance of the latter, in particular equivalences of net structures like, for instance, an analogue of the De Morgan laws. However, the operations are expected to be of some use in designing large systems from smaller ones and in their transformations. The expectation is supported by a simple example (Ex. 3.3). A particular challenge seems to come from finding conditions making possible to infer some behavioural properties of compound nets from the properties of their components. A modest sample of this issue is hinted in Remark 3.5. This is a subject of current investigations, to be published in a separate article. Besides, combining the idea presented in this paper with equations for message passing worked out in [4] is under investigation.

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