BALANCE PROPERTIES OF THE FIXED POINT
OF THE SUBSTITUTION ASSOCIATED TO QUADRATIC
SIMPLE PISOT NUMBERS

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Abstract. In this paper we will deal with the balance properties of
the infinite binary words associated to \( \beta \)-integers when \( \beta \) is a quadratic
simple Pisot number. Those words are the fixed points of the mor-
phisms of the type \( \varphi(A) = A^p B \), \( \varphi(B) = A^q \) for \( p \in \mathbb{N}, q \in \mathbb{N}, p \geq q \),
where \( \beta = \frac{p + \sqrt{p^2 - 4q}}{2} \). We will prove that such word is \( t \)-balanced
with \( t = 1 + \left\lfloor \frac{p - 1}{p + 1 - q} \right\rfloor \). Finally, in the case that \( p < q \) it is
that the fixed point of the substitution \( \varphi(A) = A^p B \), \( \varphi(B) = A^q \) is
not \( m \)-balanced for any \( m \). We exhibit an infinite sequence of pairs of
words with the unbalance property.

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1. INTRODUCTION

A Pisot number \( \beta \) is a real algebraic integer greater than 1, all of whose con-
jugates are of modulus strictly less than 1. Since \( \beta > 1 \), we can define for every
\( x > 0 \) the so-called \( \beta \)-expansion of \( x \) as a representation of the form
\[ x = x_k \beta^k + x_{k-1} \beta^{k-1} + \cdots + x_0 + x_{-1} \beta^{-1} + x_{-2} \beta^{-2} + \cdots, \]
for \( x_i \) non-negative integers satisfying certain conditions. The \( \beta \)-expansion is a
generalization of ordinary representations of real numbers in base 10 and can be
defined for all \( \beta > 1 \). Analogically to the decimal expansions, coefficients of the
\( \beta \)-expansion are found by the ‘greedy algorithm’, i.e. we find maximal \( k \in \mathbb{Z} \) such
that \( \beta^k \leq x < \beta^{k+1} \) and we set \( x_k = \left\lfloor x/\beta^k \right\rfloor \) and \( r_k = x/\beta^k - \left\lfloor x/\beta^k \right\rfloor \). For \( i \in \mathbb{Z} \),

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i < k we put \( x_i = \lfloor \beta r_{i+1} \rfloor \) and \( r_i = \beta r_{i+1} - \lfloor \beta r_{i+1} \rfloor \). This algorithm implies that the coefficients \( x_i \) are integers in \( \{0, 1, \ldots, [\beta]\} \). The term \( \beta \)-expansion of 1 is defined differently: we put \( k = 1 \) and then proceed analogically to the case \( x \neq 0 \). For more facts about \( \beta \)-expansions see Chapter 7 of [8].

The Pisot number \( \beta \) is said to be simple, if the \( \beta \)-expansion of 1 is finite, otherwise we call it non-simple. It can be shown that the simple quadratic Pisot numbers are exactly the positive roots of the polynomials \( x^2 - px - q \) for \( p \geq q \) (see [2, 5]). Consider the set \( \mathbb{Z}_\beta \) of \( \beta \)-integers, i.e. those numbers \( x \geq 0 \) whose \( \beta \)-expansion is of the form \( x = x_k \beta^k + \cdots + x_1 \beta + x_0 \).

From now on \( \beta \) is a simple quadratic Pisot number. Drawn on the real line, there are only two distances between neighbouring points of \( \mathbb{Z}_\beta \). Conversely, there are exactly two types of distances between neighbouring points of \( \mathbb{Z}_\beta \) for \( \beta > 1 \) only if \( \beta \) is a quadratic Pisot number. If we assign names \( A, B \) to the two types of distances and write down the order of distances in \( \mathbb{Z}_\beta \) on the real line, we naturally obtain an infinite word; we will denote this word by \( u \). It can be shown that the word \( u \) is a fixed point of a certain substitution \( \varphi \) (see e.g. [7]); in particular, for the simple quadratic Pisot number \( \beta \) (the root of \( x^2 - px - q \) for \( p \geq q \geq 1 \)), the generating substitution is

\[
\varphi(A) = A^p B, \quad \varphi(B) = A^q,
\]

\[
A \mapsto A^p B \mapsto (A^p B)^p A^q \mapsto \ldots
\]

A word defined over the binary alphabet \( \{A, B\} \) is said to be \( t \)-balanced (\( t \in \mathbb{N} \)) if for all pairs of factors \( w, \hat{w} \) of \( v \), which are of the same length, the difference between the number of letters \( A \) in \( w \) and \( \hat{w} \) is less or equal to \( t \). From Theorem 13 of [1] follows that there exists an integer \( t \) such that the word \( u \) is \( t \)-balanced for all simple Pisot numbers. However, the optimal bounds were known only for the following cases:

- \( p = q = 1 \): \( t = 1 \). The word \( u \) is Fibonacci word; see Chapter 2 of [8];
- \( q = 1, p > 1 \): \( t = 1 \). This case appeared in the paper [6], see Proposition 6.1;
- \( p = q = 2 \): \( t = 2 \). See Theorem 7.1 of the paper [11].

In this paper we compute the optimal bound for all values of \( p \) and \( q \) corresponding to minimal polynomials of quadratic simple Pisot numbers.

According to [1], if \( p < q \), the word \( u \) is not \( m \)-balanced for all \( m \in \mathbb{N} \). In the last section of this paper we exhibit a sequence of pairs of factors \( v^{(n)}, w^{(n)} \) of \( u \) such that \( v^{(n)} \) and \( w^{(n)} \) are of the same length and the number of letters \( A \) in \( v^{(n)} \) and \( w^{(n)} \) differ at least by \( n \).

2. Preliminaries

Let \( \mathcal{A} = \{a_1, \ldots, a_k\} \) be a finite alphabet. A concatenation of letters in \( \mathcal{A} \) is called a word. The set \( \mathcal{A}^* \) of all finite words equipped with the empty word \( \epsilon \) and the operation of concatenation is a free monoid. The length of a word
\( w = w_0 w_1 \cdots w_{n-1} \) is denoted by \( |w| = n \). One may consider also infinite words \( v = w v_1 v_2 \cdots \), the set of infinite words is denoted by \( \mathcal{A}^\mathbb{N} \). A word \( w \) is called a factor of \( v \in \mathcal{A}^* \), resp. \( \mathcal{A}^\mathbb{N} \), if there exist words \( w^{(1)} \in \mathcal{A}^* \), \( w^{(2)} \in \mathcal{A}^* \), resp. \( w^{(2)} \in \mathcal{A}^\mathbb{N} \) such that \( v = w^{(1)} w v^{(2)} \). The word \( w \) is called a prefix of \( v \), if \( w^{(1)} = \epsilon \). It is a suffix of \( v \), if \( w^{(2)} = \epsilon \).

Denote by \( |w|_{a_i} \), the number of letters \( a_i \) in the word \( w \). The balance function \( B_v \) of the infinite word \( v \) is defined by:

\[
B_v(n) = \max_{1 \leq i \leq k} \max_{w, \hat{w} \in \mathcal{L}_n(v)} \{||w|_{a_i} - |\hat{w}|_{a_i}|\},
\]

where \( \mathcal{L}_n(v) \) denotes the set of all factors of length \( n \) of the word \( v \). We say that an infinite word \( v \) is \( t \)-balanced, if for every \( i \), \( 1 \leq i \leq k \) and for every pair of factors \( w, \hat{w} \) of \( v \), \( |w| = |\hat{w}| \) we have \( |w|_{a_i} - |\hat{w}|_{a_i}| \leq t \). The infinite word is thus \( t \)-balanced if and only if its balance function is bounded by \( t \). Recall that Sturmian words are characterized by the property that they are 1-balanced (or simply balanced).

A morphism on the free monoid \( \mathcal{A}^* \) is a map \( \varphi : \mathcal{A}^* \to \mathcal{A}^* \) satisfying \( \varphi(w \hat{w}) = \varphi(w) \varphi(\hat{w}) \) for all \( w, \hat{w} \in \mathcal{A}^* \). Clearly, the morphism \( \varphi \) is determined if we define \( \varphi(a_i) \) for all \( a_i \in \mathcal{A} \).

A morphism \( \varphi \) is called a substitution, if \( \varphi(a_i) \neq \epsilon \) for all \( i = 1, 2, \ldots, k \) and if there exist \( a_i \in \mathcal{A} \) such that \( |\varphi(a_i)| > 1 \). An infinite word \( u \) is said to be a fixed point of the substitution \( \varphi \), or invariant under the substitution \( \varphi \), if

\[
\varphi(u_0) \varphi(u_1) \varphi(u_2) \cdots = u_0 u_1 u_2 \cdots \tag{1}
\]

or \( \varphi(u) = u \), after having naturally extended the action of \( \varphi \) to infinite words. Relation (1) implies that \( \varphi(u_0) = u_0 \hat{u} \) and \( \varphi^n(u) = u \) for every \( n \in \mathbb{N} \). The length of the word \( \varphi^n(A) \) grows to infinity with \( n \), therefore for every \( n \in \mathbb{N} \) the word \( \varphi^n(u_0) \) is a prefix of the fixed point \( u \), formally \( u = \lim_{n \to \infty} \varphi^n(u_0) \).

With every substitution \( \varphi \) can be associated an incidence matrix \( M_\varphi \), which is defined as

\[
(M_\varphi)_{ij} = |\varphi(a_j)|_{a_i}.
\]

From now on, we shall focus on the substitution \( \varphi \) on the alphabet \( \{A, B\} \) given by

\[
\begin{align*}
\varphi(A) &= A^p B \\
\varphi(B) &= A^q, \quad p \geq 1, q \geq 1,
\end{align*}
\tag{2}
\]

and let us denote by

\[
u = u_0 u_1 u_2 u_3 \cdots
\]

the infinite word in the alphabet \( \mathcal{A} \) invariant under \( \varphi \). The substitution \( \varphi \) has a unique fixed point, namely

\[
u = \lim_{n \to \infty} \varphi^n(A).
\]
The incidence matrix associated with the substitution (2) is thus of the form

$$M_\varphi = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}.$$  \hfill (3)

This matrix has two real eigenvalues:

$$\theta_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \theta_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

Since $q > 0$, necessarily $\theta_1 > 1$. According to [1], Theorem 13, the balance properties of the substitution (2) are determined by the absolute value of the eigenvalue $\theta_2$:

(i) if $|\theta_2| < 1$, then the balance function of the fixed point $u$ is bounded;

(ii) if $|\theta_2| \geq 1$, then the balance function of the fixed point $u$ is not bounded.

Obviously, the situation (i) corresponds to $p \geq q \geq 1$, the situation (ii) to $p < q$.

We will find the uniform bound of the balance function for the case (i) (for fixed $p$ and $q$). In the second case we will give an example of the sequence of the pairs of factors $v^{(n)}$, $w^{(n)}$ of $u$ satisfying $|v^{(n)}| = |w^{(n)}|$ and $|v^{(n)}|_{A} - |w^{(n)}|_{A} \to +\infty$ for $n \to +\infty$.

3. Basic properties of $u$ in relation to balances

In this section we state some properties of the infinite word $u$ that follow from the form of the substitution (2). Results of this section will be used for investigation of balance properties of $u$.

**Observation 3.1.** For every $n \in \mathbb{Z}$, $n \geq 2$ we have

$$\varphi^n(A) = (\varphi^{n-1}(A))^p (\varphi^{n-2}(A))^q.$$

*Proof.* The statement can be proved easily by induction on $n$. \hfill $\square$

**Observation 3.2.** For every $n \in \mathbb{N}$,

$$\varphi^{2n}(A) = vBA^p$$

$$\varphi^{2n-1}(A) = wA^pB$$

for some words $v, w \in A^*$. \hfill $\square$

*Proof.* The statement can be proved easily by induction on $n$.

**Observation 3.3.** Let $BA^kB$ be a factor of $u$. Then $k = p$ or $k = p + q$. In particular, if $A^k$ is a factor of $u$, then $k \leq p + q$.

*Proof.* It suffices to show the statement for a finite word $\varphi^n(A)$ for every $n \in \mathbb{N}$. Since for $n \in \mathbb{N}$ the word $\varphi^n(A)$ begins with $A^pB$, we obtain the result using Observations 3.1, 3.2. \hfill $\square$
Observation 3.4. Let $vB$ be a finite factor of $u$. Then there exists a unique finite factor $w$ satisfying this condition: If $vB$ is a suffix of $\varphi(\hat{w})$, then $w$ is a suffix of $\hat{w}$. Moreover, there exists a unique nonnegative integer $k$ such that $A^k vB = \varphi(w)$.

Proof. Since $vB$ is a factor of $u$, there exists $n \in \mathbb{N}$ such that $vB$ is a factor of $\varphi^n(A)$. Therefore we can find a factor $w$ in $\varphi^{n-1}(A)$ such that $vB$ is a factor of $\varphi(w)$. Moreover, since $B$ is a suffix of $vB$, there exists $w$ such that $vB$ is a suffix of $\varphi(w)$. We choose the factor $w$ of $\varphi^{n-1}(A)$ so that it has minimal length. Assume that there exist two factors $w^{(1)}$, $w^{(2)}$, $|w^{(1)}| = |w^{(2)}| = |w|$ such that $w^{(1)} = z^{(1)}A z$, $w^{(2)} = z^{(2)}B z$ for some factors $z^{(1)}$, $z^{(2)}$, $z$ satisfying $|z| < |w|$. Hence $\varphi(w^{(1)}) = \varphi(z^{(1)})A^p B \varphi(z)$, $\varphi(w^{(2)}) = \varphi(z^{(2)})A^q \varphi(z)$. Since $|z| < |w|$, then $|\varphi(z)| < |vB|$, thus at the same time $B \varphi(z)$ is a suffix of $vB$ and $A \varphi(z)$ is a suffix of $vB$. It is a contradiction, thus $w$ is unique and with respect to its minimal length $w$ satisfies the condition from this observation.

Assume that $\varphi(w) = \hat{v}B A^p v$; then $\varphi(w) = \hat{v} \varphi(A) A^p vB$, thus $A^p vB = \varphi(z)$ for some factor $z$, which contradicts minimality of $|w|$. □

Observation 3.5. For every finite word $w$ we have

$$|w|_A = |\varphi(w)|_B,$$

$$|\varphi(w)|_A = p|w|_A + q|w|_B .$$

Observation 3.6. Let $v, w$ be factors of $u$, $|v| = |w|$. Then

$$|v|_A - |w|_A = |w|_B - |v|_B .$$

4. Balances of binary infinite word in the Pisot case

In this section we will find a uniform bound of the balance function corresponding to the fixed point of the substitution $\varphi$ given by $\varphi(A) = A^p B$, $\varphi(B) = A^q$, $p \geq q \geq 1$ and we will show that this bound is optimal.

Theorem 4.1. The infinite word $u$ invariant under the morphism $\varphi : \{A, B\} \rightarrow \{A, B\}$, given by $\varphi(A) = A^p B$, $\varphi(B) = A^q$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p \geq q$ is $t$-balanced, where $t = 1 + \left[\frac{p-1}{p+1-q}\right]$.

Proof. We shall prove the theorem by contradiction. Assume that there exist an $n \in \mathbb{N}$ and two factors $v, w$ of $u$, $|v| = |w| = n$, such that $|v|_A - |w|_A \geq t + 1$. We choose minimal $n$ with this property. Therefore

$$|v|_A - |w|_A = t + 1 .$$

Since $t + 1 \geq 2$, the words $v$ and $w$ are of the form $v = A^t A$, $w = B \hat{v} B$ for some factors $\hat{v}$, $\hat{w}$ of $u$.

Moreover, Observation 3.3 implies that one of the following situations occurs:

(i) $w = B A^p B$;

(ii) $w = B A^{p+q} B$;
(iii) \( w = BApBApB; \)
(iv) \( w = BAp+qBApB; \)
(v) \( w = BApBAp+qB; \)
(vi) \( w = BAp+qBAp+qB; \)
(vii) \( w = BApB \hat{\ldots} wBApB \) for some word \( \hat{w}; \)
(viii) \( w = BAp+qB\hat{w}BApB \) for some word \( \hat{w}; \)
(ix) \( w = BApB\hat{w}BAp+qB \) for some word \( \hat{w}; \)
(x) \( v = Ap+q+1\hat{v}Ap+q+1 \) for some factor \( \hat{v}. \)

In order to show that the only possible situation is (vii) let us at first prove the following statement:

Let \( v, w \) be the factors defined above, \( n = |v| = |w|. \) Then:

(a) if \( w = BApB\hat{w}, \) then there exists a factor \( \hat{v} \) such that \( v = A^{k+1}\hat{v}; \)
(b) if \( w = \hat{w}BApB, \) then there exists a factor \( \hat{v} \) such that \( v = \hat{v}Ap^{k+1}. \)

For the proof of (a) assume that \( v = A^jB\hat{w} \) and \( j < k+1. \) Then \( |\hat{v}| = |A^{k+1-j}B\hat{w}| < 2 \) and \( |\hat{v}|_A - |A^{k+1-j}B\hat{w}|_A = |v|_A - |w|_A, \) which contradicts the minimality of \( n. \)

The proof of statement (b) is similar.

Statements (a) and (b) will be used for determination of the structure of the word \( v: \)

(i) \( v = A^{p+2}; \)

\( t + 1 = |v|_A - |w|_A = 2 \Rightarrow t = 1 \Rightarrow q = 1. \)

However, \( v = A^{p+2} \Rightarrow q \geq 2 \) according to Observation 3.3, which is a contradiction.

(ii) \( v = A^{p+q+2}. \) It is a contradiction with Observation 3.3.

(iii) \( v = A^{2p+3} \) or \( v = A^{p+1}BAp+1. \)

Since \( v = A^{2p+3} \) contradicts Observation 3.3 (\( 2p + 3 > p + q \)), then \( v = A^{p+1}BAp+1. \) Thus \( BAB = \varphi^{-1}(A^{p-1}vAp^{-1}B) \) is a factor of \( u \) according to Observations 3.3 and 3.4: occurrence of the factor \( BAB \) in the word \( u \) together with Observation 3.3 imply that \( p = 1. \)

Since \( p \geq q, \) necessarily \( q = 1 \) and thus \( u \) is the Fibonacci word, which is known to be balanced. This is a contradiction with the assumption (4).

(iv) \( v = A^{p+q+1}BAp+1 \) or \( v = A^{2p+q+3}. \) Both situations contradict Observation 3.3.

(v) \( v = A^{p+1}BAp+q+1 \) or \( v = A^{2p+q+3}. \) It is a contradiction with Observation 3.3.

(vi) \( v = A^{p+q+1}BAp+q+1 \) or \( v = A^{2p+2q+3}. \) It is a contradiction with Observation 3.3.

(vii) \( v = A^{p+1}\hat{v}Ap+1 \) for some factor \( \hat{v}. \)

(viii) \( v = A^{p+q+1}\hat{v}Ap+1 \) for some factor \( \hat{v}. \) It is a contradiction with Observation 3.3.

(ix) \( v = A^{p+1}\hat{v}Ap^{q+1} \) for some factor \( \hat{v}. \) It is a contradiction with Observation 3.3.

(x) \( v = A^{p+q+1}\hat{v}Ap^{q+1} \) for some factor \( \hat{v}. \) It is a contradiction with Observation 3.3.
Hence
\[ w = B\hat{A}p\hat{B}BApB \quad \text{for some factor } \hat{w} \tag{5} \]
and \( v = A^{p+1}\hat{v}Ap+1 \) for some factor \( \hat{v} \). From the relation (5) follows that \(|w| \geq 2p + 4\), which implies that there exist \( p + 1 \leq i \leq p + q, \ p + 1 \leq \ell \leq p + q \) such that \( v = A^{i}\hat{B}BA^{q+i} \). Since \( \ell \geq p + 1 > q \), we can define \( h = \ell - q \in \mathbb{N} \); then
\[ v = A^{i}\hat{B}BA^{q+i+h}, \quad p + 1 \leq i \leq q + p, \quad p + 1 \leq q + h \leq q + p. \tag{6} \]

Let us consider the word \( v' \) defined by the relation
\[ v' = A^{q+p}\hat{B}BA^{q}. \]

Then
\[ v'A^h = A^jv, \quad j + i = q + p. \tag{7} \]

Relations (6) together with Observation 3.3 imply that \( v'ApB \) is a factor of \( u \), and from Observation 3.4 follows that it has uniquely determined preimage \( \varphi^{-1}(v'ApB) = xA \) for some factor \( x \). Thus \( x \) is a factor of \( u \) and \( \varphi(x) = v' \).

Similarly: \( w' = A^{p}w = A^{p}\hat{B}BA^{p}B \) is a factor of \( u \) and has uniquely determined preimage \( y = \varphi^{-1}(w') \).

The following relations for the unknown integers \( j \) and \( h \) could be obtained from (6) and (7):
\[ 0 \leq j \leq q - 1, \quad p + 1 - q \leq h \leq p. \tag{8} \]

Observation 3.5 implies
\[
\begin{align*}
|x|_A &= |v'|_B = |v|_B, \\
|y|_A &= |w'|_B = |w|_B, \\
|x|_B &= \frac{1}{q} (|v'|_A - p \cdot |v|_B) = \frac{1}{q} (|v'| - (p + 1) \cdot |v|_B) \\
&= \frac{1}{q} (|v| + j - h - (p + 1) \cdot |v|_B), \\
|y|_B &= \frac{1}{q} (|w'|_A - p \cdot |w'|_B) = \frac{1}{q} (|w'| - (p + 1) \cdot |w'|_B) \\
&= \frac{1}{q} (|w| + p - (p + 1) \cdot |w|_B).
\end{align*}
\]

Thus
\[ |y|_A - |x|_A = |w|_B - |v|_B = |v|_A - |w|_A = t + 1 \tag{9} \]
(according to Observation 3.6) and
\[
\begin{align*}
|x| &= |x|_A + |x|_B = |v|_B + \frac{1}{q} (|v| + j - h - (p + 1) \cdot |v|_B), \\
|y| &= |y|_A + |y|_B = |w|_B + \frac{1}{q} (|w| + p - (p + 1) \cdot |w|_B). \tag{10}
\end{align*}
\]
Using relation (4) and Observation 3.6 we obtain the difference between lengths of the words $x$ and $y$:

$$|x| - |y| = |v|_B - |w|_B + \frac{1}{q}((|v| - |w|) - (p + 1)(|v|_B - |w|_B) + j - h - p)$$

$$= -(t + 1) + \frac{1}{q}((p + 1)(t + 1) + j - h - p).$$  \(11\)

Necessarily $|x| \in \mathbb{N}$, $|y| \in \mathbb{N}$, thus $|x| - |y| \in \mathbb{Z}$ and

$$q | (p + 1)(t + 1) + j - h - p.$$

Let us denote $V = (p + 1)(t + 1) + j - h - p$, then

$$V \equiv 0 \pmod{q},$$  \(12\)

$$|x| - |y| = -(t + 1) + \frac{V}{q}.$$  \(13\)

From (8) follows

$$-p \leq j - h \leq 2q - 2 - p.$$  \(14\)

When we substitute (14) into the definition relation for $V$, we obtain

$$(p + 1)(t + 1) - 2p \leq V \leq (p + 1)(t + 1) + 2(q - 1) - 2p.$$  \(15\)

Let $V_{\text{min}} = (p + 1)(t + 1) - 2p$. We will show that

$$V_{\text{min}} > tq.$$  \(16\)

From relations (15), (16) and (12) follows

$$V \geq (t + 1)q.$$  \(17\)

Relation (17) together with relation (13) imply $|x| - |y| \geq -(t + 1) + \frac{(t+1)n}{q} = 0$, hence

$$|x| \geq |y|.$$  \(18\)

Relation (18) allows us to define the word $\hat{x}$ as a prefix of $x$ of length $|y|$.

Thus $|\hat{x}| = |y|$ and from relation (9) we obtain

$$|y|_A - |\hat{x}|_A \geq |y|_A - |x|_A = t + 1.$$

From relations (10) and (5) follows that $|y| < |w|$. Thus words $\hat{x}$, $y$ are factors of the same length $|\hat{x}| = |y| < |w| = n$ and satisfy $|y|_A - |\hat{x}|_A \geq t + 1$, which contradicts minimality of $n$. \(\square\)
Theorem 4.2. Let $u$ be the infinite word invariant under the morphism $\varphi : \{A, B\} \rightarrow \{A, B\}$, given by $\varphi(A) = AB, \varphi(B) = A^q, p, q \in \mathbb{N}$, $p \geq q$.
Then there exist factors $v, w$ of $u$ such that $|v| = |w|$ and $|v|_A - |w|_A = t$, where $t = 1 + \left\lfloor \frac{p-1}{p+1-q} \right\rfloor$; i.e. the bound $t$ is optimal.

Proof. We will prove that for every $n, 1 \leq n \leq t$ there exist factors $v^{(n)}, w^{(n)}$ of $u$, $|v^{(n)}| = |w^{(n)}|$, such that

$$|v^{(n)}|_A - |w^{(n)}|_A = n, \quad v^{(n)} = A\hat{v}^{(n)}$$

for some factor $\hat{v}^{(n)}$,

$$w^{(n)} = \hat{w}^{(n)}B$$

for some factor $\hat{w}^{(n)}$.

By induction on $n$:

(I) $\{n = 1\}$ Let us denote

$$v^{(1)} = A, \quad w^{(1)} = B;$$

$v^{(1)}, w^{(1)}$ are factors of $u$ of the same length and they obviously satisfy (19).

(II) $\{n - 1 \rightarrow n\}$ Let us define

$$d_n = 1 + (p + 1 - q)(n - 1), \quad v^{(n)} = \varphi \left( w^{(n-1)} \right) A^{d_n}, \quad A^p w^{(n)} = \varphi \left( v^{(n-1)} A \right).$$

Then

(a) $v^{(n)}$ is a factor of $u$,

Proof. $w^{(n-1)} = \hat{w}^{(n-1)}B \Rightarrow \hat{w}^{(n-1)}BA = w^{(n-1)}A$ is a factor of $u$ due to Observation 3.3 $\Rightarrow \varphi \left( w^{(n-1)} \right) = \varphi \left( w^{(n-1)} A^p B \right)$ is a factor of $u$ $\Rightarrow \varphi \left( w^{(n-1)} A \right)$ is a factor of $u$ for every $d \leq p$. Since $d_n = 1 + (p + 1 - q)(n - 1) \leq 1 + (p + 1 - q)(t - 1) = 1 + (p + 1 - q) \left\lfloor \frac{p-1}{p+1-q} \right\rfloor \leq 1 + (p - 1) = p$,

$v^{(n)}$ is a factor of $u$. \hfill $\square$

(b) $w^{(n)}$ is a factor of $u$,

Proof.

$n = 2$: $A^p w^{(2)} = \varphi \left( v^{(1)} A \right) = \varphi \left( AA \right) = A^p B A^p B$ is a factor of $u$ according to Observation 3.3.

$n > 2$: $v^{(n-1)} = \varphi \left( w^{(n-2)} \right) A^{d_{n-1}} \Rightarrow A^p w^{(n)} = \varphi \left( v^{(n-1)} A \right) = \varphi \left( \varphi \left( w^{(n-2)} \right) A^{d_{n-1}} A \right) = \varphi \left( \varphi \left( \hat{w}^{(n-2)} B A \right) A^{d_{n-1}} A \right) = \varphi \left( \varphi \left( w^{(n-2)} A \right) A^{d_{n-1}} A \right) = \varphi \left( \varphi \left( w^{(n-2)} A \right) \right)$ is a factor of $u$. Thus $w^{(n)}$ is a factor of $u$ if $d_{n-1} = 1 + (p + 1 - q)(n - 1) < p$. However, this inequality is valid for every $n \leq t$, because $d_{n-1} \leq d_t = 1 + (p + 1 - q)(t - 1) = 1 + (p + 1 - q) \left\lfloor \frac{p-1}{p+1-q} \right\rfloor \leq p$. \hfill $\square$
(c) \(|v(n)| = |w(n)|\).

Proof. From the construction of the words \(v(n), w(n)\) and Observation 3.5 follows:

\[ |v(n)| = (p + 1) \left| w(n-1) \right|_A + q \left| w(n-1) \right|_B + d_n \]
\[ = (p + 1) \left( \left| v(n-1) \right|_A - (n - 1) \right) + q \left( \left| v(n-1) \right|_B + (n - 1) \right) + 1 + (p + 1 - q)(n - 1), \]
\[ = (p + 1) \left| v(n-1) \right|_A + q \left| v(n-1) \right|_B + 1, \]
\[ |w(n)| = (p + 1) \left( \left| v(n-1) \right|_A + 1 \right) + q \left( \left| v(n-1) \right|_B - p = (p + 1) \left| v(n-1) \right|_A + q \left| v(n-1) \right|_B + 1, \]

hence \(|v(n)| = |w(n)|\). □

(d) \(v(n), w(n)\) satisfy (19).

Proof. From the construction of the words \(v(n), w(n)\) follows:

\[ \left| v(n) \right|_A = p \left| w(n-1) \right|_A + q \left| w(n-1) \right|_B + d_n = p \left| w(n-1) \right|_A + q \left| w(n-1) \right|_B + 1 + (p + 1 - q)(n - 1), \]
\[ \left| w(n) \right|_A = p \left| v(n-1) \right|_A + q \left| v(n-1) \right|_B - p + p = p \left| v(n-1) \right|_A + q \left| v(n-1) \right|_B. \]

These relations together with Observation 3.6 imply

\[ \left| v(n) \right|_A - \left| w(n) \right|_A = p \left( \left| w(n-1) \right|_A - \left| v(n-1) \right|_A \right) + q \left( \left| w(n-1) \right|_B - \left| v(n-1) \right|_B \right) + 1(p + 1 - q)(n - 1) \]
\[ = -p(n - 1) + q(n - 1) + 1 + (p + 1 - q)(n - 1) = n. \]

The fact that \(v(n) = A\hat{v}(n)\) and \(w(n) = \hat{w}(n)B\) for some factors \(\hat{v}(n)\) and \(\hat{w}(n)\) follows directly from the definition of \(\varphi\). □

Now let us denote \(v = v(t), w = w(t)\). Then \(|v| = |w|\) and \(|v|_A - |w|_A = t\), which proves the theorem. □

Remark 4.3.

- If \(p = q\), i.e. if \(\varphi(A) = A^pB, \varphi(B) = A^p\), then \(u\) is \(p\)-balanced.
- For this type of substitution, the word \(u\) is Sturmian (i.e. \(t = 1\)), only if \(q = 1\).
5. Balance properties of the fixed point $u$

of the substitution $A \mapsto A^pB$, $B \mapsto A^q$, $p < q$

From [1], Theorem 13 follows, that if $p < q$, then $u$ is not $m$-balanced for any $m \in \mathbb{N}$. In this section we will give a specific proof of this fact, in which we will find an explicit infinite sequence of pairs of factors with certain prescribed unbalance property.

**Theorem 5.1.** The infinite word $u$ invariant under the morphism $\varphi : \{A, B\} \to \{A, B\}$, given by $\varphi(A) = A^pB$, $\varphi(B) = A^q$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p < q$ is not $n$-balanced for any $n \in \mathbb{N}$, i.e. for every $n \in \mathbb{N}$ there exist factors $v(n)$, $w(n)$ of $u$, $|v(n)| = |w(n)|$ such that $|v(n)|_A - |w(n)|_A \geq n$.

**Proof.**

$p + 1 = q$: In this case $\theta_1 = p + 1$ and $\theta_2 = -1$ is a root of unity, thus according to [1], Theorem 13, $B_u(n) = (O \cap \Omega)(\log N)$, where $O$ and $\Omega$ are Landau symbols.

We define words $v(n)$, $w(n)$ recurrently, similarly as in the proof of Theorem 4.2, but for every $n \in \mathbb{N}$:

$$v^{(1)} = A, \quad w^{(1)} = B;$$

$n \in \mathbb{N}$, $n \geq 2$:

$$v^{(n)} = \varphi \left( w^{(n-1)} \right) A, \quad A^p w^{(n)} = \varphi \left( v^{(n-1)} A \right).$$

Then $v(n)$, $w(n)$ are factors of $u$ and satisfy

$$|v(n)| = |w(n)|;$$

$$|v(n)|_A - |w(n)|_A = n;$$

$$v(n) = A \hat{v}(n)$$

for some factor $\hat{v}(n);$  

$$w(n) = \hat{w}(n) B$$

for some factor $\hat{w}(n),$

which can be proved similarly to the case $p \geq q$ (see proof of Th. 4.2).

$p \leq q - 2$: By induction. We will define a sequence of pairs of words $v(n)$, $w(n)$ satisfying

- $|v(n)| = |w(n)|$;
- $|w(n)|_B - |v(n)|_B \geq n$;
- $v(n) A$ is a factor of $u$.

(I) $\{n = 1\}$ Let us define

$$v^{(1)} = A, \quad w^{(1)} = B;$$

from Observation 3.3 follows that $v^{(1)} A = A A$ is a factor of $u$.

(II) $\{n - 1 \to n\}$ Let factors $v^{(k)}$, $w^{(k)}$ satisfy

- $|v^{(k)}| = |w^{(k)}|,$
- $|w^{(k)}|_B - |v^{(k)}|_B \geq k;$.  

• \( v^{(k)} A \) is a factor of \( u \)
  for all \( k < n \), then we will define factors \( v^{(n)} \), \( w^{(n)} \) as follows.

  At first, \( v^{(n)} \) is given by
  \[
  v^{(n)} = \varphi \left( w^{(n-1)} \right);
  \]

  then
  \[
  \left| v^{(n)} \right| = (p + 1) \left| w^{(n-1)} \right|_A + q \left| w^{(n-1)} \right|_B
  \]
  and \( v^{(n)} A \) is a factor of \( u \) with respect to Observations 3.2 and 3.3.

  Since \( \varphi \left( v^{(n-1)} A \right) = (p + 1) \left| w^{(n-1)} \right|_A + q \left| w^{(n-1)} \right|_B \),
  we have
  \[
  \left| v^{(n)} \right| - \left| \varphi \left( v^{(n-1)} A \right) \right| = (p + 1) \left( \left| w^{(n-1)} \right|_A - \left| v^{(n-1)} \right|_A - 1 \right) \\
  + q \left( \left| w^{(n-1)} \right|_B - \left| v^{(n-1)} \right|_B \right) \\
  = (p + 1) \left( \left| v^{(n-1)} \right|_A - \left| w^{(n-1)} \right|_A \right) - p - 1 \geq \\
  \geq (p + 1)(n - 1) - p - 1 \geq -p - 1 - p - 1 \geq -p.
  \]

  Let \( v^{(n-1)} A z^{(n-1)} \) be such factor of \( u \), that \( \left| \varphi \left( v^{(n-1)} A z^{(n-1)} \right) \right| \geq \left| v^{(n)} \right| + p \). Since \( A^p \) is a prefix of \( \varphi(A) \) as well as of \( \varphi(B) \) (because \( p < q \)), we can define a factor \( \hat{w}^{(n)} \) by
  \[
  A^p \hat{w}^{(n)} = \varphi \left( v^{(n-1)} A z^{(n-1)} \right);
  \]
  it is obvious that \( \left| \hat{w}^{(n)} \right| \geq \left| v^{(n)} \right| \). Factor \( w^{(n)} \) will be now defined as a prefix of \( \hat{w}^{(n)} \) of the length \( \left| v^{(n)} \right| \), thus \( \left| v^{(n)} \right| = \left| w^{(n)} \right| \).

  Relation (20) implies
  \[
  \left| A^p w^{(n)} \right| - \left| \varphi \left( v^{(n-1)} A \right) \right| = p + \left| v^{(n)} \right| - \left| \varphi \left( v^{(n-1)} A \right) \right| \geq p + (-p) = 0. \quad (21)
  \]

  From relation (21) follows that there exists a factor \( \hat{w}^{(n)} \), which satisfies
  \[
  A^p \hat{w}^{(n)} = \varphi \left( v^{(n-1)} A \right) \hat{w}^{(n)}.
  \]

  Then \( v^{(n)} \), \( w^{(n)} \) are factors of \( u \), \( \left| v^{(n)} \right| = \left| w^{(n)} \right| \),
  \[
  \left| v^{(n)} \right|_B = \left| w^{(n-1)} \right|_A, \\
  \left| w^{(n)} \right|_B \geq \left| \varphi \left( v^{(n-1)} A \right) \right|_B = \left| w^{(n-1)} \right|_A + 1,
  \]
hence
\[ |w^{(n)}_B| - |v^{(n)}_B| \geq |v^{(n-1)}_A| + 1 - |w^{(n-1)}_A| \geq n - 1 + 1 = n. \tag{22} \]

Since there exists for every \( n \in \mathbb{N} \) a pair of factors \( v^{(n)}, w^{(n)} \) of the same length satisfying (22), the theorem is proved. \( \square \)

6. Conclusions

We have described the balance properties of the infinite word \( u \) invariant under the substitution \( \varphi \) given by \( \varphi(A) = A^pB, \varphi(B) = A^q \). The main result consists in the determination of the optimal bound of the balance function of \( u \) when \( u \) corresponds to some quadratic simple Pisot number.

When \( \beta \) is quadratic non-simple Pisot number, then the substitution associated to it is of the form \( \varphi(A) = A^pB, \varphi(B) = A^qB, p > q \), and according to [1], the balance function of the corresponding word \( u \) is balanced in this case as well. Determination of its optimal bound remains to be an interesting open question.

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References


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