INTEGERS WITH A MAXIMAL NUMBER OF FIBONACCI REPRESENTATIONS

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Abstract. We study the properties of the function $R(n)$ which determines the number of representations of an integer $n$ as a sum of distinct Fibonacci numbers $F_k$. We determine the maximum and mean values of $R(n)$ for $F_k \leq n < F_{k+1}$.

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1. Introduction

Let $(F_k)_{k \geq 0}$ be the Fibonacci sequence defined by

$$F_0 = F_1 = 1, \quad F_{k+1} = F_k + F_{k-1} \quad \text{for } k \geq 1.$$ 

Every positive integer $n$ can be written as a sum of distinct Fibonacci numbers, i.e. in the form

$$n = F_{m_r} + F_{m_{r-1}} + \cdots + F_{m_1}, \quad \text{where } m_r > m_{r-1} > \cdots > m_1 \geq 1. \quad (1)$$

The expression (1) is called a representation of the number $n$ in the Fibonacci number system. The index of the maximal Fibonacci number that appears in the representation of $n$ is called the length of the representation. Every Fibonacci representation can be written in the form of a finite word $w = w_{m_r}w_{m_{r-1}}\ldots w_1$ in the alphabet $\{0, 1\}$, where $w_i = 1$ for $i = m_1, \ldots, m_r$, and $w_i = 0$ otherwise. For example the number $n = 32$ can be represented as

$$32 = 21 + 5 + 3 + 2 + 1 = F_7 + F_4 + F_3 + F_2 + F_1$$

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and this representation corresponds to the word 1001111. Due to the recurrence relation for Fibonacci numbers, different representations of the number \( n \) can be obtained by substituting the string 011 by 100 and \textit{vice versa}. All representations of 32 correspond to words 1010100, 1010011, 1001111, 111111.

The number of different Fibonacci representations of \( n \) will be denoted by \( R(n) \). Let us enumerate the first twenty values of the sequence \((R(n))_{n\geq 1}\):

\[
(R(n))_{n\geq 1} = 1, 1, 2, 1, 2, 1, 3, 2, 3, 1, 3, 2, 4, 2, 3, 3, 1, \ldots
\]  

\[ (2) \]

For a given positive integer \( n \) we can find \( k \) such that \( F_k \leq n < F_{k+1} \). It is obvious that every representation of \( n \) has length \( \leq k \). On the other hand, since

\[
F_1 + F_2 + \cdots + F_{k-2} < F_k \leq n,
\]

the lengths of every representation of \( n \) is at least \( k-1 \). Thus representations of the number \( n \) can be divided into long (having length \( k \)) and short (of length \( k-1 \)). Let us denote by \( R_1(n) \) the number of long representations of \( n \), and by \( R_0(n) \) the number of short representations of \( n \). Clearly

\[
R(n) = R_1(n) + R_0(n).
\]

If we prefix the short Fibonacci representations of \( n \) with the prefix 0, they have the same length as the long representations of \( n \). The lexicographically greatest among all such representations of the number \( n \) is called the Zeckendorf representation of \( n \) and the corresponding word in the alphabet \( \{0, 1\} \) is denoted by \( \langle n \rangle \). The distinguishing characteristic of this representation is that there are no adjacent 1’s. For example, we have \( \langle 32 \rangle = 1010100 \).

The Zeckendorf representation of a number \( n \) is a word of the form

\[
\langle n \rangle = 10^{r_1}10^{r_2}\cdots10^{r_l}, \quad \text{where} \quad r_1, \ldots, r_{l-1} \geq 1, \text{ and } r_l \geq 0.
\]

\[ (3) \]

The sum \( r_1 + r_2 + \cdots + r_l + l \) determines the length of the Zeckendorf representation of \( n \). Since the relation between the number \( n \) and the word (3) is one-to-one, we define for the simplicity of notation

\[
\varrho(r_1, \ldots, r_l) := R(n) \\
\varrho_1(r_1, \ldots, r_l) := R_1(n) \\
\varrho_0(r_1, \ldots, r_l) := R_0(n)
\]

\[ (4) \]

where \( \langle n \rangle = 10^{r_1}10^{r_2}\ldots10^{r_l} \).
It can be seen easily that \( R(n) = 1 \) if and only if \( n = F_k - 1 \) for some \( k \geq 2 \). The values of \( R(n) \) for \( n = F_k \pm j, j \leq 8 \) are given in \([3]\). The segment of the sequence \( R(n) \) between two consecutive occurrences of 1 is a palindrome \([3, 4]\), i.e.

\[
R(F_k - 1 + i) = R(F_k + 1 - 1 - i), \quad \text{for } i = 1, 2, \ldots, F_k - 1.
\]

The aim of this paper is to find the maximal and the mean values of the function \( R(n) \) for \( F_k - 1 < n < F_{k+1} - 1 \), which corresponds to the numbers \( n \) whose Zeckendorf representation has a fixed length \( k \). We determine the numbers

\[
\text{Max}(k) := \max \{ R(n) \mid n \in \mathbb{N}, F_k \leq n < F_{k+1} \}
\]

\[
= \max \left \{ \varrho(r_1, \ldots, r_l) \mid l \in \mathbb{N}, r_1, \ldots, r_{l-1} \geq 1, r_l \geq 0, l + \sum_{i=1}^{l} r_i = k \right \}.
\]

In addition, we classify the arguments of the maxima.

Let us determine several initial values of the sequence \( \text{Max}(k) \). It suffices to divide the sequence \( (R(n))_{n \geq 1} \) to blocks of length \( F_0, F_1, F_2, \ldots \) along the occurrence of consecutive 1’s and to find maximal values in these blocks, see (2). We have

\[
\begin{align*}
\text{Max}(1) &= \max \{ R(n) \mid 1 \leq n < 2 \} = R(1) = 1, \\
\text{Max}(2) &= \max \{ R(n) \mid 2 \leq n < 3 \} = R(2) = 1, \\
\text{Max}(3) &= \max \{ R(n) \mid 3 \leq n < 5 \} = R(3) = 2, \\
\text{Max}(4) &= \max \{ R(n) \mid 5 \leq n < 8 \} = R(5) = R(6) = 2, \\
\text{Max}(5) &= \max \{ R(n) \mid 8 \leq n < 13 \} = R(8) = R(11) = 3, \\
\text{Max}(6) &= \max \{ R(n) \mid 13 \leq n < 21 \} = R(16) = 4.
\end{align*}
\]  

(5)

2. Properties of the functions \( \varrho, \varrho_0, \varrho_1 \)

Berstel [1] gives an explicit formula for computing the values of functions \( \varrho, \varrho_1, \varrho_0 \) defined in (4). Denote the matrix

\[
M(r) := \begin{pmatrix}
\begin{bmatrix} r \end{bmatrix} & \begin{bmatrix} r \end{bmatrix} \\
1 & 1
\end{pmatrix}
\end{pmatrix}.
\]

**Theorem 2.1** (Berstel). Let \( r_1, \ldots, r_l \in \mathbb{Z}, r_1, \ldots, r_{l-1} \geq 1, r_l \geq 0 \). Then

\[
\begin{pmatrix}
\varrho_0(r_1, \ldots, r_l) \\
\varrho_1(r_1, \ldots, r_l)
\end{pmatrix} = M(r_1)M(r_2) \cdots M(r_l) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Since \( g(r_1, \ldots, r_l) = g_0(r_1, \ldots, r_l) + g_1(r_1, \ldots, r_l) \), we have explicit formulas for the functions \( g, g_0, g_1 \) in the following form

\[
g(r_1, \ldots, r_l) = \binom{1}{1} M(r_1) M(r_2) \cdots M(r_l) \binom{0}{1},
\]

\[
g_0(r_1, \ldots, r_l) = \binom{1}{0} M(r_1) M(r_2) \cdots M(r_l) \binom{0}{1},
\]

\[
g_1(r_1, \ldots, r_l) = \binom{0}{1} M(r_1) M(r_2) \cdots M(r_l) \binom{0}{1}.
\]

(6)

Let us now derive some recurrence relations for \( g(r_1, \ldots, r_l) \) that will be needed for determining the maximal values. If \( l = 1 \) we get directly from (6) that

\[
g(r) = \left\lfloor \frac{r}{2} \right\rfloor + 1.
\]

(7)

**Lemma 2.2.** Let \( l \in \mathbb{N} \), and let \( r_1, r_2, \ldots, r_l \in \mathbb{Z} \), \( r_1, r_2, \ldots, r_{l-1} \geq 1, \ r_l \geq 0 \). If \( r_1 \) is odd, then \( g(r_1, \ldots, r_l) = g(r_1, \ldots, r_l - 1) \).

**Proof.** It follows from (6) since for \( r_1 \) odd we have \( M(r_1) \binom{0}{1} = M(r_1 - 1) \binom{0}{1} \). \( \square \)

**Lemma 2.3.** Let \( l \in \mathbb{N} \), \( l \geq 2 \) and let \( r_1, r_2, \ldots, r_l \in \mathbb{Z} \), \( r_1, r_2, \ldots, r_{l-1} \geq 1, \ r_l \geq 0 \). If \( r_i \) is even for some \( 1 \leq i \leq l - 1 \), then

\[
g(r_1, \ldots, r_l) = g(r_1, \ldots, r_i) g(r_{i+1}, \ldots, r_l).
\]

**Proof.** For \( r_i \) even we have \( M(r_i) = M(r_i) \binom{0}{1} \binom{1}{1} \). Substituting into (6) we obtain the lemma. \( \square \)

**Lemma 2.4.** Let \( l \in \mathbb{N} \), \( l \geq 2 \), and let \( r_1, r_2, \ldots, r_l \in \mathbb{Z} \), \( r_1, r_2, \ldots, r_{l-1} \geq 1, \ r_l \geq 0 \). We have

\[
g(r_1, r_2, \ldots, r_l) = \frac{r_1 + 1}{2} g(r_2, \ldots, r_l) + g_0(r_2, \ldots, r_l), \quad \text{if } \ r_1 \text{ is odd},
\]

\[
g(r_1, r_2, \ldots, r_l) = \left( \frac{r_1}{2} + 1 \right) g(r_2, \ldots, r_l), \quad \text{if } \ r_1 \text{ is even}.
\]

**Proof.** First suppose \( r_1 \) is odd. Since

\[
(1 \ 1) M(r_1) = \frac{r_1 + 1}{2} (1 \ 1) + (1 \ 0),
\]

substituting into (6) gives the desired result. The statement for \( r_1 \) even is a consequence of Lemma 2.3 and the relation (7). \( \square \)

**Lemma 2.5.** Let \( l \in \mathbb{N} \), \( l \geq 3 \), and let \( r_1, r_2, \ldots, r_l \in \mathbb{Z} \), \( r_1, r_2, \ldots, r_{l-1} \geq 1, \ r_l \geq 0 \). If for some \( i, \ 2 \leq i \leq l - 1 \), the coefficient \( r_i \) is odd, then

\[
g(r_1, \ldots, r_l) = g(r_1, \ldots, r_i - 1) g(r_{i+1}, \ldots, r_l) + g(r_1, \ldots, r_{i-1} + 1) g_0(r_{i+1}, \ldots, r_l).
\]
Proof. Again, it suffices to verify the matrix equality
\[ M(r_{i-1})M(r_i) = M(r_{i-1})M(r_i - 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 1) + M(r_{i-1} + 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 0) \]
for \( r_i \) odd and to use (6).

The following lemma is a direct consequence of the definition of functions \( \varrho \), \( \varrho_0 \) and can be found in [4] as Lemma 1.

**Lemma 2.6** (Edson, Zamboni). Let \( l \in \mathbb{N}, l \geq 2, \) and let \( r_1, r_2, \ldots, r_l \in \mathbb{Z}, r_1, r_2, \ldots, r_l \geq 1, r_l \geq 0. \) Then

(i) \( \varrho_0(r_1, r_2, \ldots, r_l) = \varrho(r_1 - 2, r_2, \ldots, r_l), \) for \( r_1 \geq 3; \)
(ii) \( \varrho_0(2, r_2, \ldots, r_l) = \varrho(r_2, \ldots, r_l); \)
(iii) \( \varrho_0(1, r_2, \ldots, r_l) = \varrho(r_2, \ldots, r_l); \)
(iv) \( \varrho(r_1, \ldots, r_{l-1}, 1, 1, \ldots, 1) = \varrho(r_1, \ldots, r_{l-1}). \)

Clearly, \( \varrho(r_1, \ldots, r_l) \geq 1. \) However, the number of short Fibonacci representations \( \varrho_0(r_1, \ldots, r_l) \) can be equal to 0. Using the rules given in Lemma 2.6 we easily deduce that

\[ \varrho_0(r_1, \ldots, r_l) = 0 \iff r_1 = r_2 = \cdots = r_{l-1} = 1 \text{ and } r_l \in \{0, 1\}. \quad (8) \]

### 3. Lower Bound on \( \text{Max}(k) \)

In order to find the lower estimates of \( \text{Max}(k) \), let us determine the values \( \varrho(r_1, \ldots, r_l) \) on some chosen \( l \)-tuples \( (r_1, \ldots, r_l) \).

**Lemma 3.1.**

1) \( \varrho(3, 3, \ldots, 3, 4) = \varrho(1, 3, \ldots, 3, 2) = F_{2k+1} \) for \( k \geq 1. \)

2) \( \varrho(3, 3, \ldots, 3, 2) = \varrho(1, 3, \ldots, 3, 4) = F_{2k+2} \) for \( k \geq 1. \)

**Proof.** Let us first show by induction that for the \( s \)-th power of the matrix \( M(3) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) we have

\[ (M(3))^s = \begin{pmatrix} F_{2s} & F_{2s-1} \\ F_{2s-1} & F_{2s-2} \end{pmatrix}, \quad \text{for } s \in \mathbb{N}. \quad (9) \]

For \( s = 1 \) the statement is trivial. For \( s \geq 2 \) we use the induction hypothesis

\[ (M(3))^s = (M(3))^{s-1} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_{2s-2} & F_{2s-3} \\ F_{2s-3} & F_{2s-4} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_{2s} & F_{2s-1} \\ F_{2s-1} & F_{2s-2} \end{pmatrix}. \]
Note that (9) is valid also for $s = 0$ if we define $F_{-1}, F_{-2}$ in such a way that the recurrence relation is still valid, $(F_{-1} = 0, F_{-2} = 1)$. It is now easy to use (6) to find

\[ \varrho(3, 3, \ldots, 3, 4) = \begin{pmatrix} 1 & 1 \\ \end{pmatrix} \begin{pmatrix} F_{2k-2} \\ F_{2k-3} \\ F_{2k-4} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} = F_{2k+1} \]

and

\[ \varrho(1, 3, \ldots, 3, 2) = \begin{pmatrix} 1 & 1 \\ \end{pmatrix} \begin{pmatrix} F_{2k-2} \\ F_{2k-3} \\ F_{2k-4} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = F_{2k+1} \]

The relations (2) can be proved similarly.

As a corollary, we have a lower estimate on the maxima for numbers with Zeckendorf representation of odd length.

**Corollary 3.2.** Max$(2k + 1) \geq F_{k+1}$ for $k \geq 1$.

From the definition of the function $\varrho$ it follows that

\[ \varrho(2, r_1, \ldots, r_l) \geq 2 \varrho(r_1, \ldots, r_l) \]

and for $r_l > 0$ also

\[ \varrho(r_1, \ldots, r_l, 2) \geq 2 \varrho(r_1, \ldots, r_l) \]

Therefore we have the following lower estimate on the maxima for numbers with Zeckendorf representation of even length.

**Corollary 3.3.** Max$(2k + 2) \geq 2$Max$(2k - 1) \geq 2F_k$ for $k \geq 2$.

Our aim is to show that the inequalities in Corollaries 3.2 and 3.3 are in fact equalities.

### 4. Maxima of the function $R(n)$

Let us now determine the maximum of the function $R(n) = \varrho(r_1, r_2, \ldots, r_l)$, where $F_k \leq n < F_{k+1}$ and $(n) = 10^{r_1}10^{r_2} \ldots 10^{r_l}$. The $l$-tuple $r_1, \ldots, r_l \in \mathbb{Z}$ must satisfy $r_1, r_2, \ldots, r_{l-1} \geq 1$, $r_l \geq 0$ and $\sum_{i=1}^{l} r_i + l = k$. We shall not repeat these assumptions.

Let us show that Max$(k)$ is not reached on integers $n$ whose Zeckendorf representation has only one 1. More precisely, we have the following proposition.

**Proposition 4.1.** Let Max$(k) = \varrho(r_1, r_2, \ldots, r_l)$. Then $l \geq 2$ or $k \leq 5$. 

Proof. Suppose by contradiction that $k \geq 6$ and $l = 1$. Then using (7), we have $\text{Max}(k) = \varrho(k-1) = \left\lfloor \frac{k-1}{2} \right\rfloor + 1$. For $k$ even we have by Corollary 3.3

$$2F_{\frac{k-2}{2}} \leq \text{Max}(k) = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 = \frac{k}{2},$$

which is in contradiction with $2F_{i-1} > i$ for all $i \geq 3$. For $k$ odd we have by Corollary 3.2

$$F_{\frac{k+1}{2}} \leq \text{Max}(k) = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 = \frac{k+1}{2},$$

which contradicts the fact that $F_i > i$ for all $i \geq 4$. □

In the following several propositions we show that the maximum is reached on $l$-tuples of a certain specific form. The proofs are done by contradiction. Assuming that the maximal $l$-tuple does not satisfy the desired properties, we find another $l$-tuple on which the function $\varrho$ has strictly greater value.

**Proposition 4.2.** Let $\text{Max}(k) = \varrho(r_1, r_2, \ldots, r_l)$ for $k \geq 6$. Then $r_1$ is even.

Proof. Since the above Proposition 4.1 implies that $l \geq 2$, it suffices to prove that for $r_1$ odd we have

$$\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) < \varrho(r_1 + 1, r_2, \ldots, r_{l-1}, r_l - 1). \quad (10)$$

We divide the demonstration of (10) into two cases.

a) Let $r_1$ be even. Using Lemmas 2.2 and 2.4 we have

$$\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) = \varrho(r_1, r_2, \ldots, r_{l-1}, r_1 - 1)
= \left( \frac{r_1}{2} + 1 \right) \varrho(r_2, \ldots, r_{l-1}, r_1 - 1),$$

$$\varrho(r_1 + 1, r_2, \ldots, r_{l-1}, r_l - 1)
= \frac{r_1}{2} + 2 \varrho(r_2, \ldots, r_{l-1}, r_l - 1) + \varrho_0(r_2, \ldots, r_{l-1}, r_l - 1).$$

In order to obtain (10) we need to show that $\varrho_0(r_2, \ldots, r_{l-1}, r_l - 1) > 0$. Using (8), $\varrho_0(r_2, \ldots, r_{l-1}, r_l - 1) = 0$ with $r_1$ odd implies $r_2 = r_3 = \cdots = r_l = 1$. However, in this case the property (iv) of Lemma 2.6 and Proposition 4.1 give

$$\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) = \varrho(1, 1, \ldots, 1) = \varrho(r_1) < \varrho(k - 1) < \text{Max}(k),$$

which contradicts the assumption of the proposition. Thus we necessarily have $\varrho_0(r_2, \ldots, r_{l-1}, r_l - 1) > 0$ and (10) is valid.
b) Let \( r_1 \) be odd. Again we use Lemmas 2.2 and 2.4 to obtain
\[
\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) = \frac{r_1 + 1}{2} \varrho(r_2, \ldots, r_{l-1}, r_1) + \varrho_0(r_2, \ldots, r_{l-1}, r_1) + \varrho_0(r_2, \ldots, r_{l-1}, r_1),
\]
\[
\varrho(r_1 + 1, r_2, \ldots, r_{l-1}, r_l - 1) = \varrho(r_1 + 1, \ldots, r_l) = \left(\frac{r_1 + 1}{2} + 1\right) \varrho(r_2, \ldots, r_l).
\]

The validity of (10) is obvious, since \( \varrho(r_2, \ldots, r_l) > \varrho_0(r_2, \ldots, r_l) \). □

In order to find the arguments of the maxima of the function \( \varrho \), we use the matrix formula (6). First we introduce a partial ordering on non-negative matrices. Lemma 4.4 then shows that replacing a matrix in (6) by a “bigger” one increases the value of the function \( \varrho \).

Definition 4.3. Let \( X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \tilde{X} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \) be integer matrices with non-negative components. We say that \( X \) majors \( \tilde{X} \) (written \( X \succ \tilde{X} \)) if
\[
a \geq \tilde{a}, \quad b \geq \tilde{b}, \quad a + c \geq \tilde{a} + \tilde{c} \quad \text{and} \quad b + d \geq \tilde{b} + \tilde{d}.
\]

Lemma 4.4. Let \( \alpha = (1 \ 1)AXB(0) \) and \( \tilde{\alpha} = (1 \ 1)A\tilde{X}B(0) \), where
\[
A = \mathbb{I}_2 \quad \text{or} \quad A = M(r_1) \ldots M(r_s),
\]
\[
B = \mathbb{I}_2 \quad \text{or} \quad B = M(p_1) \ldots M(p_t),
\]
and \( X, \tilde{X} \) are non-negative integer matrices. If \( X \succ \tilde{X} \), then \( \alpha \succ \tilde{\alpha} \).

Proof. Denote \( (x \ y) = (1 \ 1)A \) and \( (z) = B(0) \). It is easy to see that \( x \geq y \geq 1 \) and that \( z \geq 0, u \geq 1 \). Let \( X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \tilde{X} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \) satisfy (11). Then
\[
\alpha - \tilde{\alpha} = (x \ y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} - (x \ y) \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}
\]
\[
= \left( (a - \tilde{a})x + (c - \tilde{c})y, (b - \tilde{b})x + (d - \tilde{d})y \right) \begin{pmatrix} z \\ u \end{pmatrix}
\]
\[
\geq \left( (a + c - \tilde{a} - \tilde{c})y, (b + d - \tilde{b} - \tilde{d})y \right) \begin{pmatrix} z \\ u \end{pmatrix} \geq (0 \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1. \quad \square
\]

Proposition 4.5. Let \( \varrho(r_1, r_2, \ldots, r_l) = \text{Max}(k) \). Then \( r_i \leq 5 \) for all \( i = 1, 2, \ldots, l \).

Proof. Let \( \langle n \rangle = 10^5 \cdot 10^6 \cdot \ldots \cdot 10^n \), and assume that there exists an index \( i \) such that \( r_i \geq 6 \). Denote by \( m \) the number with Zeckendorf representation \( \langle m \rangle = 10^5 \cdot 10^{i-1} \cdot 10^4 \cdot 10^3 \cdot 10^2 \cdot 10 \). Zeckendorf representations \( \langle n \rangle \) and \( \langle m \rangle \)
have the same length. Since

\[ M(r_i) = \left( \begin{array}{cc} \lfloor \frac{r_i}{2} \rfloor & \lfloor \frac{r_i}{2} \rfloor \\ 1 & 1 \end{array} \right) \prec \left( \begin{array}{cc} \lfloor \frac{r_i-3}{2} \rfloor & \lfloor \frac{r_i-3}{2} \rfloor \\ 1 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \]

we have according to Lemma 4.4

\[ R(m) = g(r_1, r_2, \ldots, r_l) < g(r_1, \ldots, r_{i-1}, r_i - 3, 2, r_{i+1}, \ldots, r_l) = R(m), \]

which contradicts the assumption of the proposition.

**Proposition 4.6.** Let \( g(r_1, r_2, \ldots, r_{i-1}, r_l) = \text{Max}(k), \) where \( k \geq 6 \) and the \( r_i \) are odd for \( i = 1, 2, \ldots, l - 1. \) Then \( r_1 \in \{1, 3\}, \ r_2, \ldots, r_{i-1} = 3, \) and \( r_l \in \{2, 4\}. \)

**Proof.** As a consequence of Proposition 4.2, the final coefficient \( r_l \) is even, and due to Proposition 4.5 it can take only values \( \{0, 2, 4\}. \) Assumption of the present proposition with Proposition 4.5 implies that \( r_1, r_2, \ldots, r_{i-1} \in \{1, 3, 5\}. \) First let us show by contradiction that 5 does not occur. Suppose the opposite, i.e. that there exists an index \( 1 \leq i \leq l - 1 \) such that \( r_i = 5. \) Let \( i \) be the maximal index with this property. Let \( s \) be the minimal non-negative integer, such that \( r_{i+s} \neq 3. \)

1) Let \( r_{i+s} = 1. \) We verify that

\[ \tilde{X} = M(5)(M(3))^{s-1}M(1) \prec (M(3))^{s+1} = X. \]

According to (9), we obtain

\[ \tilde{X} = \left( \begin{array}{cc} 3 & 2 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} F_{2s-2} & F_{2s-3} \\ F_{2s-3} & F_{2s-4} \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{cc} F_{2s+2} & F_{2s} \\ F_{2s} & F_{2s-2} \end{array} \right), \quad X = \left( \begin{array}{cc} F_{2s+2} & F_{2s+1} \\ F_{2s+1} & F_{2s} \end{array} \right). \]

Obviously \( \tilde{X} \prec X \) and using Lemma 4.4 we obtain

\[ \text{Max}(k) = g(r_1, \ldots, r_{i-1}, 5, 3, \ldots, 3, 1, r_{i+s+1}, \ldots, r_l) \]

\[ \prec g(r_1, \ldots, r_{i-1}, \underline{3}, \ldots, \underline{3}, r_{i+s+1}, \ldots, r_l), \]

which is a contradiction.

2) Let \( r_{i+s} = 2. \) Similarly as in (1) we use matrices and Lemma 4.4 to obtain the contradiction

\[ \text{Max}(k) = g(r_1, \ldots, r_{i-1}, 5, 3, \ldots, 3, 2) \prec g(r_1, \ldots, r_{i-1}, 3, \ldots, 3, 4). \]
3) Let \( r_{i+s} = 4 \). Similarly as in (1) we use matrices and Lemma 4.4 to obtain the contradiction
\[
\text{Max}(k) = g(r_1, \ldots, r_{i-1}, 5, 3, \ldots, 3, 4) < g(r_1, \ldots, r_{i-1}, 3, \ldots, 3, 2).
\]

4) Let \( r_{i+s} = 0 \). Similarly as in (1) we use matrices and Lemma 4.4 to obtain the contradiction
\[
\text{Max}(k) = g(r_1, \ldots, r_{i-1}, 5, 3, \ldots, 3, 0) < g(r_1, \ldots, r_{i-1}, 3, \ldots, 3, 2).
\]

Thus we have shown that \( r_1, \ldots, r_{l-1} \leq 3 \), i.e. all take values in \( \{1, 3\} \).

Let us now prove by contradiction that at most one of the coefficients \( r_1, \ldots, r_{l-1} \) is equal to 1. Assume that there exist indices \( i, i+s, 1 \leq i < i+s \leq l-1 \) such that \( r_i = r_{i+s} = 1 \) and \( r_{i+1} = r_{i+2} = \cdots = r_{i+s-1} = 3 \). Denote
\[
\tilde{X} = M(1)(M(3))^{s-1}M(1) = \begin{pmatrix} F_{2s-1} & F_{2s-3} \\ F_{2s} & F_{2s-2} \end{pmatrix},
\]
\[
X = (M(3))^s = \begin{pmatrix} F_{2s} & F_{2s-1} \\ F_{2s-1} & F_{2s-2} \end{pmatrix}.
\]
Since \( \tilde{X} \preceq X \), we derive that
\[
\text{Max}(k) = g(r_1, \ldots, r_{i-1}, 1, 3, \ldots, 3, 1, r_{i+s+1}, \ldots, r_l) < g(r_1, \ldots, r_{i-1}, 3, \ldots, 3, r_{i+s+1}, \ldots, r_l),
\]
which contradicts the maximality of \( g(r_1, \ldots, r_l) \). Thus at most one of the coefficients \( r_1, \ldots, r_{l-1} \) is equal to 1 and the others are equal to 3.

If \( l = 2 \), the proposition is proved. For \( l \geq 3 \) we show by contradiction that \( r_2 = \cdots = r_{l-1} = 3 \). Suppose that \( r_i = 1 \) for some \( 2 \leq i \leq l-1 \). Since
\[
(1 1)(M(3))^{i-1}M(1) = \begin{pmatrix} F_{2i} & F_{2i-2} \\ F_{2i} & F_{2i-1} \end{pmatrix},
\]
\[
(1 1)M(1)(M(3))^{i-1} = \begin{pmatrix} F_{2i} & F_{2i-2} \\ F_{2i} & F_{2i-1} \end{pmatrix},
\]
it follows that
\[
\text{Max}(k) = g(3, \ldots, 3, 1, r_{i+1}, \ldots, r_l) < g(1, 3, \ldots, 3, r_{i+1}, \ldots, r_l),
\]
which is a contradiction.

It remains to show that \( r_l \neq 0 \). But if \( r_l = 0 \), then \( r_{l-1} = 3 \). Relation \( M(3)M(0) \prec M(4) \) implies a contradiction. \( \square \)
We are now in position to state the theorem about the maximal values of $R(n)$.

**Theorem 4.7.**

\[
\begin{align*}
\max\{R(n) \mid F_{2k+1} \leq n < F_{2k+2}\} &= \text{Max}(2k+1) = F_{k+1} \quad \text{for} \quad k \geq 0, \\
\max\{R(n) \mid F_{2k+2} \leq n < F_{2k+3}\} &= \text{Max}(2k+2) = 2F_k \quad \text{for} \quad k \geq 1.
\end{align*}
\]

**Proof.** In the proof we shall make use of the following inequalities for Fibonacci numbers, which are not difficult to demonstrate.

\[
F_{x+1}F_{y+1} \leq 2F_{x+y} \quad \text{for} \quad x, y \geq 0, \tag{12}
\]

where the equality holds only if $x = 1$ or $y = 1$.

\[
2F_xF_y \leq F_{x+y+1} \quad \text{for} \quad x, y \geq 1, \tag{13}
\]

where the equality holds only if $x = y = 2$.

Since the lower bounds on the maxima of the function $R(n)$ are known from Corollaries 3.2 and 3.3, it suffices to prove inequalities

\[
\text{Max}(2k+1) \leq F_{k+1} \quad \text{and} \quad \text{Max}(2k+2) \leq 2F_k. \tag{14}
\]

Let us show it by induction on $k$. For initial values of $k$ the validity of the theorem follows from (5). Now assume that

\[
\text{Max}(2j+1) \leq F_{j+1} \quad \text{and} \quad \text{Max}(2j+2) \leq 2F_j, \quad \text{for} \quad j < k.
\]

With this induction hypothesis we want to show (14).

- Let us first show that $\text{Max}(2k+2) \leq 2F_k$.
  Let $r_1, r_2, \ldots, r_l$ be an $l$-tuple such that $\varrho(r_1, r_2, \ldots, r_l) = \text{Max}(2k+2)$ where $k \geq 2$. Proposition 4.2 implies that $r_1$ is even. Since $r_1 + r_2 + \cdots + r_l + l = 2k+2$, there must exist an $i < l$ such that $r_i$ is even. Let $i$ be the maximal $i < l$ with this property. The number $r_{i+1} + \cdots + r_l + (l - i)$ is odd, say $2m + 1$. Then $r_1 + \cdots + r_i + i = 2k + 2 - (2m + 1)$. Lemma 2.3, the induction hypothesis and inequality (12) implies

\[
\text{Max}(2k+2) = \varrho(r_1, \ldots, r_l) = \varrho(r_1, \ldots, r_i) \varrho(r_{i+1}, \ldots, r_l) 
\leq \text{Max}(2k-2m+1) \text{Max}(2m+1) = F_{k-m+1}F_{m+1} \leq 2F_k.
\]

- Now let us show the inequality $\text{Max}(2k+1) \leq F_{k+1}$.
  Let $r_1, r_2, \ldots, r_l$ be an $l$-tuple such that $\varrho(r_1, r_2, \ldots, r_l) = \text{Max}(2k+1)$ where $k \geq 2$. Suppose that besides $r_1$ there exist another $i < l$ such that $r_i$ is even and let $i$ be the maximal index $i < l$ with this property. Let us denote $r_{i+1} + \cdots + r_l + (l - i) =
2m + 1. Then \( r_1 + \cdots + r_2 + i = 2k + 1 - (2m + 1) = 2k - 2m \). Lemma 2.3, the induction hypothesis and inequality (13) implies

\[
\text{Max}(2k + 1) = \varrho(r_1, \ldots, r_i) = \varrho(r_{i+1}, \ldots, r_l)
\]

\[
\leq \text{Max}(2k - 2m) \text{Max}(2m + 1) = 2F_{k-m-1}F_{m+1} \leq F_{k+1}. \quad (16)
\]

It remains to consider the case that the \( l \)-tuple \( r_1, r_2, \ldots, r_l \) which satisfies \( \varrho(r_1, r_2, \ldots, r_l) = \text{Max}(2k + 1) \) contains all \( r_i \) odd for \( 1 \leq i \leq l - 1 \). According to Proposition 4.6 the maximal \( l \)-tuple is of the form \((1,3,\ldots,3,4),(3,\ldots,3,2),(1,3,\ldots,3,2)\), or \((3,\ldots,3,2)\). Note that for fixed length of the Zeckendorf representation only two of these are possible, namely \((1,3,\ldots,3,2)\), or \((3,\ldots,3,4)\) for length \( 1 \mod 4 \), and \((1,3,\ldots,3,4),(3,\ldots,3,2)\) for length \( 3 \mod 4 \). The values of the function \( \varrho \) for these \( l \)-tuples was determined in Lemma 3.1. Therefore the statement of the theorem is proved. \( \square \)

5. Argument of Max(\( k \))

In this section we determine the integers on which the maximum of the function \( R(n) \) is reached for a fixed length \( \sum_{i=1}^{l} r_i + l \) of the Zeckendorf representation \( \langle n \rangle = 10^{r_1} \cdots 10^{r_l} \). The proof of Theorem 4.7 allows us to determine the \( l \)-tuples \( r_1, \ldots, r_l \) representing such integers \( n \).

Suppose first that the Zeckendorf representation of \( n \) has odd length. In this case the proof of Theorem 4.7 indicates that unless equality holds in (16), all the coefficients \( r_1, \ldots, r_{l-1} \) are odd and therefore the \( l \)-tuples \( r_1, \ldots, r_{l-1}, r_l \) are of very specific form (as a consequence of Prop. 4.6).

Equality in (16) provides an exceptional \( l \)-tuple. In order to make (16) true, the relation (13) necessitates that \( k = 4 \) (hence \( m = 1 \)) and

\[
\varrho(r_1, \ldots, r_k) = \text{Max}(6) \quad \text{and} \quad \varrho(r_{k+1}, \ldots, r_l) = \text{Max}(3).
\]

Since according to the table (5) we have \( \text{Max}(6) = R(16) = \varrho(2,2) \) and \( \text{Max}(3) = R(3) = \varrho(2,2,2) \), necessarily \( l = 3 \) and \( r_1 = r_2 = r_3 = 2 \).

Corollary 5.1.

(i) \( \text{Max}(4k+3) \) is reached precisely on two arguments for \( k \geq 1 \) and on one argument for \( k = 0 \). We have \( \text{Max}(3) = \varrho(2) \), and for \( k \geq 1 \)

\[
\text{Max}(4k+3) = \varrho(1,3,\ldots,3,4) = \varrho(3,\ldots,3,2),
\]

\( k \)-times \( k \)-times

(ii) \( \text{Max}(4k+1) \) is reached precisely on two arguments for \( k \geq 3 \) or \( k = 1 \), on three arguments for \( k = 2 \), and on one argument for \( k = 0 \). We have \( \text{Max}(1) = \varrho(0), \text{Max}(9) = \varrho(1,3,2) = \varrho(3,4) = \varrho(2,2,2), \) and for \( k = 1 \)
and $k \geq 3$

$$\text{Max}(4k + 1) = g(1, 3, \ldots, 3, 2) = g\left(3, \ldots, 3, 4\right).$$

As for integers with even length of their Zeckendorf representation, proof of Theorem 4.7 requires that an $l$-tuple $r_1, \ldots, r_l$ on which the maximum of $g$ is reached must satisfy equality in (15). Relation (12) for Fibonacci numbers implies that $m - k = 1$ or $m = 1$. This can be true only if $i = 1$ or $i = l - 1$ respectively. Equality in (15) further requires that either $r_1 = 2, r_2, \ldots, r_l$ are odd and $g(r_2, \ldots, r_l)$ is maximal, or $r_l = 2$ and $g(r_2, \ldots, r_l)$ is maximal, respectively.

Corollary 5.2. Let $k \geq 3$ and let $r_1, \ldots, r_l$ satisfy $\sum_{i=1}^{l} r_i + l = 2k$. Then $g(r_1, \ldots, r_l) = \text{Max}(2k)$ if and only if

$$r_1 = 2 \quad \text{and} \quad g(r_2, \ldots, r_l) = \text{Max}(2k - 3)$$

or

$$r_l = 2 \quad \text{and} \quad g(r_1, \ldots, r_{l-1}) = \text{Max}(2k - 3).$$

Recall that the elements of the sequence $(R(n))_{n \in \mathbb{N}}$ can be grouped into palindromes $R(F_k), \ldots, R(F_{k+1} - 2)$ separated by values $R(F_{k+1} - 1) = 1$. Corollaries 5.1 and 5.2 show that up to the exceptional initial cases, the maximal value in each palindrome occurs twice (for $k$ odd) and four times (for $k$ even). The description of arguments of the maxima of $R(n)$ in the palindrome, i.e. for $n$ with fixed length of Zeckendorf representation, is given in Theorem 5.3. We need to introduce the following notation,

$$i_{2k} = \begin{cases} 
F_{k+2}F_{k-5} + F_3 + 1 & \text{for } k \text{ even,} \\
F_{k+1}F_{k-4} + F_3 + 1 & \text{for } k \text{ odd,}
\end{cases}$$

$$i_{2k+1} = \begin{cases} 
F_{k+1}F_{k-3} + 1 & \text{for } k \text{ even,} \\
F_{k}F_{k-2} + 1 & \text{for } k \text{ odd,}
\end{cases}$$

$$j_{2k} = \begin{cases} 
F_{k+1}F_{k-3} + 1 & \text{for } k \text{ even,} \\
F_{k}F_{k-2} + 1 & \text{for } k \text{ odd.}
\end{cases}$$

**Theorem 5.3.**

(i) $\text{Max}(2k + 1)$ for $k \geq 1, k \neq 4$ is reached precisely on the integers

$$F_{2k+1} - 1 + i_{2k+1}, \quad F_{2k+2} - 1 - i_{2k+1}.$$

For $k = 4$, $\text{Max}(2k + 1) = \text{Max}(9)$ is reached precisely on three integers, namely

$$F_9 - 1 + i_9 = 63, \quad F_{10} - 1 - i_9 = 79, \quad \text{and their average} \quad 71.$$
Corollaries 5.1 and 5.2 show that up to the exceptional initial cases, the maximal value in the palindrome $R$ is an integer $i \in \{1, 2, \ldots, F_{k-1} - 1\}$ such that

$$R(F_k - 1 + i_k) = R(F_{k+1} - 1 - i_k) = \text{Max}(k).$$

Without loss of generality $i_k$ is in our considerations the smaller of the two integers satisfying it. Similarly, for $k$ odd we have $i_k, j_k \in \{1, 2, \ldots, F_{k-1} - 1\}$ such that

$$R(F_k - 1 + i_k) = R(F_{k+1} - 1 - i_k) = R(F_k - 1 + j_k) = R(F_{k+1} - 1 - j_k) = \text{Max}(k).$$

We consider $i_k < j_k$ to be the two smallest of the four integers satisfying it.

We derive the compact form of $i_k$ and $j_k$ from arguments of maxima given in Corollaries 5.1 and 5.2. For that we use the relation

$$F_i + F_{i+4} + F_{i+8} + \cdots + F_{i+4(k-1)} = F_{2k+i-2}F_{2k-1}, \quad \text{for } i, k \geq 1,$$

which can be shown using $F_k = \frac{1}{\sqrt{5}}(\tau^{k+1} - \tau^{-k+1})$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio and $\tau' = \frac{1}{2}(1 - \sqrt{5})$ its algebraic conjugate.

It is interesting to study the position of the maximal values in the palindrome $R(F_k - 1), R(F_k), \ldots, R(F_{k+1} - 1)$, i.e. the position of integers $i_k, (i_k$ and $j_k)$ in the set $1, 2, \ldots, F_{k-1}$. This is described by Proposition 5.4 and illustrated in Figure 1.

**Proposition 5.4.** Let $k \geq 1$. Then

$$\lim_{k \to \infty} \frac{i_{2k+1}}{F_{2k}} = \lim_{k \to \infty} \frac{i_{2k}}{F_{2k-1}} = \frac{1}{\tau + 2}, \quad \left| i_{2k+1} - \frac{F_{2k}}{\tau + 2} \right| < \frac{1}{2};$$

$$\left| i_{2k} - \frac{\tau}{\tau + 2} \right| < \frac{1}{2};$$

$$\lim_{k \to \infty} \frac{j_{2k}}{F_{2k-1}} = \frac{\tau}{\tau + 2}, \quad \left| j_{2k} - \frac{\tau F_{2k-1}}{\tau + 2} \right| < \frac{1}{2};$$

$$\left| j_{2k} - \frac{1}{\tau + 2} \right| < \frac{1}{2}.$$
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Figure 1. Illustration of the function $R(n)$ for $n \in [20, 33]$. The values $R(F_k - 1), R(F_k), \ldots, R(F_{k+1} - 1)$ for $k = 7$ form a palindrome. Since $k \equiv 3 \mod 4$, the maximal value $\text{Max}(7)$ appears twice and these local maxima are at the integers nearest to the asymptotical position, which is marked by the vertical lines.

The proposition shows that the numbers $i_{2k+1}$ and $j_{2k}$ are the closest integers to the asymptotic position of the maximal value. Let us mention that it is slightly more complicated in case of $i_{2k}$.

Remark. Bicknell-Johnson defines in [2] a sequence $(A(q))_{q \in \mathbb{N}}$ which determines the smallest positive integer with $q$ Fibonacci representations and shows that

$$A(F_k) = F_k^2 - 1 \quad \text{and} \quad A(2F_k) = F_{k+3}F_k - 1 + (-1)^K.$$ 

Since $F_{2k+1} - 1 + i_{2k+1} = F_{k+1}^2 - 1$ and $F_{2k} - 1 + i_{2k} = F_{k+2}F_{k-1} - 1 + (-1)^{k-1}$, the result of [2] is a consequence of Theorems 4.7 and 5.3.

6. Mean value of $R(n)$

Berstel in his article [1] states an open question about the mean value of the function $R(n)$. In this section we answer his question. In particular, we determine the mean value of $R(n)$ for integers with fixed length $k$ of their Zeckendorf
representation, i.e., the value
\[ \frac{1}{F_{k-1}} \sum_{n=F_k}^{F_{k+1}-1} R(n). \]

**Proposition 6.1.** Let \( k \geq 1 \). Then
\[ \sum_{n=F_k}^{F_{k+1}-1} R(n) = \frac{1}{3} (2^k - (-1)^k). \]

**Proof.** Consider the word \( w = w_lw_{l-1} \ldots w_1 \) in the alphabet \( \{0, 1\} \) where \( w_l = 1 \). The word \( w \) is a representation of the number \( n = \sum_{i=1}^{l} w_i F_i \). Also \( w \) is a long representation of \( n \), if \( \sum_{i=1}^{l} w_i F_i < F_{l+1} \), and \( w \) is a short representation of \( n \), if \( \sum_{i=1}^{l} w_i F_i \geq F_{l+1} \). It can easily be shown that the latter occurs if and only if the word \( w \) has the prefix 1010 \( \cdots \) 1011. More precisely,
\[ \sum_{i=1}^{l} w_i F_i \geq F_{l+1} \]
if and only if
\[ w_lw_{l-1} \cdots w_1 = (10)^i11w_{l-2i-2} \cdots w_1, \quad \text{for some } i \geq 0, \ i \leq \left\lfloor \frac{l-2}{2} \right\rfloor. \]

Therefore the number of words \( w_l \cdots w_1 \) with \( w_l = 1 \) that represent an integer \( n \geq F_{l+1} \) is equal to the number of distinct suffixes \( w_{l-2i-2} \cdots w_1 \), i.e.,
\[ \sum_{i=0}^{\left\lfloor \frac{l-2}{2} \right\rfloor} 2^{l-2i-2} = \left\lfloor \frac{2^l - 1}{3} \right\rfloor. \]  \(\text{(17)}\)

Consequently, the number of words \( w_l \cdots w_1 \) with \( w_l = 1 \) which represent an integer \( n < F_{l+1} \) is equal to
\[ 2^{l-1} = \left\lfloor \frac{2^l - 1}{3} \right\rfloor. \]  \(\text{(18)}\)
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Since the sets of Fibonacci representations of distinct integers \( n \) are disjoint, we obtain

\[
\sum_{n=F_k}^{F_{k+1}-1} R_0(n) = \# \left\{ w_{k-1} \cdots w_1 \in \{0,1\}^* \mid w_{k-1} = 1, \sum_{i=1}^{k-1} w_i F_i \geq F_k \right\}
= \left\lfloor \frac{2^{k-1} - 1}{3} \right\rfloor.
\]

\[
\sum_{n=F_k}^{F_{k+1}-1} R_1(n) = \# \left\{ w_k \cdots w_1 \in \{0,1\}^* \mid w_k = 1, \sum_{i=1}^{k} w_i F_i < F_{k+1} \right\}
= 2^{k-1} - \left\lfloor \frac{2^{k-1} - 1}{3} \right\rfloor.
\]

Together we obtain

\[
\sum_{n=F_k}^{F_{k+1}-1} R(n) = 2^{k-1} - \left\lfloor \frac{2^{k-1} - 1}{3} \right\rfloor + \left\lfloor \frac{2^{k-1} - 1}{3} \right\rfloor = \frac{1}{3} \left( 2^k - (-1)^k \right). \quad \square
\]

Since \( F_k = \frac{1}{\sqrt{5}} (\tau^{k+1} - \tau^{-k+1}) \), the mean value of the function \( R(n) \) for \( F_k \leq n < F_{k+1} \) is equal to

\[
\frac{1}{\sqrt{5}} \left( \tau^k - \tau^{-k} \right) \sim \frac{\sqrt{5}}{3} \left( \frac{2}{\tau} \right)^k.
\]

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