THE GLOBALS OF PSEUDOVARIEITIES OF ORDERED SEMIGROUPS CONTAINING $B_2$ AND AN APPLICATION TO A PROBLEM PROPOSED BY PIN

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Abstract. Given a basis of pseudoidentities for a pseudovariety of ordered semigroups containing the 5-element aperiodic Brandt semigroup $B_2$, under the natural order, it is shown that the same basis, over the most general graph over which it can be read, defines the global. This is used to show that the global of the pseudovariety of level $3/2$ of Straubing-Thérien’s concatenation hierarchy has infinite vertex rank.

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1. Introduction

Three concatenation hierarchies of rational languages have been considered since the 1970’s: the dot-depth hierarchy introduced by Brzozowski [10], the Straubing-Thérien hierarchy [32, 33], and the group hierarchy presented in [19]. Pin (see [22]) has observed that all these hierarchies may be regarded as being indexed by half integers, that is numbers of the form $n$ or $n + 1/2$ where $n$ is a non-negative integer, and where each level other than level zero is obtained in the following way: the languages of level $n + 1/2$ are finite unions of products of the form $n$ or $n + 1/2$ where $n$ is a non-negative integer, and where each level other than level zero is obtained in the following way: the languages of level $n + 1/2$ are finite unions of products of the
form $L_0a_1L_1\ldots a_rL_r$, where $L_0,\ldots, L_r$ are languages of level $n$ and $a_0,\ldots, a_r$ are letters\textsuperscript{1}, and the languages of level $n+1$ are members of the Boolean closure of the class of languages of level $n+1/2$. Thus each hierarchy is completely determined by its level zero. The dot-depth hierarchy $B_n$ has the finite or cofinite languages of $A^+$ as a basis, the Straubing-Thérien hierarchy $V_n$ is based on the languages $\emptyset$ and $A^*$, and the level zero of the group hierarchy $G_n$ is obtained by taking the group languages. These three hierarchies are infinite and strict (see [11,22]). The main problem, which is open in almost all cases, is whether each level is decidable. In fact, for the Straubing-Thérien hierarchy, this problem has been solved (positively) for $n \leq 3/2$ [8,9,30] and some partial results are known for the level two [24,34,35,38]. For the other two hierarchies, the membership problem is only known to be decidable for $n \leq 1$ [14,16,17,19]. See [31] for related problems concerning the product of rational languages.

Some of the early results concerning the dot-depth and Straubing-Thérien hierarchies follow from deep theorems of Simon [30]. The problem of the effective characterization of the locally testable languages was solved independently by McNaughton [20] and by Brzozowski and Simon [12]. This characterization implies that locally testable languages are of dot-depth one. The graph-theoretic arguments which were implicit in these works have had a strong impact on language and semigroup theories, namely in Knast’s and Tilson’s results. Knast [17] obtained a complex theorem on graphs and achieved an effective characterization of the whole class of languages of dot-depth one.

Tilson [37] (inspired by work of Simon [30], Knast [16,17], Thérien and Weiss [36], and Straubing [33]) developed a theory of (small) categories as partial algebras over graphs and showed its importance to study semigroups. In particular, his Derived Category Theorem turned out to be a powerful tool to deal with semidirect products of pseudovarieties of semigroups [7,27]. In the profinite context, Weil and the first author [7] have used the derived category theorem to describe a basis of pseudoidentities of the semidirect product $V \star W$ of pseudovarieties of semigroups from a basis of semigroupoid pseudoidentities of the pseudovariety $gV$ of semigroupoids generated by the semigroups of $V$ viewed as one-vertex semigroupoids. The proof of this result, that has come to be known as the basis theorem, has a gap, as has been observed by Rhodes and Steinberg, but the theorem stands in case $W$ is locally finite or $gV$ has finite vertex-rank.

Since each level of the considered hierarchies is a positive variety of languages, the variety theorem [13,21] guarantees a one-to-one correspondence between the level $n$ of each hierarchy and a pseudovariety of ordered monoids (or semigroups in case of the dot-depth hierarchy). The problem of decidability for a level $n$ of the hierarchies is now reduced to determine whether the pseudovarieties $B_n$, $V_n$ and $G_n$ are decidable, where $B_n$, $V_n$, and $G_n$ denote respectively the pseudovarieties of ordered semigroups (monoids) associated to $B_n$, $V_n$, and $G_n$. There are interesting

\textsuperscript{1}This is the definition adopted when the empty word is considered, that is for languages of $A^*$. Otherwise, for languages of $A^+$, one takes instead finite unions of languages of the form $u_0L_1u_1\ldots L_ru_r$, where $L_1,\ldots, L_r$ are languages of level $n$ and $u_0,\ldots, u_r \in A^*$ are words with $u_0$ nonempty in case $r = 0$. 
connections between $V_n$, $B_n$ and $G_n$: $B_n = V_n \ast L^1 (n > 0)$ [32], where $L^1$ is the pseudovariety of locally trivial semigroups, and $G_n = V_n \ast G (n \geq 0)$ [23], where $G$ is the pseudovariety of all finite groups. It is not immediately clear if these results reduce the study of one hierarchy to another. However, it is known that, for each integer $n$, $B_n$ is decidable if and only if $V_n$ is decidable [33].

For a pseudovariety $V$ of finite ordered semigroups, $\ell V$ denotes the pseudovariety of all finite ordered semigroupoids whose finite one-vertex subsemigroupoids with at least one edge lie in $V$, and $gV$ is the pseudovariety of ordered semigroupoids generated by the ordered semigroups of $V$ viewed as ordered one-vertex semigroupoids. A basis of pseudoidentities of ordered semigroupoids which defines $\ell V$ can be easily obtained from a given basis for $V$. However it is not so easy to compute a basis of semigroupoid pseudoidentities for $gV$ given such a basis for $V$. Based on results of Reilly [28], Azevedo, Teixeira and the first author [5] proved that the problem may be systematically treated when the pseudovariety $V$ of semigroups contains the 5-element aperiodic Brandt semigroup $B_2$.

In Section 3, we show that a similar approach works in case $V$ is a pseudovariety of ordered semigroups containing the ordered inverse semigroup $B_2$. Indeed, given a basis $\Sigma$ of pseudoidentities of $V$, then the set consisting of the pseudoidentities of $\Sigma$ over the most general graphs over which they can be read is a basis for the global of $V$.

In Section 4, we use our result as well as techniques introduced in [6] to show that the pseudovariety $V_{\frac{3}{2}}$ of ordered semigroups, which corresponds to the level $3/2$ of the Straubing-Thérien hierarchy, has infinite vertex rank. Initially, as suggested by J.-E. Pin, our goal was to investigate, using the “basis theorem” for semidirect products, the decidability of the pseudovariety of finite semigroups $G_{\frac{3}{2}} = V_{\frac{3}{2}} \ast G$, which corresponds to the level $3/2$ of the group hierarchy. The infinity of the vertex-rank of $V_{\frac{3}{2}}$ renders this problem out of reach for the currently known range of validity of the basis theorem.

2. Preliminaries

We recall in Sections 2.1 and 2.2 some definitions and results concerning ordered semigroups and semigroupoids, free profinite semigroups and semigroupoids, and pseudovarieties of ordered semigroups and semigroupoids. For more details the reader is referred to [3, 4, 15, 22, 27]. Section 2.3 introduces basic facts about inverse semigroups and a version in the “ordered semigroups context” of a result of Reilly [28].

2.1. Ordered Semigroups

An ordered semigroup $S$ is a semigroup equipped with a partial order relation $\leq$ such that, for all $x, y, z \in S$, $x \leq y$ implies $xz \leq yz$ and $zx \leq yz$. A homomorphism of ordered semigroups $\varphi : (S, \leq) \to (T, \leq)$ is a semigroup homomorphism such that, for all $x, y \in S$, $x \leq y$ implies $\varphi(x) \leq \varphi(y)$. Semigroups are viewed as ordered
semigroups under the equality order relation. An ordered semigroup \((S, \leq)\) is an ordered subsemigroup of \((T, \leq)\) if \(S\) is a subsemigroup of \(T\) and the order on \(S\) is the restriction to \(S\) of the order on \(T\). If there is a surjective homomorphism of ordered semigroups \(\varphi: (S, \leq) \to (T, \leq)\), then we say that \((T, \leq)\) is an ordered quotient of \((S, \leq)\). The product of a family \((S_i, \leq_i)_{i \in I}\) of ordered semigroups is the ordered semigroup \(\prod_{i \in I} S_i\) where the multiplication is defined by \((s_i)_{i \in I}(t_i)_{i \in I} = (s_i t_i)_{i \in I}\) and the order is given by

\[(s_i)_{i \in I} \leq (t_i)_{i \in I} \text{ if and only if, for all } i \in I, s_i \leq_i t_i. \quad (1)\]

Throughout this paper \(X\) represents a finite non-empty set. Let \(X^+\) be the free semigroup on \(X\). Then \((X^+, =)\) is an ordered semigroup and it is the free ordered semigroup on \(X\).

A topological semigroup is a semigroup endowed with a topology such that the multiplication is continuous. Finite semigroups are viewed as discrete topological semigroups. For a finite set \(X\), we say that a topological semigroup \(S\) is \(X\)-generated if a mapping \(\eta: X \to S\) is given such that \(\eta(X)\) generates a dense subsemigroup of \(S\).

For a finite set \(X\), the projective limit of \(X\)-generated finite semigroups is denoted by \(\hat{X}^+\), and it is called the free profinite semigroup [4]. This semigroup is compact, totally disconnected and \(X\)-generated via the natural mapping \(\iota: X \to \hat{X}^+\). One may show that \(\hat{X}^+\) has the following universal property: every mapping \(\varphi: X \to S\) into a finite semigroup \(S\) can be extended to a unique continuous homomorphism \(\hat{\varphi}: \hat{X}^+ \to S\) such that \(\hat{\varphi} \iota = \varphi\). Moreover, the topological structure of \(\hat{X}^+\) is metrizable, that is \(\hat{X}^+\) may be endowed with a metric \(d\) such that \(d\) induces the given topology of \(\hat{X}^+\) [4, Prop. 7.4].

Note that the subsemigroup generated by \(\iota X\) is (isomorphic to) the free semigroup \(X^+\) so we may assume that \(X^+\) is a subsemigroup of \(\hat{X}^+\).

Let \(u, v \in \hat{X}^+\). We say that a finite ordered semigroup \((S, \leq)\) satisfies the (pseudo)identity \(u \leq v\) and we write \(S \models u \leq v\) if, for every continuous homomorphism \(\varphi: X^+ \to S\), we have \(\varphi(u) \leq \varphi(v)\).

A pseudovariety of ordered semigroups is a class of finite ordered semigroups which is closed under taking finite ordered subsemigroups, finitary products and ordered quotients.

Pin and Weil have extended Reiterman’s theorem [29] to pseudovarieties of ordered semigroups [25]. For a set \(\Sigma\) of pseudoidentities, we denote by \([\Sigma]\) the class of all finite ordered semigroups which satisfy all pseudoidentities of \(\Sigma\).

**Theorem 2.1.** Let \(V\) be a class of finite ordered semigroups. Then \(V\) is a pseudovariety of ordered semigroups if and only if there exists a set \(\Sigma\) of pseudoidentities over finite sets such that \(V = [\Sigma]\).
2.2. Ordered semigroupoids

A graph is a set \( G = V(G) \cup E(G) \) consisting of two sorts of elements, vertices and edges, endowed with two operations \( \alpha, \omega : E(G) \to V(G) \) which give respectively the beginning and end vertices of each edge. For \( a, b \in E(G) \), we say that \( a \) and \( b \) are coterminal if \( \alpha(a) = \alpha(b) \) and \( \omega(a) = \omega(b) \), and they are consecutive if \( \omega(a) = \alpha(b) \). A path in a graph \( \Gamma \) is a finite sequence \( a_1 \ldots a_n \) of edges of \( \Gamma \) such that \( a_i \) and \( a_{i+1} \) are consecutive, for \( i = 1, \ldots, n - 1 \). Graphs with one vertex may be identified with their sets of edges. A graph homomorphism is a mapping \( \varphi : G \to H \) between two graphs respecting sorts and operations. A subgraph \( H \) of \( G \) is a graph contained in \( G \) such that the inclusion \( H \hookrightarrow G \) is a graph homomorphism.

By a semigroupoid we mean a graph \( S \) endowed with a partial associative multiplication on \( E(S) \) given by: for \( s, t \in E \), \( st \) is defined if and only if \( \omega(s) = \alpha(t) \) and, then, \( \alpha(st) = \alpha(s) \) and \( \omega(st) = \omega(t) \). If \( S \) admits a local identity at each vertex, then we say that \( S \) is a category. For vertices \( c, d \in V(S) \), the hom-set \( S(c, d) \) is the set of edges \( s \in E(S) \) such that \( \alpha(s) = c \) and \( \omega(s) = d \); in case \( c = d \) we put \( S(c) = S(c, d) \).

For a semigroupoid \( S \), a binary relation \( \tau \) on \( E(S) \) is said to be compatible if, for all \( x, y \in E(S) \),

(i) if \( (x, y) \in \tau \) then \( x \) and \( y \) are coterminal;
(ii) if \( (x, y) \in \tau \) and \( x, z \) are consecutive edges, then \( (xz, yz) \in \tau \);
(iii) if \( (x, y) \in \tau \) and \( z, x \) are consecutive edges, then \( (zx, zy) \in \tau \).

A congruence \( \tau \) on a semigroupoid \( S \) is an equivalence relation on \( E(S) \) which is compatible. The quotient semigroupoid \( S/\tau \) has \( V(S/\tau) = V(S) \) and \( E(S/\tau) = E(S)/\tau \), with the induced maps \( \alpha \) and \( \omega \) and composition of edges.

We say that the semigroupoid \( S \) is an ordered semigroupoid if it is equipped with a compatible partial order relation \( \leq \) on \( E(S) \). It is important to notice that two edges must be coterminal to be comparable under \( \leq \).

For a finite graph \( \Gamma \), the free semigroupoid \( \Gamma^+ \) on \( \Gamma \) has as vertex-set \( V(\Gamma) \) and as edges the non-empty paths of \( \Gamma \).

A homomorphism of ordered semigroupoids is a map \( \varphi \) between \((S, \leq)\) and \((T, \leq)\) which respects sorts, operations and the partial order, that is, \( \varphi(x) \leq \varphi(y) \) whenever \( x \leq y \). A homomorphism of ordered semigroupoids \( \varphi : S \to T \) is full if the restriction of \( \varphi \) to each hom-set \( S(v_1, v_2) \) is surjective, \( \varphi \) is order-faithful if for every two coterminal edges \( x, y \), \( \varphi(x) \leq \varphi(y) \) implies \( x \leq y \), and it is said to be a quotient if \( \varphi \) is full and \( \varphi|_{V(S)} \) is bijective. Note that, if \( S \) is a semigroupoid and \( \tau \) is a congruence on \( S \), then the canonical homomorphism \( \eta : S \to S/\tau \) is a quotient homomorphism.

An ordered semigroupoid \( S \) is said to divide an ordered semigroupoid \( T \) if there are an ordered semigroupoid \( U \), a quotient homomorphism \( U \to S \) and an order-faithful homomorphism \( U \to T \). The product of a family \((S_i)_{i \in I}\) of semigroupoids has set of vertices \( \prod_{i \in I} V(S_i) \), set of edges \( \prod_{i \in I} E(S_i) \), and the partial operations \( \alpha, \omega \), and edge composition are defined component-wise. If the factors \( S_i \) are ordered semigroupoids, then so is the product under the order given
by (1). The coproduct of \((S_i)_{i \in I}\) is obtained by taking the disjoint union of the \(S_i\), both for the set of vertices, the set of edges, and the partial operations \(\alpha, \omega\), and edge composition. If the cofactors \(S_i\) are ordered semigroupoids then so is the coproduct under the (disjoint) union of the order relations of the cofactors.

A pseudovariety of ordered semigroupoids is a class of finite ordered semigroupoids containing the one-element semigroup, which is closed under taking divisors of ordered semigroupoids, and finitary direct products and coproducts. The pseudovariety of all finite ordered semigroupoids is denoted by \(\text{OSd}\).

An ordered semigroupoid \(S\) is viewed as an ordered semigroupoid by taking the set of edges \(S\) with both ends at an added vertex. Conversely, for an ordered semigroupoid \(S\) and a vertex \(v\) of \(S\), the set \(S(v)\) of all loops at vertex \(v\) constitutes an ordered semigroup called the local semigroup of \(S\) at \(v\).

The pseudovariety of ordered semigroupoids generated by a given pseudovariety \(V\) of ordered semigroups is called the global of \(V\) and is denoted \(\text{g}_V\). Note that \(\text{g}_V\) is the smallest pseudovariety of ordered semigroupoids whose ordered semigroups are precisely those of \(V\). The largest such pseudovariety is called the local of \(V\) and is denoted \(\ell_V\); it consists of all finite ordered semigroupoids such that all local semigroups are members of \(V\). A pseudovariety \(V\) of ordered semigroups is said to be local if \(\text{g}_V = \ell_V\).

A topological semigroupoid \(S\) is a semigroupoid endowed with a topology with respect to which the partial operations \(\alpha, \omega\), and edge multiplication are continuous. Finite semigroupoids equipped with the discrete topology become topological semigroupoids.

A topological semigroupoid is profinite if it is a projective limit of finite semigroupoids; its topology is said to be the profinite topology. For a finite graph \(\Gamma\), a topological semigroupoid \(S\) is said to be \(\Gamma\)-generated if there exists a graph homomorphism \(\varphi : \Gamma \to S\) such that \(\varphi|_{V(\Gamma)}\) is injective and the subgraph of \(S\) generated by \(\varphi(\Gamma)\) is dense.

We denote by \(\hat{\Gamma}^+\) the projective limit of all \(\Gamma\)-generated finite semigroupoids. Note that \(\hat{\Gamma}^+\) is a \(\Gamma\)-generated semigroupoid via the natural mapping \(\phi : \Gamma \to \hat{\Gamma}^+\). The semigroupoid \(\hat{\Gamma}^+\) has the usual universal property: for every homomorphism \(\varphi : \Gamma \to S\) into a profinite semigroupoid there exists a unique continuous homomorphism \(\hat{\varphi} : \hat{\Gamma}^+ \to S\) such that \(\hat{\varphi}\phi = \varphi\). Moreover, we may endow \(\hat{\Gamma}^+\) with a metric \(d\) such that the topology induced by \(d\) is the profinite topology [15].

By a \(V\)-pseudoidentity over a graph \(\Gamma\) we mean an ordered pair of the form \((x \leq y, \Gamma)\) where \(x\) and \(y\) are coterminal edges of \(\hat{\Gamma}^+\). A finite ordered semigroupoid \(S\) satisfies the pseudoidentity \((x \leq y, \Gamma)\) and we write \(S \models (x \leq y, \Gamma)\) if, for each continuous homomorphism of semigroupoids \(\varphi : \hat{\Gamma}^+ \to S\), we have \(\varphi(x) \leq \varphi(y)\).

Given a set \(\Sigma\) of pseudoidentities over finite graphs it is readily checked that the class \(V = [\Sigma]\), consisting of all finite ordered semigroupoids \(S\) which satisfy all pseudoidentities from \(\Sigma\), is a pseudovariety of ordered semigroupoids. Then we also say that the pseudovariety \(V\) is defined by \(\Sigma\) or that \(\Sigma\) is a basis of pseudoidentities for \(V\). The analog of Reiterman’s Theorem in this context states that
every pseudovariety of ordered semigroupoids has a basis of pseudoidentities. The
proof of the ordered case can be readily obtained from the unordered case [7,15]
using the necessary adaptations to accommodate the order as in [25].

Following [1], we say that a pseudovariety V of semigroupoids has vertex-rank
or v-rank n if n is the smallest non-negative integer such that V admits a basis of
pseudoidentities over graphs with at most n vertices. If no such integer n exists,
then we say that V has infinite v-rank.

2.3. Inverse semigroups

In this subsection, we recall and extend some results on inverse semigroups.
The reader unfamiliar with this topic may wish to consult [18] for a much more
thorough introduction.

A semigroup S is inverse if, for every s ∈ S there is a unique inverse s−1 ∈ S
such that ss−1s = s and s−1ss−1 = s−1. The natural order on a inverse monoid S
is defined as follows:

\[ x \leq y \text{ if and only if } x = ey \text{ for some } e = e^2 \in S. \]

An inverse semigroup endowed with the natural order is an ordered semigroup and
it is said to be an ordered inverse semigroup.

Inverse semigroups considered as algebras with the binary operation of multi-
plication and the unary operation of inversion form the variety I defined by the
equations

\[ xx^{-1} = x, \quad (x^{-1})^{-1} = x, \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}. \]

Note that homomorphisms of (inverse) semigroups respect the natural order on
inverse semigroups.

Denote by X−1 a set disjoint from X and in one-to-one correspondence
x ↦→ x−1 with X. We call X−1 the set of formal inverses of elements of X. For x ∈ X, we
let (x−1)−1 = x. There is a free inverse semigroup over X denoted by FI X. It is
well known that

\[ FI X \simeq (X \cup X^{-1})^+ / \tau \]

where \( \tau \) is the congruence on \((X \cup X^{-1})^+\) generated by the set

\[ \{(uu^{-1}u, u), (uu^{-1}zz^{-1}, zz^{-1}uu^{-1}) : u, z \in (X \cup X^{-1})^+\}. \]

Let w ∈ (X ∪ X−1)*. The content of w, denoted c(w), is the set of x ∈ X such
that either x or x−1 occur in w. A word w is said to be reduced if w does not
contain factors xx−1 or x−1x for any x ∈ X. If w = y_1 \ldots y_n (y_i ∈ X ∪ X−1) is
reduced then 1 and every word y_i \ldots y_j (1 ≤ i ≤ j ≤ n) are segments of w, and 1
and every segment of w which begins at y_1 are initial segments of w. Let K_X be
the set of all words of (X ∪ X−1)* of the form

\[ w = a_1a_1^{-1} \ldots a_na_n^{-1}g \]
where $a_1, \ldots, a_n, g$ are reduced words, no $a_i$ is an initial segment of any $a_j$ ($j \neq i$) and $g$ is an initial segment of $a_1$.

**Theorem 2.2.**

1. For every $w \in FI_X$, there is $u \in K_X$ such that $w = u\tau$.
2. Let $u = a_1a_1^{-1} \ldots a_m a_m^{-1} g$ and $v = b_1 b_1^{-1} \ldots b_n b_n^{-1} h$ be words of $K_X$. Then $u\tau = v\tau$ if and only if $\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_n\}$ and $g = h$.

We say that $u \in K_X$ is a canonical form of $u\tau \in FI_X$. By abuse of notation, we may also use $u$ for $u\tau$.

Let $I$ be a finite set. The set $I \times I \cup \{0\}$ with multiplication given by

$$(i, j)(i', j') = \begin{cases} (i, j') & \text{if } i' = j \\ 0 & \text{otherwise} \end{cases}$$

and $0a = a0 = 0$, for any $a \in I \times I \cup \{0\}$ is the $I \times I$ aperiodic Brandt semigroup and is denoted by $B_I$ or $B_{IJ}$. All aperiodic Brandt semigroups are inverse. Let $\mathcal{B}$ be the subvariety of the variety $\mathcal{I}$ of all inverse semigroups generated by $B_2$.

The following is well known.

**Lemma 2.3.** Let $I$ be an arbitrary set. Then $B_I$ belongs to $\mathcal{B}$.

Let $a \in FI_X$ have canonical form

$$a = a_1a_1^{-1} \ldots a_n a_n^{-1} g$$

and define

$$S[a] = \bigcup_{i=1}^{n} \{b \neq 1 : b \text{ is an initial segment of } a_i\}.$$ 

Note that, if $S[a] = \{b_1, \ldots, b_k\}$ then $a = (b_1 b_1^{-1} \ldots b_k b_k^{-1} g)\tau$.

Reilly [28] defines two relations $\gamma_a$ and $\delta_a$ on $Y = X \cup X^{-1}$ as follows:

- $(x, y) \in \gamma_a$ if and only if $x, y \in S[a]$ or there exists $u \in FI_X \cup \{1\}$ such that $ux^{-1}y \in S[a]$ or $uy^{-1}x \in S[a]$.
- $\delta_a$ is the reflexive and transitive closure of $\gamma_a$.

Since $\gamma_a$ is a symmetric relation it follows that $\delta_a$ is an equivalence relation.

If $a_i = a_{i1} \ldots a_{im}$ for $i = 1, \ldots, n$ and $g = g_1 \ldots g_r$ with $a_{ij}, g_k \in Y$, then we put $s(a) = a_{i1}\delta_a$ and

$$e(a) = \begin{cases} s(a) & \text{if } g = 1 \\ g^{-1}\delta_a & \text{otherwise.} \end{cases}$$

Note that, for all $i, j = 1, \ldots, n$, $a_{ij}, a_{j1} \in S[a]$ so $(a_{i1}, a_{j1}) \in \delta_a$.

We say that a variety $\mathcal{V}$ of inverse semigroups satisfies $u \leq v$, with $u, v \in FI_X$ if, for every homomorphism $\varphi : FI_X \to S$ with $S \in \mathcal{V}$, we have $\varphi u \leq \varphi v$. 

The following result generalizes the necessary condition of Theorem 3.3 of [28].

**Proposition 2.4.** Let \( a, b \in FI_X \). If \( B \) satisfies \( b \leq a \) then the following conditions hold:

1. \( c(a) \subseteq c(b) \);
2. \( \delta_a \subseteq \delta_b \);
3. \( s(a) \subseteq s(b) \);
4. \( e(a) \subseteq e(b) \).

**Proof.** Let us show that \( c(a) \subseteq c(b) \). If there is \( x \in c(a) \) which is not in \( c(b) \) then, taking the homomorphism \( \varphi : FI_X \rightarrow B_2 \) given by \( \varphi(x) = 0 \), and \( \varphi(y) = (1, 1) \) for every \( y \in X \setminus \{ x \} \), we have \( \varphi(a) = 0 \) and \( \varphi(b) = (1, 1) \), which contradicts our hypothesis.

Let \( I = Y/\delta_b \) and let us define \( \theta : X \rightarrow B_I \) such that \( \theta x = (x\delta_b, x^{-1}\delta_b) \). Then there exists a homomorphism \( \hat{\theta} : FI_X \rightarrow B_I \) which extends \( \theta \).

Since \( B \) satisfies \( b \leq a \), by hypothesis, and \( B_I \in B \) by Lemma 2.3, we have \( \hat{\theta} b \leq \hat{\theta} a \).

First we will show that \( \hat{\theta} b \neq 0 \). Suppose that \( b = b_1 b_1^{-1} \ldots b_m b_m^{-1} g \) is a canonical form of \( b \) and, for each \( i, \ b_i = b_{i1} \ldots b_{ip_i} \), is the reduced form of \( b_i \). Let \( 1 \leq j < p_i \).

Since \( b_1 \ldots b_{i,j+1} \in S[b] \), it follows that \( (b_i^{-1}, b_{i,j+1}) \in \gamma_b \leq \delta_b \). This allows us to obtain

\[
\hat{\theta}(b_{i,j} b_{i,j+1}) = (\theta b_{i,j})(\theta b_{i,j+1}) = (b_{i,j} \delta_b, b_{i,j}^{-1} \delta_b)(b_{i,j+1} \delta_b, b_{i,j+1}^{-1} \delta_b)
\]

\[
= (b_{i,j} \delta_b, b_{i,j+1} \delta_b) \neq 0.
\]

By [28, Lem. 2.2], this implies that \( \hat{\theta} b_i \neq 0 \) for every \( i \), thus \( \hat{\theta} (b_i b_i^{-1}) \neq 0 \), and so

\[
\hat{\theta}(b_i b_i^{-1}) = (\theta b_{i1}) \ldots (\theta b_{ip_i}) (\theta b_{ip_i})^{-1} \ldots (\theta b_{i1})^{-1}
\]

\[
= (b_{i1} \delta_b, b_{i1} \delta_b)
\]

\[
= (s(b), s(b)).
\]

If \( g \neq 1 \) then \( g \) is an initial segment of \( b_1 \) so \( \hat{\theta} b = \hat{\theta} g \neq 0 \). If \( g = 1 \) then

\[
\hat{\theta}(b) = \hat{\theta}(b_1 b_1^{-1}) \hat{\theta}(b_2 b_2^{-1}) \ldots \hat{\theta}(b_m b_m^{-1}) = (s(b), s(b)) \neq 0.
\]

Since \( \hat{\theta} b \leq \hat{\theta} a \) and \( \hat{\theta} b \neq 0 \), we have \( \hat{\theta} a = \hat{\theta} b \).

Let us show that \( \delta_a \subseteq \delta_b \). Let \( x, y \in Y \) such that \( (x, y) \in \gamma_a \). Suppose that \( a = a_1 a_1^{-1} \ldots a_n a_n^{-1} h \) is a canonical form of \( a \) and let \( a_1 = a_{11} \ldots a_{1p_1} \) (\( a_{i1} \in X \cup X^{-1} \)) be the reduced form of \( a_1 \). If \( x \in S[a] \) then \( \hat{\theta}(xx^{-1}aa^{-1}) = \hat{\theta}(aa^{-1}) \) which implies that \( x\delta_b = a_{11} \delta_b \). So, if \( x, y \in S[a] \), then we have

\[
x\delta_b = a_{11} \delta_b = y\delta_b.
\]
Suppose now that there exists \( u \in FI_X \cup \{ 1 \} \) such that \( ux^{-1}y \in S[a] \). We have 
\[
\hat{\theta}(ux^{-1}y)(ux^{-1}y)^{-1}a = \hat{\theta}a \text{ and } \hat{\theta}a \neq 0 \text{ so }
\]
\[
0 \neq \hat{\theta}(x^{-1}y) = (x^{-1}b, x\delta_b)(y\delta_b, y^{-1}\delta_b)
\]
which implies that \( x\delta_b = y\delta_b \). This shows that \( \gamma_a \subseteq \delta_b \), thus \( \delta_a \subseteq \delta_b \).

Let us show that \( s(a) \subseteq s(b) \) and \( e(a) \subseteq e(b) \). First, since \( \hat{\theta}a = \hat{\theta}b \neq 0 \), we must have \( a_1\delta_b = b_1\delta_b \), which implies that \( s(a) \subseteq s(b) \) by (ii).

Suppose next that \( g = g_1 \ldots g_r \) and \( h = h_1 \ldots h_s \) are reduced forms of \( g \) and \( h \), respectively. If \( g \neq 1 \) and \( h \neq 1 \), then
\[
0 \neq \hat{\theta}b = \hat{\theta}g = (g_1\delta_b, g_1^{-1}\delta_b) \ldots (g_s\delta_b, g_s^{-1}\delta_b) = (g_1\delta_b, g_s^{-1}\delta_b).
\]
Similarly, \( \hat{\theta}a = (h_1\delta_b, h_s^{-1}\delta_b) \). Since \( \hat{\theta}a = \hat{\theta}b \), we have \( g_1^{-1}\delta_b = h_s^{-1}\delta_b \), so \( e(a) \subseteq e(b) \) again by (ii). If \( g = 1 \) and \( h \neq 1 \) then \( \hat{\theta}a = (h_1\delta_b, h_s^{-1}\delta_b) \) and \( \hat{\theta}b = (b_1\delta_b, b_1\delta_b) \). Since \( \hat{\theta}a = \hat{\theta}b \), we have \( h_s^{-1}\delta_b = b_1\delta_b \), thus \( e(a) \subseteq e(b) \). The case \( g \neq 1 \) and \( h = 1 \) is similar. Finally, if \( g = 1 = h \) then \( e(a) = s(a) \subseteq s(b) = e(b) \). \( \square \)

3. Pseudoidentities satisfied by the globals of pseudovarieties containing \( B_2 \)

In this section we extend some results of Section 5 of [5] from pseudovarieties of semigroups to pseudovarieties of ordered semigroups.

Let \( X \) be a finite set and \( u \in \tilde{X}^+ \). Let \( \rho_u \) be the equivalence relation over \( Y = X \cup X^{-1} \) generated by the relation
\[
\{(x^{-1}, y) : xy \text{ is a factor of } u \}.
\] (2)
Since factors of length 2 can be recognized by finite semigroups, the correspondence \( u \in \tilde{X}^+ \rightarrow \rho_u \in \mathcal{P}(Y^2) \) defines a continuous map, where \( \mathcal{P}(Y^2) \) is viewed as a discrete space.

The content of \( u \), \( c(u) \), is the set of \( x \in X \) such that \( x \) is a factor of \( u \), that is \( u = u_1 x u_2 \) for some \( u_1, u_2 \in X^* \).

**Definition 3.1.** [5] For \( u \in \tilde{X}^+ \) and \( Y = X \cup X^{-1} \), let \( A_u \) be the graph defined by
\[
V(A_u) = Y / \rho_u
\]
\[
A_u (x \rho_u, y \rho_u) = \{ z : z \in X, (x, z) \in \rho_u, (z^{-1}, y) \in \rho_u \}.
\]
Note that each \( z \in X \) gives one edge and \( E(A_u) = X \).

This definition of \( A_u \) introduces some edges \( x \notin c(u) \) which do not play any role when we test pseudoidentities over \( A_u \), but it simplifies some technical arguments.

**Lemma 3.2.** Let \( X \) be a finite set. If \( u \in X^+ \) then \( \rho_u = \delta_u \).
Proof. Suppose that \( u = x_1 \ldots x_n \) with \( x_i \in X, \ i = 1, \ldots, n \). Then \( u = uu^{-1}u \) is a canonical form of \( u \in FI_X \). It follows that \((x, y) \in \gamma_u \) if and only if \( x = y = x_1 \) or (exactly) one the two-letter words \( x^{-1}y \) and \( y^{-1}x \) is a factor of \( u \). This shows that \( \gamma_u \) contains the relation (2) and is contained is the reflexive symmetric closure of (2). Hence \( \gamma_u \) and (2) generate the same equivalence relation. \( \square \)

For a finite set \( X \) and \( u \in \hat{X}^\ast \), we denote by \( t_1(u) \) the unique letter which is a suffix of \( u \), that is, \( u = u_1t_1(u) \) for some \( u_1 \in X^\ast \). Dually, \( i_1(u) \) is the unique letter which is a prefix of \( u \), which means that \( u = i_1(u)u_2 \) for some \( u_2 \in X^\ast \).

**Theorem 3.3.** Let \( X \) be a finite set and \( u, v \in \hat{X}^\ast \). The following conditions are equivalent:

(a) \( B_2 \) satisfies \( u \leq v \).

(b) We have \( \rho_v \subseteq \rho_u \), \((i_1(u), i_1(v)) \in \rho_u \) and \((t_1(u)^{-1}, t_1(v)^{-1}) \subseteq \rho_u \).

(c) There is a graph homomorphism \( \theta : A_v \to A_u \) such that

(i) \( \theta(i_1(v)\rho_v) = i_1(u)\rho_u \) and \( \theta(t_1(v)^{-1}\rho_v) = t_1(u)^{-1}\rho_u \), and

(ii) for every \( z \in A_v(x_\rho_y, y_\rho_x) \), \( \theta(z) = z \).

Proof. First, we prove that (a) is equivalent to (b). Recall that \( u \) and \( v \) are limits of sequences of words, say \((u_n)\) and \((v_n)\), respectively. By continuity of \( \rho \), \( i_1 \), and \( t_1 \), we may assume that, for every \( n \), \( \rho_{u_n} = \rho_u \), \( \rho_{v_n} = \rho_v \), \( i_1(u_n) = i_1(u) \), \( i_1(v_n) = i_1(v) \), \( t_1(u_n) = t_1(u) \), and \( t_1(v_n) = t_1(v) \).

As \( B_2 \) is finite, there is \( p \) such that, for each \( n \geq p \), \( B_2 \) satisfies the pseudoidentities \( u = u_n \) and \( v = v_n \). By Lemma 3.2 and Proposition 2.4, if \( B_2 \) satisfies the inequality \( u_n \leq v_n \) then Condition (b) holds for \( u_n \) and \( v_n \). Thus if \( B_2 \) satisfies \( u \leq v \) then Condition (b) holds.

To prove the converse, let \( \eta : \hat{X}^\ast \to B_2 \) be an arbitrary continuous homomorphism of ordered semigroups. If \( \eta(u) = 0 \), the minimum of \( B_2 \), then obviously \( \eta(u) \leq \eta(v) \). So, assume that \( \eta(u) \neq 0 \). By continuity of \( \eta \), we may assume that \( \eta(u_n) = \eta(u) \) for every \( n \). It is clear that Condition (b) holds for \( u_n \) and \( v_n \). Our goal is to show that \( \eta(u_n) = \eta(v_n) \) and thus \( \eta(u) = \eta(v) \), by continuity of \( \eta \). Hence we may assume that \( u, v \in X^\ast \).

Suppose that \( xy \) is a factor of \( v \). Since \( \rho_v \subseteq \rho_u \), then there are an index \( n \) and \( z_1, z_2, \ldots, z_{2n} \in X \) such that the words

\[ z_0z_1, z_2z_1, z_2z_3, \ldots, z_2n^2z_{2n-1}, z_2n^2z_{2n+1}, \]

are factors of \( u \), where \( z_0 = x \) and \( z_{2n+1} = y \). If \( \eta(z_i) = (r_i, s_i) \) with \( r_i, s_i \in \{1, 2\} \), then \( \eta(u) \neq 0 \) implies that \( s_0 = r_1 = s_2 = r_3 = \cdots = s_{2n} = r_{2n+1} \). Since \( v \) is a word, it follows that \( \eta(v) \neq 0 \), and so \( \eta(u) \) and \( \eta(v) \) lie in the same \( \mathcal{J} \)-class. Furthermore, in this case a similar process may be used to show that \((i_1(u), i_1(v)) \in \rho_u \) implies that \( \eta(u) \) and \( \eta(v) \) are \( \mathcal{R} \)-related, and \((t_1(u)^{-1}, t_1(v)^{-1}) \in \rho_u \) implies that \( \eta(u) \) and \( \eta(v) \) are \( \mathcal{L} \)-related. Hence we have \( \eta(u) = \eta(v) \).
Let us show that (b) is equivalent to (c). Suppose that Condition (b) holds. Since \( \rho_v \subseteq \rho_u \), we may define a map \( \varphi : A_v \to A_u \) by
\[
\varphi(x\rho_v) = x\rho_u, \quad \text{for } x\rho_v \in V(A_v), \\
\varphi(z) = z, \quad \text{for } z \in A_v(x\rho_v, y\rho_v).
\]
It is clear that \( \varphi \) is a graph homomorphism. By (b), it is immediate that \( \varphi(i_1(v)\rho_u) = i_1(v)\rho_u = i_1(u)\rho_u \) and \( \varphi(t_1(v)^{-1}\rho_u) = t_1(v)^{-1}\rho_u = t_1(u)^{-1}\rho_u \).

Conversely, suppose that \( \theta : A_v \to A_u \) is a graph homomorphism which verifies the conditions of (c). For \( x \in X \), we have
\[
x \in A_u(x\rho_u, x^{-1}\rho_u), \quad x \in A_u(\theta(x\rho_v), \theta(x^{-1}\rho_v))
\]
and \( x \) gives one edge in \( A_u \), then \( x\rho_u = \theta(x\rho_v) \). It follows that, if \( (x, y) \in \rho_v \) then \( \theta(x\rho_v) = \theta(y\rho_v) \), that is \( x\rho_u = y\rho_u \). Thus \( \rho_v \subseteq \rho_u \).

If \( x \in i_1(v)\rho_v \) then \( x\rho_u = \theta(x\rho_v) = \theta(i_1(v)\rho_v) = i_1(u)\rho_v \) so \( i_1(v)\rho_v \subseteq i_1(u)\rho_u \). Similarly, we have \( t_1(v)^{-1}\rho_v \subseteq t_1(u)^{-1}\rho_u \).  

For \( u \in X^+ \), let \( \varphi_u : A_u \to X \) be the natural graph homomorphism, where the finite set \( X \) is viewed as a graph with one vertex. Then there is a unique continuous homomorphism of semigroupoids \( \widehat{\varphi_u} : \widehat{A_u} \to X^+ \) such that \( \widehat{\varphi_u} \) extends \( \varphi_u \), that is the following diagram commutes:
\[
\begin{array}{ccc}
A_u & \xrightarrow{\phi_u} & \widehat{A_u}^+ \\
\varphi_u \downarrow & & \downarrow \widehat{\varphi_u} \\
X & \xrightarrow{\varphi} & X^+
\end{array}
\]

By Proposition 2.3 of [2], \( \widehat{\varphi_u} \) is faithful. We define the content of \( w \in \widehat{A_u}^+ \) as being the content of \( \widehat{\varphi_u}(w) \).

**Lemma 3.4.** Let \( u \in X^+ \) and let \( \widehat{\varphi_u} \) be the faithful homomorphism of semigroupoids described above. Then \( \widehat{\varphi_u} \) restricted to \( E(\widehat{A_u}^+) \) is injective and \( u = \widehat{\varphi_u}(\tilde{u}) \) for some \( \tilde{u} \in A_{\tilde{u}}(i_1(u)\rho_u, t_1(u)^{-1}\rho_u) \).

**Proof.** Let \( \varphi = \varphi_u \). Note that, if \( w = a_1 \ldots a_n (a_i \in X) \) is a finite path on \( A_u \) then \( \widehat{\varphi}(w) = w \in X^+ \).

Let us show that \( \widehat{\varphi} \) is injective. Suppose that \( \widehat{\varphi}(w_1) = \widehat{\varphi}(w_2) \) for some \( w_1, w_2 \in E(\widehat{A_u}^+) \). Let \( (s_n) \) and \( (r_n) \) be sequences of finite paths on \( A_u \) which converge to \( w_1 \) and \( w_2 \) respectively. Since the functions \( \alpha, \omega \) and content are continuous, we may assume that, for all \( n, \alpha(w_1) = \alpha(s_n), \alpha(w_2) = \alpha(r_n), \omega(w_1) = \omega(s_n), \omega(w_2) = \omega(r_n), c(w_1) = c(s_n), \) and \( c(w_2) = c(r_n) \). As \( \widehat{\varphi} \) is continuous, the sequences of words \( \langle \widehat{\varphi}(s_n) \rangle \) and \( \langle \widehat{\varphi}(r_n) \rangle \) converge to \( \widehat{\varphi}(w_1) \) and \( \widehat{\varphi}(w_2) \), respectively. Since \( i_1 \) and \( t_1 \) are continuous, we may assume that, for all \( n, i_1(\widehat{\varphi}(s_n)) = i_1(\widehat{\varphi}(w_1)) \),
Let Corollary 3.5. for all \( n \)

\[
i_1(\hat{\varphi}(r_n)) = i_1(\hat{\varphi}(w_2)), \quad t_1(\hat{\varphi}(s_n)) = t_1(\hat{\varphi}(w_1)) \quad \text{and} \quad t_1(\hat{\varphi}(r_n)) = t_1(\hat{\varphi}(w_2)).
\]

Therefore, for all \( n \), we have

\[
i_1(\hat{\varphi}(s_n)) = i_1(\hat{\varphi}(w_1)) = i_1(\hat{\varphi}(w_2)) = i_1(\hat{\varphi}(r_n)), \quad \text{and}
\]

\[
t_1(\hat{\varphi}(s_n)) = t_1(\hat{\varphi}(w_1)) = t_1(\hat{\varphi}(w_2)) = t_1(\hat{\varphi}(r_n)).
\]

It follows that \( \alpha(u_1) = \alpha(s_n) = i_1(\hat{\varphi}(s_n))\rho_n = i_1(\hat{\varphi}(r_n))\rho_n = \alpha(r_n) = \alpha(w_2) \) and, similarly, \( \omega(w_1) = \omega(w_2) \). This shows that \( w_1 \) and \( w_2 \) are coterminal hence \( w_1 = w_2 \), since \( \hat{\varphi} \) is faithful.

Finally, we show that \( u \in \hat{\varphi}(\overline{A}_u^+) \). Suppose that \( u = a_1 \ldots a_n \) with \( a_i \in X \).

For each \( i < n \), \( a_ia_{i+1} \) is a factor of \( u \) so \( (a_i^{-1}, a_{i+1}) \in \rho_u \), so that \( a_i \) and \( a_{i+1} \) are consecutive edges, that is \( \omega(a_i) = \alpha(a_{i+1}) \). Hence u may be viewed as a finite path and \( \hat{\varphi}(u) = u \), as we have observed above. If \( u \) is the limit of a sequence \( (u_n) \) of words then, by continuity of the functions involved, we may assume that \( c(u_n) = c(u), i_1(u_n) = i_1(u), t_1(u_n) = t_1(u) \), and \( \rho_{u_n} = \rho_u \) (so \( A_{u_n} = A_u \)) for all \( n \). By compactness of \( \overline{A}_u^+ \), there is a subsequence of \( (u_n) \) which converges to some \( \tilde{u} \in \overline{A}_u^+ \). Since \( \hat{\varphi} \) is continuous and \( \hat{\varphi}(u_n) = u_n \) for all \( n \), we conclude that \( \hat{\varphi}(\tilde{u}) = u \).

By Lemma 3.4, for a given \( u \in \overline{X}^+ \), there is a unique \( \tilde{u} \in \tilde{E}(\overline{A}_u^+) \) such that \( \overline{\varphi}_u(\tilde{u}) = u \) so we will abuse notation and denote \( \tilde{u} \) by \( u \).

Let \( u, v \in \overline{X}^+ \) and let \( \theta : A_v \to A_u \) be a homomorphism of graphs. By the universal property of \( \overline{A}_u^+ \) there exists a unique continuous homomorphism \( \hat{\theta} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
A_v & \xrightarrow{\Phi_v} & \overline{A}_v^+ \\
\theta \downarrow & & \hat{\theta} \downarrow \\
A_u & \xrightarrow{\Phi_u} & \overline{A}_u^+
\end{array}
\]

**Corollary 3.5.** Let \( u, v \in \overline{X}^+ \) be such that \( B_2 \) satisfies \( u \leq v \), and let \( \hat{\varphi}_u \) be the homomorphism of graphs described in Lemma 3.4. Then there exists a unique edge \( v' \in A_u(i_1(u)\rho_u, t_1(u)^{-1}\rho_u) \) such that \( \overline{\varphi}_u(v') = v \).

**Proof.** By Lemma 3.4, there is (a unique) \( \tilde{v} \in A_v(i_1(v)\rho_v, t_1(v)^{-1}\rho_v) \) such that \( \overline{\varphi}_v(\tilde{v}) = v \), where \( \overline{\varphi}_v : \overline{A}_v^+ \to \overline{X}^+ \) is a homomorphism of semigroupoids described in Lemma 3.4. By Theorem 3.3, there exists a homomorphism of graphs \( \theta : A_v \to A_u \) which satisfies

\[
\theta(i_1(v)\rho_v) = i_1(u)\rho_u, \quad \theta(t_1(v)^{-1}\rho_v) = t_1(u)^{-1}\rho_u
\]

(3)

\[
\theta(z) = z, \text{ for every } z \in E(A_v).
\]

(4)
Since \( \varphi_v(z) = z = \varphi_u(\theta(z)) \) for every \( z \in E(A_u) \), the following diagram commutes:

\[
\begin{array}{ccc}
A_v^+ & \xrightarrow{\theta} & A_u^+ \\
\downarrow{\varphi_v} & & \downarrow{\varphi_u} \\
X^+ & & X^+
\end{array}
\]

Let \( v' = \hat{\theta}(\hat{v}) \). Then \( v' \in A_u(i_1(u)\rho_u, t_1(u)^{-1}\rho_u) \), by (3), and

\[\varphi_u(v') = \varphi_u(\hat{\theta}(\hat{v})) = \varphi_v(\hat{v}) = v.\]

To complete the proof it is suffices to recall that \( \hat{\varphi_u} \) is faithful. \( \square \)

Taking into account Lemma 3.4 and Corollary 3.5, we will abuse notation and denote \( \hat{\theta}(\hat{v}) \) by \( v \).

Let \( S \) be a semigroupoid. If \( |V(S)| > 1 \) then the consolidated semigroup \( S_{cd} \) is the set \( S_{cd} = E(S) \cup \{0\} \), with the multiplication defined by

\[ss' = \begin{cases} s \cdot s' & \text{if } \omega s = \alpha s', \\ 0 & \text{otherwise,}\end{cases}\]

and \( 0 \cdot a = a \cdot 0 = 0 \), for every \( a \in S_{cd} \). From this point, we will omit the \( \cdot \) to represent the operation in the consolidated semigroup. If \( |V(S)| = 1 \) then the consolidated semigroup of \( S \) is \( S \) itself, viewed as a semigroup.

If \( S \) is an ordered semigroupoid, then \( S_{cd} \) is an ordered semigroup under the order given by \( s \leq s' \) if \( s \leq s' \) in \( S \), and \( 0 \leq a \) for every \( a \in S_{cd} \) in case \( 0 \in S_{cd} \).

The following lemma is adapted from [5].

**Lemma 3.6.** Let \( u, v \in \widehat{X}^+ \) and suppose that the ordered semigroup \( B_2 \) satisfies the pseudoidentity \( u \leq v \). Then a finite semigroupoid \( S \) satisfies \((u \leq v, A_u)\) if and only if \( S_{cd} \) satisfies \( u \leq v \).

**Proof.** By Theorem 3.3 there exists a graph homomorphism \( \theta : A_v \rightarrow A_u \) that satisfies Condition (3) of Corollary 3.5. Thus, by Lemma 3.4 and Corollary 3.5, \( u \) and \( v \) represent edges of \( A_v^+ \) from \( i_1(u)\rho_u \) to \( t_1(u)^{-1}\rho_u \). Hence \( (u \leq v, A_u) \) is indeed a semigroupoid pseudoidentity.

Let \( S \) be a finite semigroupoid and let \((u_n)\) and \((v_n)\) be sequences of words of \( X^+ \) converging respectively to \( u \) and \( v \) in \( \widehat{X}^+ \). We may assume that, for all \( n \),

\[i_1(u_n) = i_1(u), t_1(u_n) = t_1(u), i_1(v_n) = i_1(v), t_1(v_n) = t_1(v), A_{u_n} = A_u, A_{v_n} = A_v, S_{cd} \text{ satisfies } u = u_n, v = v_n, \text{ and } S \text{ satisfies } (u = u_n, A_u) \text{ and } (v = v_n, A_u).\]

This reduces the proof to the case \( u, v \in \widehat{X}^+ \). Since \( B_2 \) satisfies \( u \leq v \), it is clear that \( c(v) \subseteq c(u) \). Without loss of generality, we may assume that \( X = c(u) \).

Suppose first that \( S_{cd} \models u \leq v \) and consider an arbitrary semigroupoid homomorphism \( \varphi : A_u^+ \rightarrow S \). Define a homomorphism \( \eta : X^+ \rightarrow S_{cd} \) by taking \( \eta(z) = \varphi(z) \) for each \( z \in X \).
Since $u$ and $v$ may be viewed as paths from $i_1(u)\rho_u$ to $t_1(u)^{-1}\rho_u$ in $A_u$, then $\varphi(u)$ and $\varphi(v)$ are paths in $S$, so $\eta(u), \eta(v) \neq 0$. Since $\eta(u) \leq \eta(v)$, it follows that $\varphi(u) \leq \varphi(v)$. This shows that $S \models (u \leq v, A_u)$.

Conversely, suppose that $S \models (u \leq v, A_u)$ and let $X^+ \to S_{cd}$ be an arbitrary homomorphism. If $\eta(u) = 0$ then $\eta(u) \leq \eta(v)$.

If $\eta(u) \neq 0$ then $\eta(u)$ may be viewed as a path in $S$. Let us construct a graph homomorphism $\varphi : A_u \to S$ such that $\varphi(x) = \eta(x)$ for every $x \in E(A_u)$. Let $x\rho_u$ be a vertex of $A_u$. Suppose now that $x, y \in X^{-1}$ and $(x, y) \in \rho_u$. If $x, y \in X$ then $x = y$ or there are a positive integer $n$ and $z_1, \ldots, z_{2n-1} \in X$ such that, for $k = 1, \ldots, n$, the words $z_{2k-1}z_{2k-2}, \ z_{2k-1}z_{2k}$ are factors of $u$, where $z_0 = x$ and $z_{2n} = y$. Since $\eta(u) \neq 0$, it follows that

$$\eta(z_{2k-1}z_{2k-2}), \ \eta(z_{2k-1}z_{2k}) \neq 0,$$

so $\eta(z_{2k-1}z_{2k-2})$ and $\eta(z_{2k-1}z_{2k})$ may be viewed as paths in $S$. Thus we have the chain of equalities $\alpha(\eta(z_0)) = \omega(\eta(z_1)) = \alpha(\eta(z_2)) = \omega(\eta(z_3)) = \ldots = \omega(\eta(z_{2n-1})) = \alpha(\eta(z_{2n}))$, so $\alpha(\eta(x)) = \alpha(\eta(y))$. Similarly, if $x \in X$ and $y \in X^{-1}$ then $\alpha(\eta(x)) = \omega(\eta(y^{-1}))$, and $x, y \in X^{-1}$ implies that $\omega(\eta(x^{-1})) = \omega(\eta(y^{-1}))$.

This shows that

$$\varphi(x\rho_u) = \begin{cases} \alpha(\eta(x)), & \text{if } x \in X \\ \omega(\eta(x^{-1})), & \text{if } x^{-1} \in X \end{cases}$$

and $\varphi(x) = \eta(x)$, for each $x \in E(A_u)$, defines a map $\varphi : A_u \to S$. Moreover, it is immediate that $\varphi$ is a graph homomorphism. Let $\overline{\varphi} : A_u^+ \to S$ be the unique semigroupoid homomorphism which extends $\varphi$. As $S \models (u \leq v, A_u)$, we have $\overline{\varphi}(u) \leq \overline{\varphi}(v)$. Since $\eta(w) = \overline{\varphi}(w)$, for every edge $w$ of $A_u$, we have $\eta(u) \leq \eta(v)$. This proves that $S_{cd}$ satisfies $u \leq v$.

Theorem 3.7. Let $V$ be a pseudovariety of ordered semigroups containing $B_2$. If $V = \{ [u_i \leq v_i] \}_{i \in I}$ then $gV = \{ [u_i \leq v_i, A_{u_i}] \}_{i \in I}$.

Proof. For $i \in I$, let $V_i = [u_i \leq v_i]$. In the proof of Proposition 1.2 of [27] it has been shown that $S \in gV_i$ if and only if $S_{cd} \in V_i$. Then, by Lemma 3.6, $S \in gV_i$ if and only if $S$ satisfies $(u_i \leq v_i, A_{u_i})$, that is $gV_i = \{ [u_i \leq v_i, A_{u_i}] \}$.

Since $S \in g(\bigcap_{i \in I} V_i)$ if and only if $S_{cd} \in V_i$ for every $i \in I$, and only if $S \in gV_i$ for every $i \in I$, and only if $S \in \bigcap_{i \in I} gV_i$, we have $g(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} gV_i$.

4. The category $C_m$

In this section we apply similar techniques to those which can be found, in the context of “unordered” semigroups, in [6]. They will serve to show that the pseudovariety of ordered semigroups $V_2^+ = \{ [v^\omega uv^\omega \leq u^\omega : c(v) \subseteq c(u)] \}$ [26] has infinite vertex rank.

---

More precisely, [27] deals with monoids and categories instead of semigroups and semigroupoids, but the arguments are even simpler in our case.
For $n \geq 2$, let $C_n$ be the category generated by the graph $\Gamma_n$ described by the diagram of Figure 1 subject to the following list $\mathcal{L}_n$ of relations, where $z_i = x_i \ldots x_n x_1 \ldots x_{i-1}$ ($1 \leq i \leq n$), and the index addition is performed modulo $n$:

(R0) $e_i x_i = x_i = x_i e_i + 1$, $1 \leq i \leq n$;
(R1) $e_i y_i = y_i = y_i e_i$, $1 \leq i \leq n$;
(R2) $e_i^2 = e_i$, $1 \leq i \leq n$;
(R3) $y_i^2 = y_i$, $1 \leq i \leq n$;
(R4) $z_i = z_i y_i = y_i z_i = z_i$, $1 \leq i \leq n$;
(R5) $z_i x_i x_i + 1 = x_i x_i + 1$, $1 \leq i \leq n$;
(R6) $(y_i x_i \ldots y_i x_i - 1)_{=2}^{2} = y_i x_i \ldots y_i x_i - 1$, $1 \leq i \leq n$;
(R7) $(x_i y_i + 1 \ldots x_i y_i - 1)_{=2}^{2} = x_i y_i + 1 \ldots x_i y_i - 1$, $1 \leq i \leq n$;
(R8) $y_i^2 x_i x_i + 1 y_i + 1 = x_i x_i + 1$, $1 \leq i < n$;
(R9) $x_i - 1 y_i x_i y_i - 1 = x_i - 1 x_i$, $1 \leq i < n$;
(R10) $y_i^2 x_i x_i y_i - 1 = x_i z_i$;
(R11) $y_i^2 x_i x_i y_i - 1 \ldots x_i x_i y_i - 1 = x_i z_i$.

Many of these relations are naturally viewed as simplifying rules for paths over the graph $\Gamma_n$ and we will therefore refer to them as rules.

To simplify notation, from hereon, when writing an expression of the form $\varepsilon_1 \delta_1 \ldots \varepsilon_j \delta_j$ or $\varepsilon_i \ldots \varepsilon_j$, we mean that the omitted factors, represented by the dots, are taken for the indices describing the shortest path from $i$ to $j$ in the cycle $1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow 1$.

It is immediately verified that the edges of $C_n$ have the form

$$w = \varepsilon_{i}^{(0)} x_i \ldots \varepsilon_{n}^{(0)} x_n \left( \prod_{k=1}^{s} \varepsilon_{i}^{(k)} x_i \varepsilon_{2}^{(k)} \ldots \varepsilon_{n}^{(k)} \right) \varepsilon_{1}^{(s+1)} x_{1} \varepsilon_{2}^{(s+1)} \ldots \varepsilon_{j-1}^{(s+1)} \varepsilon_{j}^{(s+1)}$$

for some $i, j \in \{1, \ldots, n\}$, $s \geq 0$ and $\varepsilon_{i}^{(k)} \in \{\varepsilon_i, y_i, y_i^2\}$, for $k = 0, \ldots, s + 1$ and $t = 1, \ldots, n$. The following is easy to establish as a consequence of the definition of $z_i$ (note that $z_i x_i = x_i z_i$) and rule (R4) of $\mathcal{L}_n$.

Lemma 4.1. Let $i, j \in \{1, \ldots, n\}$. For $w \in C_n(i, j)$, $z_i w = z_i x_i \ldots x_{j-1} = w z_j$.

We endow $C_n$ with the following binary relation: if $u, v$ are coterminal edges of $C_n$ then we set

$$u \leq v \quad \text{if} \quad u = v \quad \text{or} \quad u = z_{\alpha(u)} x_{\alpha(u)} \cdots x_{\omega(u) - 1}.$$
Note that the relation ≤ is a partial order. By Lemma 4.1, the partial order ≤ is compatible with multiplication (on the left and on the right), and so \((C_n, \leq)\) is an ordered category.

For \(1 \leq i \leq n\), let \(A_i = \{0_i, 1_i\}\), and put \(A = \bigcup_{i=1}^{n} A_i\). Let \(P_n\) be the semigroup \((A \cup \{0\})^+ / \Upsilon\) where \(\Upsilon\) is the congruence generated by the set consisting of the following list \(\mathcal{L}'_n\) of relations, where \(2_i\) denotes \(1_i1_i\) and \(\zeta_i = 0_i \ldots 0_i0_1 \ldots 0_i\), for \(1 \leq i \leq n\):

\[
\begin{align*}
(R_A) & \quad a_ib_j = 0, \quad 1 \leq i \leq n, j \neq i, j \neq i + 1; \\
(R_B) & \quad 0a_i = 0 = a_i0, \quad 1 \leq i \leq n; \\
(R_{\mathcal{A}}) & \quad 01_i = 1_i = 1_i0_i, \quad 1 \leq i \leq n; \\
(R_{\mathcal{B}}) & \quad 0_i0_i = 0_i, \quad 1 \leq i \leq n; \\
(R_{\mathcal{C}}) & \quad 1_i1_i = 1_i1_i, \quad 1 \leq i \leq n; \\
(R_{\mathcal{D}}) & \quad \zeta_i^2 = \zeta_i1_i = 1_i\zeta_i = \zeta_i, \quad 1 \leq i \leq n; \\
(R_{\mathcal{E}}) & \quad \zeta_i0_{i+1}0_{i+2} = 0_i0_{i+1}0_{i+2}, \quad 1 \leq i \leq n; \\
(R_{\mathcal{F}}) & \quad (1_i \ldots 1_i2_i)0_i = 1_i \ldots 1_i0_i, \quad 1 \leq i \leq n; \\
(R_{\mathcal{G}}) & \quad (0_i1_i \ldots 1_i)^2 = 0_i1_i \ldots 1_i, \quad 1 \leq i \leq n; \\
(R_{\mathcal{H}}) & \quad 2_i1_i0_{i+2} = 0_i0_{i+1}0_{i+2}, \quad 1 \leq i \leq n; \\
(R_{\mathcal{I}}) & \quad 0_i0_i1_i2_i = 0_i0_i0_{i+1}1_i, \quad 1 \leq i < n; \\
(R_{\mathcal{J}}) & \quad 0_i2_i0_i2_i = 0_i0_i0_{i+1}0_i; \\
(R_{\mathcal{K}}) & \quad 2_i0_i1_i \ldots 1_i2_i = 0_i0_i\zeta_i.
\end{align*}
\]

Note that, if \(u, v \in A^+ \setminus \Upsilon\) and \((u, v) \in \Upsilon\) then there are \(i, j \in \{1, \ldots, n\}\) such that \(i_1(u), i_1(v) \in A_i\) and \(t_1(u), t_1(v) \in A_j\).

Let us define the norm of \(w \in A^+\), denoted \(\|w\|\) in the following way: if \(w = a_i \in A_i\) then \(\|w\| = 1\), if \(w = w_1w_2\) with \(w_1, w_2 \in A^+\) then

\[
\|w\| = \left\{ \begin{array}{ll}
\|w_1\| + \|w_2\| & \text{if } t_1(w_1), i_1(w_2) \in A_i \text{ for some } i \\
\|w_1\| + \|w_2\| & \text{if } t_1(w_1) \in A_i, i_1(w_2) \in A_j \text{ with } i \neq j.
\end{array} \right.
\]

Note that \(\|w\|\) counts the minimal number of factors in a factorization of \(w\) into elements of \(A^+_n\).

We also define the norm of \(u\Upsilon \in P_n \setminus \{0\Upsilon\}\), denoted by \(\|u\Upsilon\|\), as being the minimum of the set \(\{\|w\| : w \in u\Upsilon\}\).

An edge \(w\) of the category \(C_n\) can be completely described in terms of \(y_1, \ldots, y_n\) and \(e_1, \ldots, e_n\) as it is made precise in the following.

**Lemma 4.2.** Let \(\chi : \Gamma^+_n \rightarrow A^+\) be the homomorphism of semigroupoids defined by

\[
\chi(e_i) = 0_i, \quad \chi(y_i) = 1_i, \quad \chi(x_i) = 0_i0_{i+1},
\]

and let \(\eta_1 : \Gamma^+_n \rightarrow C_n\) be the canonical projection. Then the following hold:

(a) The homomorphism \(\chi\) restricted to \(E(\Gamma^+_n)\) is injective.

(b) For all paths \(u, v \in \Gamma_n\), if \(\eta_1(u) = \eta_1(v)\) then \((\chi(u), \chi(v)) \in \Upsilon\).

(c) \(\chi(\Gamma^+_n) \subseteq A^+ \setminus \Upsilon\).
Proof. (a) Let \( u \) and \( v \) be paths in \( \Gamma_n \) such that \( \chi(u) = \chi(v) \). Let us show that \( u = v \).

If \( \chi(u) \in A_i^+ \) for some \( i \in \{1, \ldots, n\} \), then \( u = \varepsilon_1 \ldots \varepsilon_s \) and \( v = \delta_1 \ldots \delta_t \) with \( s, t \geq 1 \) and \( \varepsilon_j, \delta_k \in \{e_i, y_i\} \), for \( 1 \leq j \leq s \) and \( 1 \leq k \leq t \). Since \( |\chi(e_i)| = |\chi(y_i)| = 1 \), it must be \( s = t \) and \( \varepsilon(j) = \delta(j) \) for \( 1 \leq j \leq s \). But this implies that \( \varepsilon(j) = \delta(j) \) for \( 1 \leq j \leq s \) and, consequently, \( u = v \).

Let \( k \geq 1 \). Suppose that \( \|\chi(u)\| = k + 1 \). It follows that the paths \( u \) and \( v \) contain \( k \) edges from the set \( \{x_i : i = 1, \ldots, n\} \), that is

\[
u = u^{(i)} v^{(j+1)} \ldots x_{i+k-1} u^{(i+k)}, \text{ and } v = u^{(i)} x_i v^{(j+1)} \ldots x_{i+k-1} u^{(i+k)},
\]

for some \( 1 \leq i \leq n \), and \( w(j), w(k) \in \{e_j, y_j\} \) for \( j = i, \ldots, i + k \). Since \( \chi(u) = \chi(v) \), then \( \chi(u) = \chi(v) \) and \( \chi(w(j)) = \chi(w(k)) \), and so \( w(j) = w(k) \), for every \( j \), as it is shown above. So we conclude that \( u = v \).

(b) Let \( u \) and \( v \) be paths in \( \Gamma_n \) such that \( \eta_1(u) = \eta_1(v) \). Suppose that \( u = v \) is the rule \( (R_m) \) of \( L_n \) for some \( m \in \{0, \ldots, 11\} \). If \( 1 \leq m \leq 3 \) then \( \chi(u) = \chi(v) \) is the rule \( (R_m) \) of \( L_n \). If \( m = 0 \) then \( \chi(u), \chi(v) \in \Upsilon \) by rule \( (R_0) \). If \( 4 \leq m \leq 11 \) then \( \chi(u), \chi(v) \in \Upsilon \) by rules \( (R_1), (R_2) \) and \( (R_3) \).

Suppose now, without loss of generality, that \( v \) may be obtained from \( u \) by applying a single rule \( p = q \) of \( L_n \), that is \( u = u^{(1)} pu^{(2)} \) and \( v = u^{(1)} qu^{(2)} \), where \( u^{(i)} (i = 1, 2) \) are appropriate paths in \( \Gamma_n \). Since \( \chi \) is a homomorphism, \( \Upsilon \) is a congruence, and \( (\chi(p), \chi(q)) \in \Upsilon \), it follows that \( (\chi(u), \chi(v)) \in \Upsilon \).

(c) Let \( u \) be a path in \( \Gamma_n \). Since \( \chi(u) \in (A^+ \ldots A_{i-1}^+) + A_i^+ \ldots A_n^+ \) for some \( 1 \leq i \leq n \), we cannot apply rules \( (R_A) \) or \( (R_B) \) to \( \chi(u) \), so \( \chi(u) \notin \Upsilon \).

We note that in the proof of the Lemma 4.2(a), we show that if \( u \) is a path in \( \Gamma_n \) and \( \chi(u) \) may be factorized as \( \chi(u) = u^{(1)} u_{i+1}^{(2)} \) for some \( i \), then there are appropriate paths \( u^{(1)} \) and \( u^{(2)} \) such that \( u = u^{(1)} u^{(2)} \).

Let \( \eta_2 : A^+ \rightarrow P_n \) be the homomorphism defined by \( \eta_2(u) = u\Upsilon \). Note that \( \eta_2 \) is surjective. By Lemma 4.2(b), there is a homomorphism of semigroupoids \( \psi : C_n \rightarrow P_n \) such that the following diagram commutes.

\[ \Gamma_n \xrightarrow{\chi} A^+ \]

\[ C_n \xrightarrow{\psi} P_n \]

\[ \eta_1 \quad \eta_2 \]

\[ \psi \]

**Lemma 4.3.** Let \( \psi : C_n \rightarrow P_n \) be the homomorphism of semigroupoids defined above. The following hold:

(a) \( \psi|_{E(C_n)} \) is injective;
(b) \( \psi(E(C_n)) = P_n \setminus \{0\} \), where \( 0 \) also denotes \( 0\Upsilon \).

**Proof.** (a) Let us show that \( \psi|_{E(C_n)} \) is injective. First, we consider a new list \( L''_n \) of rules which contains rules \( (R_A) \) and \( (R_B) \) of \( L'_n \) and all rules \( \chi(p) = \chi(q) \) where
each } p = q \text{ is a rule of } \mathcal{L}_n. \text{ It is easy to see that the rules of the list } \mathcal{L}_n'' \text{ also generate the congruence } \Upsilon. 

Let } u \text{ and } v \text{ be paths in } \Gamma_n \text{ such that } \psi(\eta_1(u)) = \psi(\eta_1(v)), \text{ that is } (\chi(u), \chi(v)) \in \Upsilon. \text{ Note that } \chi(u), \chi(v) \notin 0 \Upsilon, \text{ by Lemma 4.2(c). Let us show that } \eta_1(u) = \eta_1(v). \text{ If } \chi(u) = \chi(v) \text{ is a rule of } \mathcal{L}_n'' \text{ then there is a rule } p = q \text{ of } \mathcal{L}_n \text{ such that } \chi(u) = \chi(p) \text{ and } \chi(v) = \chi(q). \text{ By Lemma 4.2(a), we have } u = p \text{ and } v = q. \text{ Since } \eta_1(p) = \eta_1(q), \text{ it follows that } \eta_1(u) = \eta_1(v). 

Let us assume, without loss of generality, that } \chi(v) \text{ may be obtained from } \chi(u) \text{ by applying a single rule } \chi(p) = \chi(q) \text{ of } \mathcal{L}_n'' \text{, say with } i_1(\chi(p)) \in A_i \text{ and } t_1(\chi(p)) \in A_j \text{ for some } i \text{ and } j. \text{ This means that we may factorize } \chi(u) = u(t^{(i)})a^p \chi(p)b\eta \text{ and } \chi(v) = u(t^{(i)})a^p \chi(q)b\eta \text{ where } t_1(u(t^{(i)})) = 0_{i-1} \text{ or } u(t^{(i)}) \text{ is the empty word, } a^p \in A_i^+, b^p \in A_j^+. \text{ Arguing as in proof of Lemma 4.2(a) we obtain paths } a \text{ and } b \text{ in } \Gamma_n \text{ such that } \chi(a) = a^p \text{ and } \chi(b) = b^p. \text{ As it has been observed above, there are appropriate paths } u^{(1)} \text{ and } u^{(2)} \text{ such that } u = u^{(1)}x_i^00i_1a^p00_i1u^{(2)} \text{ and } v = v^{(1)}x_i^00i_1a^p00_i1u^{(2)}. \text{ Since } \eta_1(p) = \eta_1(q), \text{ it follows that } \eta_1(u) = \eta_1(v). 

If } a^p \text{ is the empty word then } i_1(\chi(p)) = i_1(\chi(q)) = 0. \text{ Since } \chi(u) \text{ and } \chi(v) \text{ do not have factors of the form } 0_{i-1}0a_i1 \text{ with } a_i1 \in A_i+1, \text{ then } p \text{ and } q \text{ must begin with the same edge } e_i, \text{ that is } p = q \text{ is the rule } (R_2) \text{ of } \mathcal{L}_n. \text{ Without loss of generality, we may assume that } 

\chi(u) = u^{(1)}0_{i-1}0_{i}0_{0}0_{i+1}u^{(2)} \text{ and } \chi(v) = u^{(1)}0_{i-1}0_{i}0_{0}0_{i+1}u^{(2)} 

\text{with } b^p = b^p0_i \in A_i^+0_i. \text{ So there are appropriate paths } b, u^{(1)} \text{ and } u^{(2)} \text{ such that } u = u^{(1)}x_i^00i_1a^p00_i1u^{(2)} \text{ and } v = u^{(1)}x_i^00i_1a^p00_i1u^{(2)}. \text{ By rule } (R_0) \text{ of } \mathcal{L}_n, \text{ we have } \eta_1(u) = \eta_1(v). \text{ If } b^p \text{ is the empty word, the same argument serves to prove that } \eta(u) = \eta(v). 

(b) Let us show that } \psi(E(C_n)) = P_n \setminus \{0\}. \text{ Let } w \text{ be a representative of a } \Upsilon \text{-class of } P_n \setminus \{0\}. \text{ Suppose that } \|w\| = k + 1. \text{ By rules } (R_1') \text{ and } (R_2'), \text{ } w \text{ is } \Upsilon \text{-related with the word } v \text{ which is obtained from } w \text{ by introducing, between consecutive letters } b_i \in A_i \text{ and } b_{i+1} \in A_i+1 \text{ such that } b_i0_{i+1} \text{ is a factor of } w, \text{ the factor } 0_{i+1}. \text{ It is obvious that } \|v\| = k + 1 \text{ and so we may write } v = u(v)^{0(i)}0_{i+1}(v)^{(i+1)} \ldots 0_{i+k-1}(v)^{(i+k)} \text{ with } v^{(i)} \in A_i^+. \text{ Proceeding as above, we are able to find paths } w^{(r)} \text{ such that } \chi(w^{(r)}) = v^{(r)}. \text{ It follows that } v = \chi(w^{(i)})x_i^0u^{(r)}(v)^{(i+1)} \ldots x_i^0u^{(r)}(v)^{(r+k)} \text{ and let } \psi(y_i) \psi(y_{i+1}) \neq 0 \text{ in } (C_n)_{cd} \text{ with } \psi(y_i) \psi(y_{i+1}) = (1,1_{i+1})/\Upsilon \neq 0 \text{ in } P_n. \text{ Note that the homomorphism } \psi \text{ defined in Lemma 4.3 does not induce a homomorphism } (C_n)_{cd} \rightarrow P_n \text{ since } y_iy_{i+1} = 0 \text{ in } (C_n)_{cd} \text{ while } \psi(y_i)^2 = (1,1_{i+1})/\Upsilon \neq 0 \text{ in } P_n.
Throughout this paper we are almost always identifying each Υ-class with a representative of that Υ-class.

Let \( u', v' \in P_n \). We let \( u' \leq v' \) if either \( u' = 0 \) or there exist \( u, v \in C_n \) such that \( u' = \psi(u) \), \( v' = \psi(v) \) and \( u \leq v \). Taking into account the definition of \( \leq \) in \( C_n \), we obtain \( u' \leq v' \) if and only if \( u' = 0 \) or there are \( i, j \in \{1, \ldots, n\} \) such that \( i_1(u'), i_1(v') \in A_i, t_1(u'), t_1(v') \in A_j \) and

\[
\begin{align*}
  u' &= \zeta_i 0_{i+1} \ldots 0_j \quad \text{or} \quad u' = v'.
\end{align*}
\]

By Lemma 4.3, the relation \( \leq \) is a partial order compatible with multiplication. Therefore \((P_n, \leq)\) is an ordered semigroup and \( \psi : C_n \to P_n \) is an order-faithful homomorphism of semigroupoids.

For \( i, j \in \{1, \ldots, n\} \), let \( P_n(i, j) \) be the set

\[
\{ w \in P_n \setminus \{0\} : 0_i w 0_j = w \}.
\]

If \( i = j \), then we write \( P_n(i) \) instead of \( P_n(i, i) \). Note that \( P_n(i) \) is a submonoid of \( P_n \) with identity \( 0_i \).

Some basic properties of the relation \( \Upsilon \) are stated in the next lemma.

**Lemma 4.4.** Let \( 1 \leq i, j \leq n \), \( w \in P_n(i) \), \( t \in P_n(i, j) \). Then

(a) \( \zeta_i t = \zeta_i 0_{i+1} \ldots 0_j = 0_i \ldots 0_{j-1} \zeta_j = t \zeta_j \);

(b) \( \zeta_i w = \zeta_i = w \zeta_i \);

(c) \( w 0_{i+1} 0_{i+2} = 0, 0_{i+1} 0_{i+2} \) and \( 0_{i-2} 0_{i-1} w = 0_{i-2} 0_{i-1} 0_i \);

(d) \( 1, 0_{i+1} 1_{i+2} = 0, 0_{i+1} 0_{i+2} \).

**Proof.** All the properties can be easily proven from the definition of \( P_n \). To illustrate this, we prove (c) and (d).

(c) We have

\[
\begin{align*}
  w 0_{i+1} 0_{i+2} &= w 0_i 0_{i+1} 0_{i+2} = w \zeta_i 0_{i+1} 0_{i+2} = \zeta_i 0_{i+1} 0_{i+2} = 0, 0_{i+1} 0_{i+2}.
\end{align*}
\]

by rule \((R_{5'})\) and (b). The other equality is similar.

(d) By rule \((R_{1'})\) and (c), we have

\[
\begin{align*}
  1, 0_{i+1} 1_{i+2} &= 1, 0_i 0_{i+1} 0_{i+2} = 0, 0_{i+1} 0_{i+2} 1_{i+2} = 0, 0_{i+1} 0_{i+2}.
\end{align*}
\]

For \( i = 1, \ldots, n \), let \( B_i = 0^*_i 1, 0^*_i \) and \( C_i = A^*_i 1, A^*_i 1, A^*_i \). The next result describes two rational languages recognized by the natural homomorphism from \( A^* \) to \( P_n \).
Corollary 4.6. Let \( \eta_2 : A^+ \to P_n \) be the homomorphism defined in Lemma 4.3. Let \( L_1 \) and \( L_2 \) be the rational languages

\[
\begin{align*}
L_1 &= (B_1 \cup 0_1^+)B_2 \ldots B_n(B_1 \ldots B_n)^{*}A_1^+ \\
L_2 &= (B_1 \cup 0_1^+)B_2 \ldots B_n(B_1 \ldots B_n)^{*}C_1 \ldots C_n(B_1 \ldots B_n)^{*}(B_1 \cup 0_1^+).
\end{align*}
\]

Then \( \eta_2^{-1} \eta_2(L_i) = L_i \), for \( i = 1, 2 \).

Proof. Let \( L \in \{ L_1, L_2 \} \) and \( w \in L \). Then we cannot apply rules \((R_4'), (R_B), (R_{a}), (R_{b}), (R_{c}), (R_{d})\) and \((R_{i1})\) to \( w \). On the other hand, if we apply rules \((R_{iv}), (R_{iv'}), (R_{iv''})\) and \((R_{i2})\), then we obtain another word of \( L \).

Now, if \( u \in \eta_2^{-1} \eta_2(L) \) then there exists \( w \in L \) such that \( (w, u) \in \Upsilon \), hence \( u \in L \). The inclusion \( L \subseteq \eta_2^{-1} \eta_2(L) \) is clear. \( \square \)

By abuse of language, we say that \( u \in A^+ \) is a factor of \( w \in P_n \setminus \{0\} \) when \( u \Upsilon \) is a factor of \( w \).

For \( a, b, c, d \in \{0, 1, 2\} \), we define the following elements of \( P_n(1) \):

\[
\begin{align*}
w_1(a, b) &= a_11_2 \ldots 1_nb_1, \quad \text{and} \quad w_2(c, d) = c_12_2 \ldots 2_nd_1.
\end{align*}
\]

Some other elements of \( P_n(1) \) may be obtained as products of \( w_1(a', b') \) and \( w_2(c', d') \) for some \( a', b', c', d' \in \{0, 1, 2\} \):

\[
\begin{align*}
w' &= 0_11_2 \ldots 1_n1_1 \ldots 1_n0_1 = w_1(0, 1)w_1(0, 0); \\
w_3(a, d) &= a_11_2 \ldots 1_n2_1 \ldots 2_nd_1 = w_1(a, 1)w_2(1, d); \\
w_4(c, d) &= c_12_2 \ldots 2_n1_1 \ldots 1_nd_1 = w_2(c, 1)w_1(0, d), \quad \text{and} \\
w_5(a, d) &= a_11_2 \ldots 1_n2_1 \ldots 2_n1_1 \ldots 1_nd_1 = w_1(a, 0)w_2(2, 1)w_1(0, d).
\end{align*}
\]

Corollary 4.6. Let \( a, d \in \{0, 1\} \) and \( b, c \in \{0, 1, 2\} \). In \( P_n(1) \), we have

\((a)\) \( w_5(a, d) \neq \zeta_1 \) and \( w_2(c, d) \neq \zeta_1 \); 
\((b)\) \( w_1(a, b) \neq \zeta_1 \) and \( w_1(a, b) \neq w_2(c, d) \); 
\((c)\) If \( i \) is a factor of \( w_1(a, b) \) then \( i = 1 \) and \( b = 2 \).

Proof. (a) The word \( \zeta_1 \in A^+ \) is not in \( L_2 \) and the word \( w_5(a, d) \) belongs to \( L_2 \) so \( w_5(a, d) \neq \zeta_1 \) in \( P_n(1) \), by Lemma 4.5. On the other hand, we have \( w_5(a, d) = w_1(a, b)w_2(c, d)w_1(a', d) \) for some \( b, a' \in \{0, 1, 2\} \) such that \( b + c = 2 \) and \( d + a' = 1 \). Hence, if \( w_2(c, d) = \zeta_1 \) then \( w_5(a, d) = \zeta_1 \), by Lemma 4.4(b), and the latter we have already shown to fail.

(b) Since the word \( w_1(a, b) \) belongs to \( L_1 \) and the words \( w_2(c, d) \) and \( \zeta_1 \) are not in \( L_1 \), we have \( w_1(a, b) \neq w_2(c, d) \) and \( w_1(a, b) \neq \zeta_1 \), by Lemma 4.5.

(c) If \( i \) is a factor of \( w_1(a, b) \) then there are \( u^{(1)}, u^{(2)} \in A^* \) such that \( w_1(a, b) = u^{(1)}2u^{(2)} \) in \( P_n(t) \). Since the word \( w_1(a, b) \) belongs to \( L_1 \), the same happens with the word \( u^{(1)}2u^{(2)} \) by Lemma 4.5. So it must be \( i = 1 \), \( 2u^{(2)} \in 2_1A_1^* \), and \( b = 2 \). \( \square \)

For \( i = 1, \ldots, n \), let \( F_i = \{0_i, 1_i, 2_i\} \).
Lemma 4.7. Let m be a positive integer. The number of $\Upsilon$-classes which contain words $w$ such that $\|w\| \leq m$ is finite.

Proof. Suppose that $w\Upsilon \neq \emptyset$. Then by rules $(R_1')$, $(R_2')$ and $(R_3')$ there are $i, j \in \{1, \ldots, n\}$ and a word $u \in (F_1 \ldots F_{i-1})^*F_i \ldots F_j$ such that $\|w\Upsilon\| = \|u\|$ and $(w, u) \in \Upsilon$. As the set \{u \in (F_1 \ldots F_{i-1})^*F_i \ldots F_j : \|u\| \leq m\} is finite, the lemma follows.

In the following, we characterize all elements of $P_n(1)$.

Notice that, for every $b \in \{0, 1, 2\}$, $k = 1, 3, 5$, and $l = 2, 3, 4, 5$,

$$w_k(2, b) = \zeta_1 = w_l(b, 2).$$

In the set \{0, 1, 2\} we define an operation $\oplus$ in the following way: for $t \in \{0, 1, 2\}$, $0 \oplus t = t \oplus 0 = 1 \oplus 1 = 2$ and $2 \oplus t = t \oplus 2 = 2$. Then \{0, 1, 2\} is a commutative monoid with zero 2 and identity 0.

Lemma 4.8. The monoid $P_n(1)$ consists of the elements

\[0, 1, 1, 2, 1, \zeta_1, w_1(a, b), w_2(c, d), w'_1, w_3(a, d), w_4(c, d), w_5(a, d)\]

with $a, d \in \{0, 1\}$ and $b, c \in \{0, 1, 2\}$, where 0 is the identity and $\zeta_1$ is the zero.

Proof. Let $M$ be the set of the elements (5) of $P_n(1)$. From the definitions of $P_n$ and $P_n(1)$, and Lemma 4.4(b), we obtain the following equalities: for every $a, b \in \{0, 1, 2\}$ and $c \in \{1, 2\}$,

$$w_0 = w_{01} = w, \quad \zeta_1 w = w_{\zeta_1} = \zeta_1, \quad \text{for every } w \in M,$$
$$a_1b_1 = (a \oplus b)_1, \quad c_1w'_1 = w_1(c, 0), \quad w'_1c_1 = w_1(0, c),$$
$$c_1w_k(a, b) = w_k(a \oplus c, b) \quad \text{and} \quad w_k(a, b)c_1 = w_k(a, b \oplus c), \quad \text{for } 1 \leq k \leq 5.$$

Let $a, d, a', d' \in \{0, 1\}$ and $b, c, b', c' \in \{0, 1, 2\}$. Then we may establish the partial multiplication table indicated in Table 1.

Since $w'_1$, $w_3(a', d')$, $w_4(c', d')$ and $w_5(a', d')$ can be written as products of $w_1(a, b)$ and $w_2(c, d)$, the above shows that $M$ is closed under multiplication and thus $M$ is a monoid.

Let $u\Upsilon \in P_n(1)$. By rules $(R_1')$, $(R_2')$, and $(R_3')$, there is a word $v \in (F_1 \ldots F_n)^*F_1$ such that $v = u$ in $P_n(1)$, so we may assume that $u \in (F_1 \ldots F_n)^*F_1$. Note that $\|u\| = kn + 1$ for some $k \geq 0$. If $k = 0$ then $u = a_1$ with $a \in \{0, 1, 2\}$, thus $u \in M$. Suppose that $\|u\| = n + 1$, and $u \neq \zeta_1$ in $P_n(1)$. Then $u = a_1(c(2))_2 \cdots (c(n))_n b_1$ with $a, b, c(i) \in \{0, 1, 2\}$ for $2 \leq i \leq n$. If there is $j$ such that $c(j) = 0$ then $0_{j-1}0_10_{j+1}$ is a factor of $u$, so $u = \zeta_1$ in $P_n(1)$, by Lemma 4.4. It follows that all $c_i$ are non-zero. If $c(j) = 1$ then

$$u = w_1(a, b) = a_1(1)_2 \ldots (1)_n b_1.$$
Table 1. Partial multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>$w_1(a, b)$</th>
<th>$w_2(c, d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1(a', b')$</td>
<td>$w_1(a', b)$, if $b' \oplus a = 1, a' \oplus b \neq 0$ [\zeta_1\text{, otherwise}]</td>
<td>$w_2(a', d)$, if $b' \oplus c = 2$ [\zeta_1\text{, otherwise}]</td>
</tr>
<tr>
<td>$w_2(c', d')$</td>
<td>$w_4(c', b)$, if $d' \oplus a = 1, b' \neq 2$ [\zeta_1\text{, otherwise}]</td>
<td>[\zeta_1]</td>
</tr>
<tr>
<td>$w'$</td>
<td>$w_1(0, b)$, if $a = 1, b \neq 0$ [\zeta_1\text{, otherwise}]</td>
<td>$w_3(0, d)$, if $c = 2$ [\zeta_1\text{, otherwise}]</td>
</tr>
<tr>
<td>$w_3(a', d')$</td>
<td>$w_5(a', b)$, if $d' \oplus a = 1, b' \neq 2$ [\zeta_1\text{, otherwise}]</td>
<td>[\zeta_1]</td>
</tr>
<tr>
<td>$w_4(c', d')$</td>
<td>$w_4(c', b)$, if $d' \oplus a = 1, b' \neq 2$ [\zeta_1\text{, otherwise}]</td>
<td>[\zeta_1]</td>
</tr>
<tr>
<td>$w_5(a', d')$</td>
<td>$w_5(a', b)$, if $d' \oplus a = 1, b' \neq 2$ [\zeta_1\text{, otherwise}]</td>
<td>[\zeta_1]</td>
</tr>
</tbody>
</table>

with $a \in \{0, 1\}$, by rules $(R_0^+)$, $(R_7^-)$, and Lemma 4.4. If $c^{(i)} = 2$ then $u = w_2(a, b) = a_12_2\ldots2_n b_1$.

By rule $(R_b^+)$, it must be $b \neq 2$.

If $k > 1$, then $u$ may be written as a product of factors of $P_n(1)$ with length $n + 1$, namely factors of the form $\zeta_1$, $w_1(a, b)$, and $w_2(c, d)$. By the first part of the proof, $P_n(1) = M$.

The following lemma is a consequence of Lemmas 4.7 and 4.8.

Lemma 4.9. For all $i, j \in \{1, \ldots, n\}$, the set $P_n(i, j)$ is finite.

Proof. If $w \in P_n(i, j)$ then either $||w|| < 2n$ or $w = uw'v$ with $||u||, ||v|| < n$ and $w' \in P_n(1)$. Since $P_n(1)$ is finite by Lemma 4.8, it follows that $P_n(i, j)$ is finite, by Lemma 4.7.

Lemma 4.10. Let $u \in P_n(1) \setminus \{2_1\}$. If there is $1 \leq i \leq n$ such that $2_i$ is a factor of $u$ then $u^2 = \zeta_1$.

Proof. First we claim that $w$ has a factor of the form 2, if and only if $w \in K$ where

$$K = \{2_1, \zeta_1, w_1(a, 2), w_2(c, d), w_3(a, d), w_4(c, d), w_5(a, d) : a, d \in \{0, 1\}; c \in \{0, 1, 2\}\}.$$ 

In fact, it is clear that all elements of $K$ have a factor $2_i$ for some $i \in \{1, \ldots, n\}$. On the other hand, if $w \notin K$ then $w \in \{0_1, 1_1, w', w_1(a, b) : a, b \in \{0, 1\}\}$. It is obvious that $0_1$ and $1_1$ do not have the factor $2_i$ and, by Corollary 4.6(c), there is no $i$ such that $2_i$ is a factor of $w_1(a, b)$ (with $a, b \in \{0, 1\}$). Finally if $w'$ has a
factor 2, for some i then the same happens with $1_i w' = w_1(1, 0)$, which contradicts the above.

Now, using Table 1 given in the proof of Lemma 4.8, it is easy to verify that $w^2 = \zeta_i$ for every $w \in K \setminus \{2_i\}$. □

**Lemma 4.11.** Suppose $u\Upsilon \in P_n(k) \setminus \{\zeta_k\}$. Then $2_j$ is a factor of $u\Upsilon$ if and only if $2_j$ is a factor of the word $u$.

**Proof.** Since $u\Upsilon \in P_n(k) \setminus \{\zeta_k\}$, none of the rules $(R_A)$, $(R_B)$, $(R_4')$, $(R_5')$, $(R_6)$, $(R_{10'})$, $(R_{11'})$ may be applied to an element of the congruence class $u\Upsilon$. For the remaining rules, it suffices to observe that, if one side has $2_j$ as a factor, then so does the other. □

The next result plays a useful role in the sequel.

**Lemma 4.12.** Let $1 \leq j \leq n$ and $w \in P_n(j) \setminus \{2_j\}$. If there is $1 \leq i \leq n$ such that $2_i$ is a factor of $w$ then $w^3 = \zeta_j$.

**Proof.** For $j = 1$, this is a consequence of Lemmas 4.10 and 4.4. Suppose that $j \neq 1$. Let $u\Upsilon \in P_n(j)$ such that the word $2_j$ does not belong to $u\Upsilon$. Since $2_i$ is a factor of $u\Upsilon$, there are words $u^{(1)}, u^{(2)} \in u\Upsilon$ such that $u^{(1)}2_ju^{(2)} \in u\Upsilon$. We may write the word $u^{(1)}2_ju^{(2)}$ as $v^{(1)}v^{(2)}v^{(3)}$ with $v^{(1)} \in A_1^+ \ldots A_n^+$, $v^{(2)} \in (A_1^+ \ldots A_n^+)^*A_1^+$, and $v^{(3)} \in A_2^+ \ldots A_n^+$. It is obvious that $2_i$ is a factor of $v^{(1)}$, $v^{(2)}$, or $v^{(3)}$ so, by Lemmas 4.4 and 4.10, and rules $(R_1')$, $(R_2')$, and $(R_4')$ we have

$$u^3\Upsilon = (v^{(1)}0_1v^{(2)}v^{(3)})^3\Upsilon$$

$$= (v^{(1)}(v^{(2)}v^{(3)}v^{(1)})0_1v^{(2)}v^{(3)}v^{(1)})\Upsilon$$

$$= (v^{(1)}\zeta_i v^{(2)}v^{(3)})\Upsilon = \zeta_j\Upsilon,$$

which completes the proof. □

As a consequence of Lemma 4.12, the only idempotents of $P_n(i)$ which contain 2, as a factor are 2, and $\zeta_i$.

**Lemma 4.13.** The set of idempotents of $P_n(i)$ is $\{u^3 : u \in P_n(i)\}$. Moreover, for every $u \in P_n(i) \setminus \{0_i, 1_i, 2_i\}$, we have $u^3 = \zeta_i$ or $u = a_11_{i+1} \ldots 1_{i-1}b_i$ with $a + b = 1$.

**Proof.** Let $u \in P_n(i)$. If there is $j$ such that $2_j$ is a factor of $u$ then either $u^3 = u = 2_i$ or $u^3 = \zeta_i$, by Lemma 4.12.

If there is $j \in \{1, \ldots, n\}$ such that $0_j0_{j+1}0_{j+2}$ is a factor of $u^2$ then $u^2 = \zeta_i$, by Lemma 4.4, and so $u^3 = \zeta_i$. If $u^2$ does not have factors of the form $2_j$ or $0_j0_{j+1}0_{j+2}$ then, either $u \in \{0_i, 1_i\}$, or there is a word $v \in a_i1_{i+1} \ldots 1_{i-1}(1_i \ldots 1_{i-1})^*b_i$, for some $a, b \in \{0, 1\}$ with $a + b = 1$, such that $u = v$ in $P_n(i)$. In the first case, $u^3 \in \{0_i, 2_i\}$, and in the latter case, $u^2 = u = a_i1_{i+1} \ldots 1_{n}1_1 \ldots 1_{i-1}b_i$ in $P_n(i)$, by rules $(R_1')$, $(R_6')$ and $(R_7')$.

Finally, suppose that $\|u\| > 1$, then $u$ does not have factors of form $2_k$, $u^2$ does not have factors of the form $0_k0_{k+1}0_{k+2}$, but there is $j$ such that $2_j$ is a factor of $u^2$. □
Under these conditions, there is a word \( v \in a_11_{i+1} \ldots 1_{i-1}(1_{i-1} \ldots 1_1)^*b_i \), for some \( a, b \in \{0, 1, 2\} \) with \( a + b \in \{1, 2\} \), such that \( u = v \) in \( P_n(i) \). Let us show that \( a + b \) cannot be 1. Indeed, if \( a + b = 1 \) then \( v = v^2 = a_i1_{i+1} \ldots 1_{n} b_i \) in \( P_n(i) \), and so there are words \( 1_1 \ldots 1_{i-1}a_i' \) and \( b_i'1_{i+1} \ldots 1_{n}0_1 \), with \( a + a' = 1 \) and \( b + b' = 1 \), such that

\[
1_1 \ldots 1_{i-1}a_i'vb_i'1_{i+1} \ldots 1_{n}0_1 = w_1(1, 0)
\]

in \( P_n(1) \), by rules \((R_1')\), \((R_6')\), and \((R_7')\). Now, by Lemma 4.4, since \( u^6 = v^6 = v \) in \( P_n(i) \),

\[
w_1(1, 0) = 1_1 \ldots 1_{i-1}a_i'vb_i'1_{i+1} \ldots 1_{n}0_1 = 0_11_1 \ldots 1_{i-1}a_i'v'1_{i+1} \ldots 1_{n}0_1 = 0_11_1 \ldots 1_{i-1}a_i'v'1_{i+1} \ldots 1_{n}0_1 = 0_1 = 1,
\]

which contradicts Corollary 4.6. If \( a + b = 2 \) then \( u^2 = v^2 = \zeta_i \), by rule \((R_6')\), and so \( u^4 = \zeta_i \). This shows that \( \{u^3 : u \in P_n(i)\} \) is a set of idempotents. It is immediate that this set contains all idempotents of \( P_n(i) \). \(\square\)

In the following, we list the idempotents of the monoid \( P_n(i) \).

**Corollary 4.14.** Let \( 1 \leq i \leq n \). Then the idempotents of \( P_n(i) \) are

\[
0_i, \ 2_i, \ \zeta_i \text{ and } a_i1_{i+1} \ldots 1_{n}1_{i} \ldots 1_{i-1}b_i
\]

with \( a + b = 1 \).

**Proof.** By rules \((R_1')\), \((R_2')\), \((R_3')\), \((R_4')\), \((R_6')\), and \((R_7')\), and by definition of \( P_n(i) \), the elements of (6) are idempotents. By Lemma 4.13, (6) lists all idempotents of \( P_n(i) \). \(\square\)

We recall that \( V_{\frac{\pi}{2}} \) denotes the pseudovariety of ordered semigroups defined by the pseudoidentities

\[
u^c \varphi v^c \leq u^c
\]

where \( c(v) \subseteq c(u) \). It is clear that \( B_2 \in V_{\frac{\pi}{2}} \) so, by Theorem 3.7, the global of \( V_{\frac{\pi}{2}} \) is defined by the pseudoidentities of the form

\[
(v^c \varphi v^c \leq u^c, A_{u^c v^c v^c})
\]

with \( c(v) \subseteq c(u) \). We next prove that \( C_n \) fails a specific pseudoidentity of this form.

**Proposition 4.15.** Let \( n \geq 2 \). Then \( C_n \) does not belong to \( gV_{\frac{\pi}{2}} \).
Proof. Consider the graph $\Gamma_n$ described in Figure 1 and let $u = y_1x_1 \ldots y_nx_n$ and $v = y_1'x_1 \ldots y_n'x_n$ be finite paths in $\Gamma$. Note that the graphs $A_{w^\infty uv^\infty}$ and $\Gamma_n$ are identical up to the name of the vertices. Since $c(v) \subseteq c(u)$, it follows that $gV_2^w$ satisfies the pseudoidentity $(w^\infty vw^\infty \leq w^\infty, \Gamma_n)$, by Theorem 3.7.

Let us show that $C_n$ fails this pseudoidentity, and consequently, $C_n$ does not lie in $gV_2^w$. Arguing by contradiction, suppose that $C_n$ satisfies the pseudoidentity in question. Evaluate the graph $\Gamma_n$ in $C_n$ through a graph homomorphism which maps the edges $x_i$ and $y_i$ of $\Gamma_n$ to the corresponding edges $x_i$ and $y_i$ of the category $C_n$. We obtain two edges $u$ and $v$ of $C_n$ such that

$$\psi(u^\omega) = \psi(u) = w_1(1,0) \quad \text{and} \quad \psi(u^\omega uv^\omega) = \psi(uvu) = w_5(1,0),$$

where $\psi$ is the order-faithful homomorphism defined in Lemma 4.3. Therefore we must have $w_5(1,0) \leq w_1(1,0)$. By definition of $\leq$ in $P_n$, it must be either $w_5(1,0) = \zeta_1$ or $w_5(1,0) = w_1(1,0)$. Since $w_5(1,0) \neq \zeta_1$ by Corollary 4.6(a), we deduce that $w_5(1,0) = w_1(1,0)$. As a consequence we have $\zeta_1 = w_5(1,0)w_4(2,0) = w_1(1,0)w_4(2,0) = w_5(1,0)$, which contradicts Corollary 4.6. \hfill $\square$

The following result is a decisive property of $C_n$ related with the pseudovariety $gV_2^w$.

**Proposition 4.16.** Let $\Gamma$ be a finite graph and let $u, v$ be loops at the same vertex of $\Gamma^+$ such that $c(v) \subseteq c(u)$. Then every subcategory of $C_n$ with $n-1$ vertices satisfies $(w^\infty vw^\infty \leq w^\infty, \Gamma)$.

**Proof.** Let $T$ be a subcategory of $C_n$ with $n-1$ vertices. Let $u$ and $v$ be loops at the same vertex of $\Gamma^+$ such that $c(v) \subseteq c(u)$. By Lemmas 4.3 and 4.9, $C_n$ is finite and so there are edges $u'$ and $v'$ of $\Gamma^+$ such that $u = u'$ and $v = v'$ in $T$, and we may assume that $u$ and $v$ are edges of $\Gamma^+$. Let $\eta: \Gamma^+ \to T$ be a homomorphism of semigroupoids and let $\psi$ be the order-faithful homomorphism defined in Lemma 4.3. We denote by $\varphi$ the homomorphism $\psi \circ \eta$ as depicted in the following commutative diagram:

$$\begin{array}{ccc}
\Gamma^+ & \xrightarrow{\eta} & T & \xrightarrow{\psi} & C_n \\
\varphi & & & & \\
& & & \downarrow & \\
P_n & & & & \\
\end{array}$$

First, we recall that, by Lemma 4.13, $\varphi(u^\infty) = \varphi(u^3)$. Since $u^3vu^3$ is a path in $\Gamma$, we have $\varphi(u^3vu^3) \neq 0$. Let $k = \alpha(yu)$, so $k \in \{1, \ldots, n\}$. Since $T$ has $n-1$ vertices, there is $j \in \{1, \ldots, n\}$, with $j \neq k$, such that, for all $x \in c(u)$, neither $t_1(\varphi(x))$ nor $t_1(\varphi(x))$ belongs to $A_j$. If $\varphi(u^3) = \zeta_k$ then $\varphi(u^3vu^3) = \zeta_k$. If $\varphi(u^3) \neq \zeta_k$ then, by Corollary 4.14, the idempotent $\varphi(u^3)$ satisfies the relation

$$\varphi(u^3) \in \{0_k, 2_k, a_k1_{k+1} \ldots 1_n1_1 \ldots 1_{k-1}b_k : a + b = 1\}.$$
If \( \varphi(u^3) \in \{0_k, 2_k\} \) then \( \varphi(u^3v^3) = \varphi(u^3) \) since \( c(v) \subseteq c(u) \). Suppose that \( \varphi(u^3) = a_k1_{k+1} \cdots 1_{n_1}1_{k+1} \cdots 1_{k-1}b_k \), with \( a + b = 1 \), and \( \varphi(u^3v^3) \neq \zeta_k \). By Lemma 4.13, \( \varphi(u^3) = \varphi(u) \) and, by Corollary 4.6(c), \( \varphi(u) \) has no factors \( 2_i \) for every \( i \in \{1, \ldots, n\} \), since there are \( a' \) and \( b' \) such that \( 1_1 \cdots 1_{k-1}a'_k \varphi(u)b'_k1_{k+1} \cdots 1_n0_1 = w_1(1,0) \). Hence for every \( x \in c(u) \) there is no \( i \in \{1, \ldots, n\} \) such that \( 2_i \) is a factor of \( \varphi(x) \).

Now, by Lemma 4.11, \( 2_i \) is a factor of \( \varphi(uvw) \) if and only if \( 2_i \) is a factor of every representative of the \( \Upsilon \)-class \( \varphi(uvw) \). Since \( \varphi(uvw) \) is a product of factors of the form \( \varphi(x) \), with \( x \in c(u) \), none of which has \( 2_i \) as a factor by the preceding paragraph, the only way \( 2_i \) may appear as a factor of \( \varphi(uvw) \) is if \( uvw \) has a factor \( x'y \), with \( x, y \in c(u) \), such that \( t_1(\varphi(x)) = 1_i = i_1(\varphi(y)) \). Since \( j \) is a not a vertex of \( T \), this cannot happen for \( i = j \). Hence \( 2_j \) is not a factor of \( \varphi(uvw) \).

Let \( z = 1_1 \cdots 1_{k-1}b_k \varphi(v)a_k1_{k+1} \cdots 1_n1_1 \in P_n(1) \). Since

\[
\zeta_k \neq \varphi(uvw) = a_k1_{k+1} \cdots 1_nz1_2 \cdots 1_{k-1}b_k,
\]

then \( z \neq \zeta_1 \) and \( 2_j \) is not a factor of \( z \), so \( z \in \{1_1, w_1(1,1)\} \) by Lemma 4.8. Therefore \( \varphi(uvw) = a_k1_{k+1} \cdots 1_n1_1 \cdots 1_{k-1}b_k \), by rules \((R_1)\), \((R_{f'})\), and \((R_7)\).

We have shown that \( \varphi(u^3v^3) = \zeta_k \) or \( \varphi(u^3v^3) = \varphi(u^3) \). It follows that \( \varphi(u^3v^3) \leq \varphi(u^3) \) in \( P_n \). As \( \psi \) is an order-faithful homomorphism, we conclude that \( T \) satisfies the pseudoidentity \( (u^3v^3u^\omega \leq u^\omega, \Gamma) \).

We may now prove the main result of this section.

**Theorem 4.17.** The pseudovariety \( g\mathcal{V}_{\frac{1}{2}} \) has infinite \( v \)-rank.

**Proof.** Let us assume that \( g\mathcal{V}_{\frac{1}{2}} \) has finite \( v \)-rank \( r \), that is, \( g\mathcal{V}_{\frac{1}{2}} \) admits a basis of pseudoidentities \( \Sigma \) over graphs with at most \( r \) vertices. We claim that the category \( C_{r+1} \) belongs to \( g\mathcal{V}_{\frac{1}{2}} \). Indeed, when we verify the pseudoidentities in \( \Sigma \), we only work with subcategories of \( C_{r+1} \) with at most \( r \) vertices. By Proposition 4.16, such subcategories belong to \( g\mathcal{V}_{\frac{1}{2}} \), hence \( C_{r+1} \) satisfies the pseudoidentities of \( \Sigma \) and then it lies in \( g\mathcal{V}_{\frac{1}{2}} \), which is in contradiction with Proposition 4.15.

**References**


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