CORRIGENDUM: COMPLEXITY OF INFINITE WORDS ASSOCIATED WITH BETA-EXPANSIONS

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Abstract. We add a sufficient condition for validity of Proposition 4.10 in the paper Frougny et al. (2004). This condition is not a necessary one, it is nevertheless convenient, since anyway most of the statements in the paper Frougny et al. (2004) use it.

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1. Introduction

The aim of this note is to correct the mistake contained in our paper [2]. We shall use the notation of the paper and refer to the statements included in it.

We were pointed out [1] a counterexample to assertion (1) of Theorem 6.2 in the paper. The assertion says that the complexity of the fixed point $u_\beta$ of the canonical substitution $\varphi_\beta$ associated with a simple Parry number $\beta$ with the Rényi expansion $d_\beta(1) = t_1 t_2 \cdots t_{m-1} 1$ is affine, namely $C(n) = (m-1)n+1$. This statement is however true only under the condition used for assertion (2) of the theorem, namely that the Rényi expansion $d_\beta(1) = t_1 t_2 \cdots t_m$ satisfies

\[
t_1 = t_2 = \cdots = t_{m-1} \quad \text{or} \quad t_1 > \max\{t_2, \ldots, t_{m-1}\}. \quad (*)
\]

The mistake occurred due to a slip in the proof of Proposition 4.10. We show in this note that under the additional condition $(*)$ the proposition is valid.

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The corrected version of Proposition 4.10 of [2] is stated here as Proposition 2.2. At the end of this note we explain which statements of the paper [2] need to be equipped with condition (*), as well.

Let us mention that the condition (*) in Proposition 2.2 may be weakened. Nevertheless, we have chosen the condition in the form (*), since anyway most of the statements in the paper [2] use it.

2. PROOF OF PROPOSITION 4.10 OF [2]

In order to prove Proposition 2.2 we need the following lemma.

Lemma 2.1. Let $t_1 > \max\{t_2, \ldots, t_{m-1}\}$. Let $w$ be a right special factor of $u_3$ with at least 3 distinct right extensions $X, Y, Z$, such that $w$ contains a non-zero letter, $wX$ is a left special factor and $X \neq 0$. Then there exists a word $\tilde{w}$ which is a right special factor of $u_3$ with at least 3 distinct right extensions $\tilde{X}, \tilde{Y}, \tilde{Z}$ such that $\tilde{w}\tilde{X}$ is a left special factor, $\tilde{X} \neq 0$, and $wX = \varphi(\tilde{w}\tilde{X})$.

Proof. The word $w$ can be written as $w = w^pU^q$, where $U \neq 0$ and $p \geq 0$. Thus $U^qX, U^qY, U^qZ$ are factors of $u_3$. Since at least one of $X, Y, Z$ is $\geq 2$, we can derive from Lemma 4.5 of [2] and condition $t_1 > \max\{t_2, \ldots, t_{m-1}\}$ that $p < t_1$. Since $w^pU$ is a left special factor, according to (ii) of Lemma 3.7 there exists a left special factor $\tilde{w}$ such that $w^pU = \varphi(\tilde{w})$. Now

$$
wX = \varphi(\tilde{w})0^pX$$
$$wY = \varphi(\tilde{w})0^pY$$
$$wZ = \varphi(\tilde{w})0^pZ$$

are distinct factors of $u_3$. Hence there must exist distinct letters $\tilde{X}, \tilde{Y}, \tilde{Z}$ such that $\tilde{w}\tilde{X}, \tilde{w}\tilde{Y}, \tilde{w}\tilde{Z}$ are also factors of $u_3$. Moreover, since $X \neq 0$ and $p < t_1$, we have $\varphi(\tilde{X}) = 0^pX$, where $\tilde{X} \neq 0$. As $\varphi(\tilde{w}\tilde{X}) = wX$ is a left special factor, (ii) of Lemma 3.7 implies that $\tilde{w}\tilde{X}$ is a left special factor, which completes the proof. \( \square \)

The following statement is the same as in Proposition 4.10 of [2], except the additional condition (*).

Proposition 2.2. Let $d_3(1)$ satisfies the condition (*). Then for every maximal left special factor $v = v_0v_1 \cdots v_l$ containing a letter $v_i \neq 0$ there exists a maximal left special factor $w$ and an $s \in \{t_1, t_2, \ldots, t_{m-1}\}$ such that $v = \varphi(w)^s$.

Proof. Let $j = \max\{ i \mid v_i \neq 0 \}$. According to Lemma 3.7 there exists a left special factor $w = w_0w_1 \cdots w_l$ such that $v_0v_1 \cdots v_j = \varphi(w_0)\varphi(w_1) \cdots \varphi(w_l)$ and thus

$$v = v_0v_1 \cdots v_j0^s = \varphi(w_0)\varphi(w_1) \cdots \varphi(w_l)0^s,$$

where $s = k - j$.

Since $v$ is maximal, we can use Observation 4.2 and Corollary 4.6 to derive that $s \in \{t_1, t_2, \ldots, t_{m-1}\}$. 


It remains to show that $w$ is a maximal left special factor of $u_β$. Assume that $w$ is not maximal. We distinguish two cases according to which part of condition (*) is satisfied.

- Let $t_1 = t_2 = \cdots = t_{m-1} =: t$. Since $w$ is not maximal, then according to Lemma 4.9 there exists a left special factor $wX$, where $X \neq m - 1$ or a left special factor $w(m-1)0$. However, then (ii) of Lemma 3.7 implies that $\varphi(wX) = \varphi(w)0^t(X + 1)$, resp. $\varphi(w(m-1)0) = \varphi(w)0^{t-1}1$, is also a left special factor. Since $s = t$, the factor $v$ is a proper prefix of both of them, which is a contradiction with the maximality of $v$.

- Let $t_1 > \max\{t_2, \ldots, t_{m-1}\}$. Since $v = \varphi(w)0^s$ is a maximal left special factor of $u_β$ and $w$ is not maximal, there exists a letter $X$ such that $wX$ is again a left special factor. Lemma 3.7 implies that $\varphi(wX)$ is also a left special factor. Since $v = \varphi(w)0^s$ may not be a proper prefix of $\varphi(wX)$, the condition $t_1 > \max\{t_2, \ldots, t_{m-1}\}$ implies $X \neq 0$.

The maximality of the left special factor $v = \varphi(w)0^s$ implies also existence of distinct letters $Y^*, Z^*$ such that $\varphi(w)0^sY^*, \varphi(w)0^sZ^*$ are factors of $u_β$ and but they are not left special. There must exist distinct letters $Y, Z$ such that $wY, wZ$ are factors of $u_β$ but not left special.

We have thus shown that $w$ is a right special factor with at least 3 distinct right extensions $X \neq 0, Y, Z$, where $wX$ is a left special factor. Repeated use of Lemma 2.1 leads to a right special factor $w^{(0)} = 0^q$, for $q \geq 1$, which has at least 3 distinct right extensions $X^{(0)} \neq 0, Y^{(0)}, Z^{(0)}$, such that $w^{(0)}X^{(0)}$ is a left special factor of $u_β$. Lemma 4.5 implies that $X^{(0)} = 1$ and $q = t_1$. At least one letter among $Y^{(0)}, Z^{(0)}$ is non-zero, say $Y^{(0)}$. Then $Y^{(0)} \geq 2$, but then $w^{(0)}Y^{(0)} = 0^{t_1}Y^{(0)}$ is due to Lemma 4.5 not a factor of $u_β$, which is a contradiction.

\[\square\]

3. Conclusions

Proposition 4.10 was used in [2] for proving Corollary 4.11, second implication of Theorem 4.12, assertion (1) of Theorem 6.2 and Corollary 6.3. Therefore condition (*) should be added in the mentioned statements as well.

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References