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# FINITE PRESENTABILITY OF STRONGLY FINITE DILATORS *

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**Abstract.** In this paper, we establish the following results: (i) every strongly finite dilator is finitely presentable in the category of endofunctors on the category of ordinals; (ii) a dilator $F$ is strongly finite if and only if $F$ is finitely presentable in the category of dilators.

**Mathematics Subject Classification.** 03F15.

## 1. INTRODUCTION

In [3], Girard introduced the theory of dilators as a tool by which he attempted to extend several results in subrecursion theory and proof theory (*cf.* [3, 4] and [2]). He considered dilators as a notion abstracted from various kinds of proof theoretic ordinal notation systems. In fact, some notation systems can be viewed as certain dilators. For example, Weiermann in [6] showed that ordinal notation systems with Aczel–Buchholz–Feferman functions give rise to dilators.

On the other hand, since dilators are defined as endofunctors on **ON** (the category of ordinals) which commute with direct limits and pull-backs[2], one can study dilators from category theoretical point of view. Let **Ford** denote the category of finite ordinals. Johnstone [5] showed that **DIL** (the category of dilators) is essentially equivalent to a certain full subcategory of the atomic topos $Sh(\mathbf{Ford}^{op})$,

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[2] What is called "colimit" in category theory in general is called "limit" in [3]; the usual concept of "limit" of category theory is called "inverse limit" in [3]; the term "pull-back" as used in category theory is retained in [3]. We follow [3].

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where $\mathbf{Ford}^{op}$ is regarded as an atomic site with its canonical Grothendieck topology (*cf.* Sect. 1 of [5]). He also took notice of representable functors from **Ford** to **Set** and characterized prime dilators as the representable functors (*cf.* Sect. 3 of [5]).

In this paper, we establish a relationship between finitely presentable dilators and strongly finite dilators, which are a generalization of prime dilators. Considering the property of finite presentability and the relationship between prime dilators and representable functors in [5], one can easily conjecture that each prime dilator is finitely presentable in **DIL**, or in a certain full subcategory of $\mathbf{ON}^{\mathbf{ON}}$ (the category of endofunctors on **ON**). Our aim is to prove an extension of this fact by establishing the following results:

(i) every strongly finite dilator is finitely presentable in $\mathbf{ON}^{\mathbf{ON}}$;
(ii) a dilator $F$ is strongly finite if and only if $F$ is finitely presentable in **DIL**.

These results directly follow from the ideas of [5].

From Theorem 4.3.10 in [3] (Th. 2.9 in the next section), it follows that to construct all dilators through direct limits in **DIL**, it is sufficient to provide the set of all strongly finite dilators. The fact (ii) above implies that, to construct all dilators through direct limits in **DIL**, it is necessary to provide the set of all strongly finite dilators.

The results of this paper can be proved with no difficulty. However, we hope it is not unworthy to establish the above facts, since these facts give a guarantee that the definition of dilators is reasonable and that we can regard each dilator as a natural extension of an ordinal.

Proof theoretic ordinal notation systems express a lot of information abstracted from proof schemata which can in turn be used to estimate the logical strength of the corresponding formal systems. Thus, since dilators correspond to proof theoretic ordinal notation systems, their characterization gives rise to an analysis of the information that can be expressed by such notation systems.

## 2. Preliminaries

In this section, we provide some definitions and results about dilators, almost all of which are contained in [3].

**Definition 2.1.** **ON** (**OL**) denotes the category with ordinals (linearly ordered sets) as objects and strictly increasing functions as morphisms. Composition in **ON** (**OL**) is function composition.

We abbreviate the term "strictly increasing function(s)" by s.i.f.(s). We use the following symbols: $x, y, \cdots, a, b, \cdots$ for ordinals; $f, g, \cdots, \phi, \psi, \cdots$ for functions; $F, G, \cdots$ for functors; $T, U, \cdots$ for natural transformations. We often use $n, m, \cdots$ to denote natural numbers (*i.e.*, ordinals in $\omega$). For any ordinal $x$, $E_x$ denotes the identity function on $x$.

We abbreviate the term "$f$ is a morphism from an object $a$ to an object $b$" by $f : a \to b$. We denote the composition of $f : a \to b$ and $g : b \to c$ by $gf$ or by $g \cdot f$. We denote the range of an s.i.f. $f$ by $rg(f)$.

**Definition 2.2.** (1) A partially ordered set $I = (I, \leq_I)$ is said to be *directed* if for any $a$, $b \in I$ there exists a $c \in I$ with $a \leq_I c$ and $b \leq_I c$.

(2) Let $I = (I, \leq_I)$ be a (non-empty) directed set and let $\mathcal{C}$ be a category. A family $(x_i, f_{ij})_{i,j \in I}$ is called a *direct system* in $\mathcal{C}$ if $(x_i, f_{ij})_{i,j \in I}$ satisfies the following conditions:

(i) for any $i \in I$, $x_i$ is an object in $\mathcal{C}$;
(ii) for any $i, j \in I$ with $i \leq_I j$, $f_{ij}$ is a morphism in $\mathcal{C}$ from $x_i$ to $x_j$;
(iii) for any $i \in I$, $f_{ii}$ is the identity morphism on $x_i$;
(iv) for any $i, j, k \in I$ with $i \leq_I j$ and $j \leq_I k$, $f_{ik} = f_{jk} f_{ij}$.

(3) Let $(x_i, f_{ij})_{i,j \in I}$ be a direct system in $\mathcal{C}$. A family $(x, f_i)_{i \in I}$, where $x$ is an object in $\mathcal{C}$ and $f_i : x_i \to x$ in $\mathcal{C}$, is called a *direct limit* of $(x_i, f_{ij})_{i,j \in I}$ in $\mathcal{C}$ if $(x, f_i)_{i \in I}$ satisfies the following conditions.

(i′) For any $i, j \in I$ with $i \leq_I j$, $f_i = f_j f_{ij}$.
(ii′) If $(y, g_i)_{i \in I}$ is any family that satisfies (i′), then there exists a unique morphism $h : x \to y$ such that $g_i = h f_i$ for any $i \in I$.

**Proposition 2.3.** *In the categories* **ON** *and* **OL**, *the condition* (ii′) *in Definition* 2.2 *can be replaced by the following:*

(ii″) $x = \bigcup_{i \in I} rg(f_i)$, *i.e., every point in $x$ is in the range of some $f_i$.*

**Theorem 2.4.** (Th. 1.4.1 in [3]). *In* **OL**, *all direct systems have direct limits.*

**Theorem 2.5.** (Th. 1.3.9 in [3]). *In the categories* **ON** *and* **OL**, *for any object $x$, there exists a direct system $(x_i, f_{ij})_{i,j \in I}$, where every $x_i$ is a natural number, and a family $(f_i)_{i \in I}$ such that $(x, f_i)_{i \in I}$ is a direct limit of $(x_i, f_{ij})_{i,j \in I}$.*

**Definition 2.6.** A functor $F : \mathbf{ON} \to \mathbf{ON}$ is called a *dilator* if $F$ commutes with every direct limit and every pull-back in **ON**.

**Definition 2.7.** A dilator $F$ is said to be *strongly finite* if there are only finitely many dilators $G$ and natural transformations from $G$ to $F$. A dilator $F$ is said to be *prime* if $F$ has no natural transformation to $F$ except the identity natural transformation $E_F$ and $T_0 : \mathbf{0} \to F$. Here, $\mathbf{0}$ denotes the dilator which assigns to every ordinal the ordinal 0, and $T_0$ denotes the obvious natural transformation $\mathbf{0} \to F$.

Note that every prime dilator is strongly finite.

**Definition 2.8.** (1) **DIL** denotes the category consisting of all dilators and their natural transformations.

(2) **SFD** denotes the full subcategory of **DIL** determined by the strongly finite dilators.

Note that in **ON** (or **DIL**), for any direct system $(x_i, f_{ij})_{i,j \in I}$, if there exists a direct limit of $(x_i, f_{ij})_{i,j \in I}$, then such a direct limit is unique.

**Theorem 2.9.** (Th. 4.3.10 in [3]). *Every dilator is the direct limit of a direct system consisting of strongly finite dilators in* **DIL**.

**Definition 2.10.** [1] In a category $\mathcal{C}$, an object $a$ is *finitely presentable* if for any direct limit $(x, f_i)_{i\in I}$ of a direct system $(x_i, f_{ij})_{i\in I}$ in $\mathcal{C}$ and for any $f : a \to x$, there exists an $i_0 \in I$ satisfying the following conditions:

(i) there exists a morphism $g : a \to x_{i_0}$ such that $f_{i_0} \cdot g = f$;
(ii) for any $g, h : a \to x_{i_0}$ with $f_{i_0} \cdot g = f_{i_0} \cdot h = f$, there exists an $i_1 \in I$ such that $i_0 \leq_I i_1$ and $f_{i_0 i_1} \cdot g = f_{i_0 i_1} \cdot h$.

**Remark 2.11.** If all morphisms in $\mathcal{C}$ are monic, then Definition 2.10(ii) is always satisfied.

**Example 1**. The finitely presentable objects in **Set** and **OL** are the finite sets and the finite linearly ordered sets, respectively. The finitely presentable objects in **ON** are the natural numbers.

**Example 2**. Let $\mathcal{A}$ be a small category. Then every hom-functor is a finitely presentable object of the functor category $\mathbf{Set}^{\mathcal{A}}$.

By a fact in Section 3 of [5] and the Example 2 above, one can consider that every prime dilator is finitely presentable in **DIL**, or in a certain full subcategory of $\mathbf{ON}^{\mathbf{ON}}$. In the following section, we establish finite presentability of all strongly finite dilators.

## 3. Finite presentability of strongly finite dilators

In this section, we first define "associated representations" of dilators. Every dilator has a unique denotation system: for any ordinal $x$ and any $z \in F(x)$, there is a minimal natural number $n$, a unique ordinal $z_0 \in F(n)$ and a unique $f : n \to x$ such that $F(f)(z_0) = z$. So, we can denote $z$ by $(z_0; x_0, \cdots, x_{n-1}; x)_F$, where $f(i) = x_i$ for any $i < n$. We call such an expression the *associated representation* (of $F$).

**Lemma 3.1.** (Rem. 2.3.13 in [3]). *Let $F$ be a dilator and let $(\cdots)_F$ be the associated representation of $F$. For any $z = (z_0; x_0, \cdots, x_{n-1}; x)_F \in F(x)$, any $y \geq x$ and $g : x \to y$, $F(g)(z)$ is given by $(z_0; g(x_0), \cdots, g(x_{n-1}); y)_F$.*

For every dilator $F$, we define the *range* of $F$ as follows:

$$rg(F) = \{(z; n);\ z = (z; 0, 1, \cdots, n-1; n)_F\}.$$

The finite presentability of strongly finite dilators will be established by using the properties of their ranges.

**Lemma 3.2.** (Th. 4.3.2 in [3]). *A dilator $F$ is strongly finite if and only if $rg(F)$ is finite.*

**Lemma 3.3.** (1) Let $(F_i, T_{ij})_{i,j\in I}$ be a direct system in $\mathbf{ON}^{\mathbf{ON}}$ and let $(F, T_i)_{i\in I}$ be the direct limit of $(F_i, T_{ij})_{i,j\in I}$ in $\mathbf{ON}^{\mathbf{ON}}$. Then, for any ordinal $x$, $(F(x), T_i(x))_{i\in I}$ is the direct limit of the direct system $(F_i(x), T_{ij}(x))_{i,j\in I}$ in $\mathbf{ON}$.

(2) The same fact holds also in the case where $(F_i, T_{ij})_{i,j\in I}$ is contained in $\mathbf{DIL}$ and $(F, T_i)_{i\in I}$ is the direct limit of $(F_i, T_{ij})_{i,j\in I}$ in $\mathbf{DIL}$.

*Proof.* (1) By Theorem 2.4, there exists a limit $(\alpha_x, \phi_i^x)_{i\in I}$ in $\mathbf{OL}$ of the system $(F_i(x), T_{ij}(x))_{i,j\in I}$. Since there exists an s.i.f. $\psi : \alpha_x \to F(x)$ with $\forall i \in I(T_i(x) = \psi\phi_i^x)$, $\alpha_x$ is well-ordered. So, we can regard $\alpha_x$ as an object of $\mathbf{ON}$.
We define a functor $G \in \mathbf{ON}^{\mathbf{ON}}$ as follows: for each $x \in \mathbf{ON}$, $G(x) = \alpha_x$; and for each $f : x \to y$ in $\mathbf{ON}$, $G(f)$ is the s.i.f. $\psi_f : \alpha_x \to \alpha_y$ which is uniquely determined by $\forall i \in I(\psi_f\phi_i^x = \phi_i^y F_i(f))$. Moreover, for each $i \in I$, we define a natural transformation $U_i : F_i \to G$ by $\forall x \in \mathbf{ON}(U_i(x) = \phi_i^x)$. One can easily check that $G$ and $U_i$ are well defined, and $(G, U_i)_{i\in I}$ is the limit of $(F_i, T_{ij})_{i,j\in I}$ in $\mathbf{ON}^{\mathbf{ON}}$. Therefore, $\alpha_x = G(x) = F(x)$ and $\phi_i^x = U_i(x) = T_i(x)$, and hence, $(F(x), T_i(x))_{i\in I}$ is the limit of $(F_i(x), T_{ij}(x))_{i,j\in I}$ in $\mathbf{ON}$.

(2) If $(F_i, T_{ij})_{i,j\in I}$ is contained in $\mathbf{DIL}$ and $(F, T_i)_{i\in I}$ is the direct limit of $(F_i, T_{ij})_{i,j\in I}$ in $\mathbf{DIL}$, then the functor $G$ defined in the same way as above is a dilator (see the proof of Th. 4.1.3(i) in [3]).

□

**Theorem 3.4.** *Every strongly finite dilator is a finitely presentable object of* $\mathbf{ON}^{\mathbf{ON}}$.

*Proof.* Let $F$ be a strongly finite dilator. We show that if $(G, T_i)_{i\in I}$ is the direct limit of a direct system $(G_i, T_{ij})_{i,j\in I}$ and if $T$ is a natural transformation from $F$ to $G$, then there exist some $N \in I$ and $U : F \to G_N$ such that $T_N U = T$. (Note that all natural transformations in $\mathbf{ON}^{\mathbf{ON}}$ are monic. See Rem. 2.11.) By Theorem 3.2, $rg(F)$ is finite. Let $rg(F) = \{(x_1; n_1), \cdots, (x_k, n_k)\}$. For each $l \le k$, let $y_l$ denote $T(n_l)(x_l)$. By Lemma 3.3 $(G(n_l), T_i(n_l))_{i\in I}$ is the direct limit of $(G_i(n_l), T_{ij}(n_l))_{i,j\in I}$ in $\mathbf{ON}$. So, by Proposition 2.3, there exist some $i_l \in I$ and $a_l \in G_{i_l}(n_l)$ such that $T_{i_l}(n_l)(a_l) = y_l$. For each $l \le k$, we fix such an $i_l \in I$ and such an $a_l \in G_{i_l}(n_l)$. Thus, since $(G_i, T_{ij})_{i,j\in I}$ is a direct system, there is an $N \in I$ for which there exists a natural transformation $T_{i_l N} : G_{i_l} \to G_N$ for each $l \le k$. We let $b_l$ $(l = 1, \cdots, k)$ be $T_{i_l N}(n_l)(a_l)$.

For an ordinal $x$ we define a function $U(x)$ from $F(x)$ to $G_N(x)$ as follows: for any $z \in F(x)$, we define $U(x)(z) = G_N(f)(b_l)$, where $l$ and $f : n_l \to x$ are uniquely determined by $z = F(f)(x_l) = (x_l; f(0), \cdots, f(n_l - 1); x)_F$. We prove the following facts: (i) $U(x)$ is an s.i.f. for any $x \in \mathbf{ON}$; (ii) $U$ is a natural transformation; and (iii) $T_N U = T$.

*Proof of* (i). Let $z$ and $z'$ be ordinals in $F(x)$ satisfying the following: $z < z'$; $z = F(f)(x_l)$ for some $f : n_l \to x$; and $z' = F(g)(x_m)$ for some $g : n_m \to x$. Since $T(x)$ is an s.i.f., we have $T(x)(z) < T(x)(z')$. So, since $G(f)(y_l) = G(f)(T(n_l)(x_l)) = T(x)(F(f)(x_l)) = T(x)(z)$ and $G(g)(y_m) = G(g)(T(n_m)(x_m)) = T(x)(F(f)(x_m))$

$= T(x)(z')$, it holds that $G(f)(y_l) < G(g)(y_m)$. Therefore, by the naturality of $T_N$,

$$\begin{aligned} T_N(x)(G_N(f)(b_l)) &= G(f)(T_N(n_l)(b_l)) = G(f)(y_l) \\ &\quad < G(g)(y_m) = G(g)(T_N(n_m)(b_m)) \\ &= T_N(x)(G_N(g)(b_m)). \end{aligned}$$

Since $T_N(x)$ is an s.i.f., $U(x)(z) = G_N(f)(b_l) < G_N(g)(b_m) = U(x)(z')$.

*Proof of* (ii). If $\psi$ is a morphism in **ON** that maps $x$ to $x'$, and if $z \in F(x)$ satisfies the condition that $z = F(f)(x_l)$ for some $f : n_l \to x$, then $U(x')(F(\psi)(z)) = U(x')(F(\psi)(F(f)(x_l))) = U(x')(F(\psi f)(x_l))$. By the definition of $U(x')$ and by Lemma 3.1, $U(x')(F(\psi f)(x_l)) = G_N(\psi f)(b_l)$. Therefore, $U(x')(F(\psi)(z)) = G_N(\psi f)(b_l) = G_N(\psi)(G_N(f)(b_l)) = G_N(\psi)(U(x)(z))$.

*Proof of* (iii). If $z \in F(x)$ satisfies the condition that $z = F(f)(x_l)$ for some $f : n_l \to x$, then

$$\begin{aligned} T_N(x)(U(x)(z)) &= T_N(x)(G_N(f)(b_l)) = G(f)(T_N(n_l)(b_l)) \\ &= G(f)(y_l) = G(f)(T(n_l)(x_l)) = T(x)(F(f)(x_l)) = T(x)(z). \end{aligned}$$

□

We next prove that every strongly finite dilator is a finitely presentable object of **DIL**, and *vice versa*. This result implies that, to construct all dilators through direct systems, it is necessary and sufficient to have all strongly finite dilators.

**Theorem 3.5.** *A dilator $F$ is strongly finite if and only if $F$ is finitely presentable in* **DIL**.

*Proof.* ($\Leftarrow$) Let $F$ be a dilator that is a finitely presentable object of **DIL**. By Theorem 2.9, there exists a direct system $(F_i, T_{ij})_{i,j\in I}$ consisting of strongly finite dilators $F_i$ and a family $(T_i)_{i\in I}$ such that $(F, T_i)_{i\in I}$ is the direct limit of $(F_i, T_{ij})_{i,j\in I}$. Since $F$ is finitely presentable, there exists an $i_0 \in I$ and a natural transformation $U : F \to F_{i_0}$ such that $U \cdot T_{i_0} = E_F$, where $E_F$ is the identity natural transformation on $F$. So, $T_{i_0}(x)U(x) = E_F(x) = E_{F(x)}$ (the identity function on $F(x)$) for any $x \in$ **ON**. Since $T_{i_0}(x)$ and $U(x)$ is an s.i.f., it holds that $T_{i_0}(x) = U(x) = E_{F(x)}$ and that $F_{i_0}(x) = F(x)$. Moreover, for any $f : x \to y$ and any $z \in F(x)$, $F(f)(z) = U(y)F(f)(z) = F_{i_0}(f)U(x)(z) = F_{i_0}(f)(z)$. So, we have that $F = F_{i_0}$, *i.e.*, $F$ is a strongly finite dilator.

($\Rightarrow$) By Lemma 3.3 (2), we can prove in the same way as that of Theorem 3.4.

□

**Definition 3.6.** A full subcategory **P** of a category $\mathcal{C}$ is said to be *dense* if for any $\mathcal{C}$-object $x$ there is a direct system in **P** whose limit is $x$.

Theorem 2.9 implies that **SFD** is dense in **DIL**, but **PD** (the category of all prime dilators and natural transformations) is not. By Theorem 3.5, if $F$ is a strongly finite dilator and if $(F_i, T_{ij})_{i,j\in I}$ is a direct system with the direct limit $(F, T_i)_{i\in I}$, then $F$ is contained in $\{F_i\}_{i\in I}$. Therefore, if **P** is dense in **DIL**, then $\mathbf{P} \supseteq \mathbf{SFD}$.

**Corollary 3.7.** **SFD** *is the unique minimal dense subcategory of* **DIL**.

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