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ELENA BARCUCCI
SARA BRUNETTI
FRANCESCO DEL RISTORO

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SUCCESSION RULES AND DECO POLYOMINOES *

Elena Barcucci[1], Sara Brunetti[1]
and Francesco Del Ristoro[1]

Abstract. In this paper, we examine the class of "deco" polyominoes and the succession rule describing their construction. These polyominoes are enumerated according to their directed height by factorial numbers. By changing some aspects of the "factorial" rule, we obtain some succession rules that describe various "deco" polyomino subclasses. By enumerating the subclasses according to their height and width, we find the following well-known numbers: Stirling numbers of the first and second kind, Narayana and odd index Fibonacci numbers. We wish to point out how the changes made on the original succession rule yield some new succession rules that produce transcendental, algebraic and rational generating functions.

AMS Subject Classification. 05B50, 05A15, 05A10.

1. Introduction

In this paper, we use the ECO (Enumeration of Combinatorial Objects) method [1] for enumerating some "deco" polyominoes subclasses [2]. The method is based on the following idea: given a class C of combinatorial objects and a parameter p on C, we consider the set $C_n = \{X \in C : p(X) = n\}$; we define an *operator* θ that constructs each object $Y \in C_n$ from another object $X \in C_{n-1}$ such that every $Y \in C_n$ is obtained from only one $X \in C_{n-1}$. Therefore, we have a recursive construction of the elements of C; from this, in turn, we can sometimes deduce a functional equation verified by the generating function of C. We can describe the construction by means of a generating tree whose vertices correspond to the

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[1] Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, 50134 Firenze, Italy; e-mail: barcucci@dsi.unifi.it & brunetti@dsi.unifi.it & fdr@dsi.unifi.it

objects. The objects having the same dimension with respect to the parameter p are at the same level and the sons of an object correspond to the objects obtained from it. We label each vertex by the number of its sons. We call this value the *fertility* of the node (and of the corresponding object). If the labels of the sons of a node labelled (k) only depend on the value k, we can represent the growing process of a generating tree by means of the following notation (called *succession rule*)

$$\begin{cases} (b) \\ (k) \to (c_1)(c_2)\dots(c_k), \end{cases}$$

where (b) is the label of the root and c_i is the label of the i-th son of a node labelled (k). In such a way, a succession rule can succintly represent a generating tree.

The aim of this paper is to show how variations on succession rules can influence the nature of the corresponding generating function. In Section 2, we describe the class of deco polyominoes that are enumerated by factorial numbers with respect to their directed height and we examine the succession rule which describes the construction of the class obtained by using the ECO method. In the subsequent sections, we illustrate some succession rules obtained by changing the previous one: the polyomino classes related to these rules are deco polyomino subclasses, enumerated by the Bell, Catalan and odd index Fibonacci numbers. We also study the classes according to their directed height and width, and we find some other well-known numbers. As a result we provide some new combinatorial interpretations of the relations that link these sequences of numbers to the previous ones. In treating the succession rules, we can virtually forget the combinatorial objects themselves. All the succession rules we studied are of the kind

$$\begin{cases} (2) \\ (k) \to (c_1)(c_2)\dots(c_k), \end{cases}$$

where $2 \le c_i \le c_{i+1} \quad (i = 1, \dots, k-1)$ and $c_k = k+1$. By simply making some changes in the succession rules, we find some generating functions that are very different from the original ones and also highly vary among themselves, since they are transcendental (Sects. 2, 3), algebraic (Sect. 4) and rational (Sect. 5).

2. The factorial succession rule

We introduce some definitions. Consider the $\mathbb{R}^2$ plane; a *cell* is a unitary square $[i, i+1] \times [j, j+1]$, $i, j \in \mathbb{N}$, and a *polyomino* is a connected set of pairs of cells having one side in common. The polyominoes are defined up to a translation. We can obtain a *directed* polyomino by starting out from a cell called source and by adding other cells in predetermined directions, such East and North, that is, to the right of or over existing cells. In this way, a polyomino grows in a preferred direction. A *column* (*row*) is the intersection of a polyomino with an infinite vertical (horizontal) strip $[i, i+1] \times \mathbb{R}$ ($\mathbb{R} \times [j, j+1]$). A *directed column-convex* polyomino is a directed

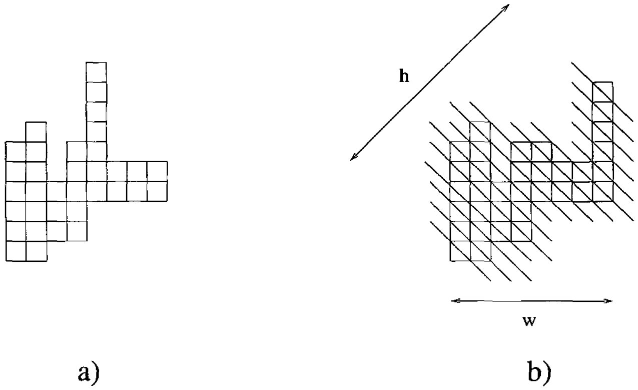

FIGURE 1. a) A directed column-convex polyomino. b) A deco polyomino of height 16 and width 8.

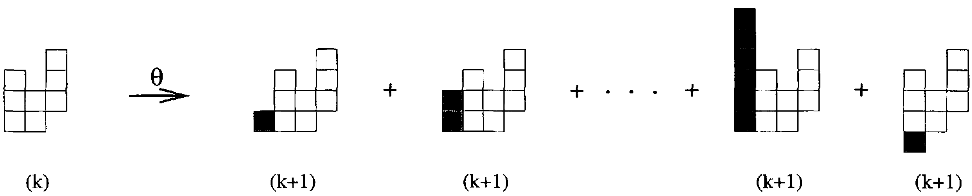

FIGURE 2. The operator θ applied to a deco polyomino of height 6.

polyomino whose columns are connected (see Fig. 1a). Finally, the *directed height* of a directed polyomino is the number of lines orthogonal to the preferred direction that go through the cell centers (henceforth we call it simply the height), its *vertical height* is the number of rows and its *width* is the number of columns. We examine a particular class of polyominoes, called *deco* (after the French *dernière colonne*: last column), defined as the set of directed column-convex polyominoes having height h, only reached in their last column (the rightmost column) [2] (see Fig. 1b). We denote this class by $\mathcal{D}_h$ and its cardinality by d_h. It is possible to construct $\mathcal{D}_h$ by means of the ECO method [1]. We define the operator $\theta : \mathcal{D}_{h-1} \to 2^{\mathcal{D}_h}$, where $2^{\mathcal{D}_h}$ denotes the power set of $\mathcal{D}_h$, as follows: for $P \in \mathcal{D}_{h-1}$, the elements of $\theta(P)$ are:

1) the polyomino obtained by adding a cell at the bottom of the first (leftmost) column of P;
2) the polyominoes obtained by adding a new column of height less than h to the left of P, in such a way the bottoms of the first two columns lie at the same level (Fig. 2).

It is easy to prove that, by performing θ over $\mathcal{D}_{h-1}$, we get all the elements of $\mathcal{D}_h$ and that each element is obtained only once. From the definition of the operator, it follows that, in the first case, there are d_{h-1} possibilities of adding a cell, while in the second, there are $(h-1)d_{h-1}$ possibilities. Consequently, we have the relation

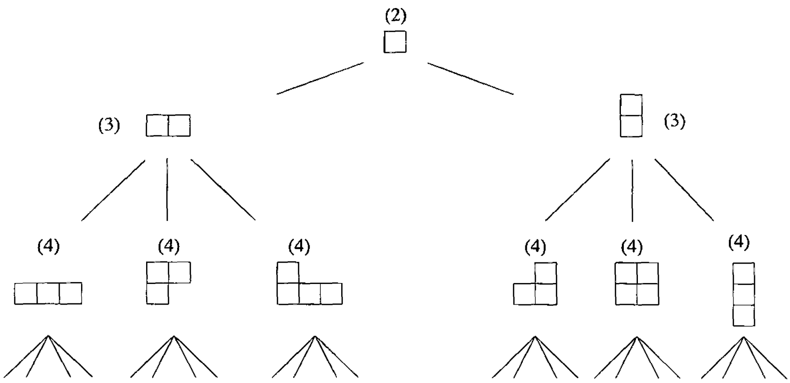

FIGURE 3. The generating tree of deco polyominoes.

$d_h = hd_{h-1}$, whose starting condition is $d_1 = 1$. As a result, the number of deco polyominoes of height h is $h!$.

We can represent this recursive construction by means of a generating tree (see Fig. 3): each vertex corresponds to an object and its label indicates the number of its sons.

Clearly, a polyomino of height $h-1$ has fertility $k = h$, and each of its k children has height h, hence fertility $k+1$. In other words, the generating tree for deco polyominoes can be succintly represented by

$$\begin{cases} (2) \\ (k) \rightarrow (k+1)\dots(k+1) \end{cases}$$

(where (2) represents the root label), abbreviated to

$$\begin{cases} (2) \\ (k) \rightarrow (k+1)^k. \end{cases} \tag{2.1}$$

Let $d_{h,w}$ be the number of deco polyominoes having height h and width w. From construction of $\mathcal{D}_h$, it follows that:

$$d_{h,w} = d_{h-1,w} + (h-1)d_{h-1,w-1}$$

with the conditions:

$$\begin{aligned} d_{h,1} &= 1, \text{ for } h \geq 1, \\ d_{h,w} &= 0, \text{ for } h < w \text{ or } w < 1. \end{aligned}$$

By setting $d'_{h,h-w+1} = d_{h,w}$, we obtain

$$d'_{h,h-w+1} = d'_{h-1,h-w} + (h-1)d'_{h-1,h-w+1}$$

which we rewrite as:

$$d'_{h,i} = d'_{h-1,i-1} + (h-1)d'_{h-1,i}$$

with conditions

$$d'_{h,h} = 1, \text{ for } h \geq 1,$$
$$d'_{h,i} = 0, \text{ for } i < 1 \text{ or } i > h.$$

This relation defines the *Stirling numbers of the first kind* (which count permutations of S_h by their cycle number i) so that, using notations from [7]:

$$d_{h,w} = d'_{h,h-w+1} = \left[\begin{array}{c} h \\ h-w+1 \end{array} \right].$$

Since $d_h = \sum_{w=1}^{h} d_{h,w}$, we provide a combinatorial interpretation of the formula relating Stirling numbers of the first kind and factorial numbers [7]:

$$\sum_{w=1}^{h} \left[\begin{array}{c} h \\ w \end{array} \right] = h!.$$

The factorial succession rule describes the construction of deco polyominoes. In the following sections, we show how we find a recursive description of a deco polyominoes subclass by making some changes in the succession rule.

3. The succession rule $(k) \rightsquigarrow (k)^{k-1}(k+1)$

Let us examine the following succession rule:

$$\left\{ \begin{array}{l} (2) \\ (k) \to (k)^{k-1}(k+1). \end{array} \right. \tag{3.1}$$

We obtain it by changing the factorial rule. The rule (3.1) defines a generating tree, which can be seen as a subtree of the generating tree of $\mathcal{D}$. In Figure 4, we show how we obtain the subtree by eliminating the appropriate branches. In this way, we only produce some objects of $\mathcal{D}$. Let $\mathcal{R}$ be the class of directed column-convex polyominoes P whose last column touches the upper side of the smallest rectangle containing P (Fig. 5). We observe that $\mathcal{R}$ is a subset of deco polyominoes. Let $\mathcal{R}_k$ denote the set of elements of $\mathcal{R}$ having vertical height $k-1$. Let us take a polyomino P in $\mathcal{R}_k$. Among the polyominoes of $\theta(P)$, exactly k belong to $\mathcal{R}$. One of them has vertical height k, and the others have vertical height $k-1$ (Fig. 6). Hence, the restriction of the operator θ to $\mathcal{R}$ is described by the succession rule (3.1). In the following example we show how to build a polyomino P having height 9 and width 5, step by step (see Fig. 7). The label in every cell represents the fertility that P assumes at the step the cell is added to it. P^i denotes the polyomino obtained after i steps, the starting point being P^0, the polyomino made up of only one cell labelled 2:

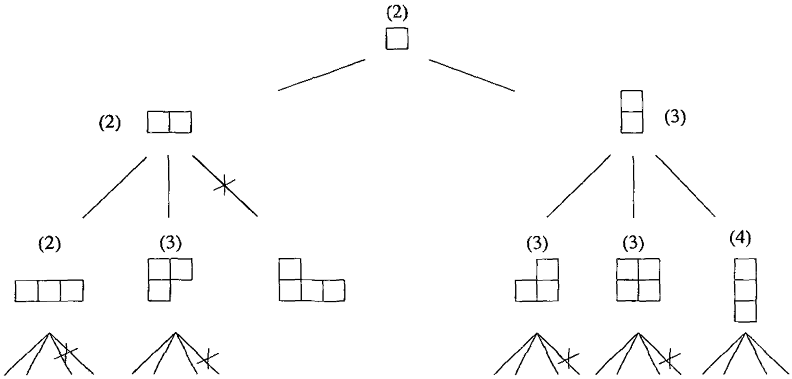

FIGURE 4. The generating tree of $\mathcal{R}$.

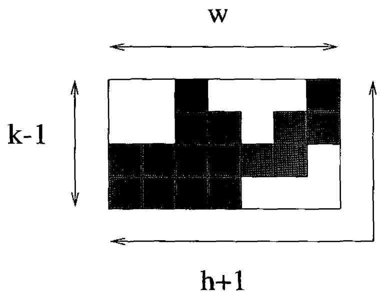

FIGURE 5. A polyomino of $\mathcal{R}$ having directed height h, vertical height $k-1$ and width w.

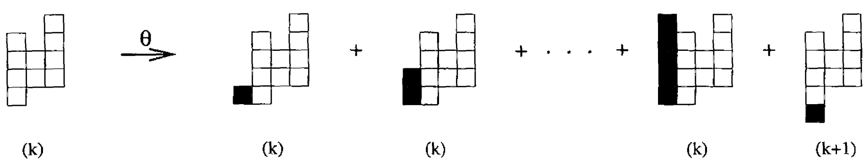

FIGURE 6. The operator θ applied to a polyomino of $\mathcal{R}$ having vertical height 5 and height 7.

step 1: a cell is added under the first one; P^1 is a column made up of two cells whose fertility is 3 (the new cell is labelled 3);
step 2: as in the previous step a cell is added under P^1 so that the fertility of P^2 is 4;

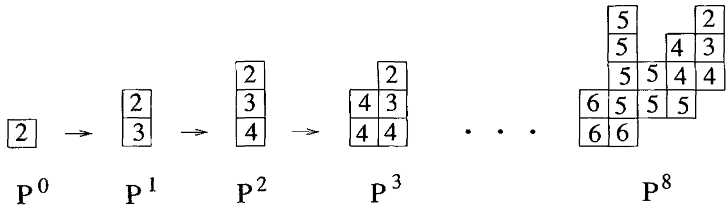

FIGURE 7. Constructing a polyomino P step by step.

step 3: a column of height 2 (*i.e.*, shorter than k) is added next to the only existing column and so the fertility of P^3 is 4 again.

⋮

step 8: a column of height 2 is added next to the first column and so the fertility of P is 6.

It can be seen that the fertility is the difference between the height and width of P plus two (see Fig. 5), and it only increases when we add a cell under the first column.

In order to enumerate the objects of the class, we denote the number of polyominoes of $\mathcal{R}$ having height $h-1$ (the minimal element of $\mathcal{R}$ has height equal to 1) and whose fertility is equal to k, by $r_{h-1}^{(k)}$. It follows that:

$$r_{h-1}^{(k)} = (k-1)r_{h-2}^{(k)} + r_{h-2}^{(k-1)} \tag{3.2}$$

with initial conditions

$$r_{h-1}^{(k)} = 0, \text{ for } k < 2, \text{ or } h < k-1, \text{ and } r_1^{(2)} = 1.$$

The *Stirling numbers of second kind* (which count partitions of $\{1, \dots, h\}$ in $k-1$ blocks) satisfy the relation (3.2), so that, by using the notation in [7], we obtain:

$$r_{h-1}^{(k)} = \left\{ \begin{matrix} h \\ k-1 \end{matrix} \right\}. \tag{3.3}$$

Since $k = h - w + 2$ and $h \geq w$ (see Fig. 5), the number $r_{h-1,w}$ of polyominoes in $\mathcal{R}$ having height $h-1$ and width w is

$$r_{h-1,w} = \left\{ \begin{matrix} h \\ h-w+1 \end{matrix} \right\}. \tag{3.4}$$

We can now enumerate the objects of this class by summing up all the numbers of polyominoes in $\mathcal{R}$ having height h, but different fertilities. The number r_{h-1} of

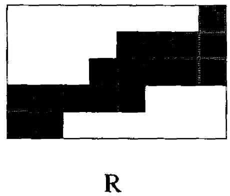

FIGURE 8. A polyomino of $\mathcal{P}$.

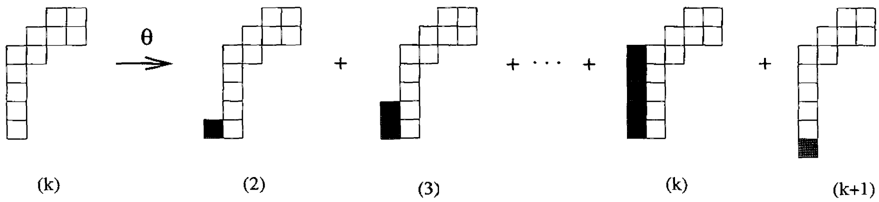

FIGURE 9. The operator θ applied to a polyomino of $\mathcal{P}$ having height 10.

polyominoes whose height is equal to $h-1$ is:

$$r_{h-1} = \sum_{k\geq 2} r_{h-1}^{(k)} = \sum_{k\geq 2} \left\{ \begin{array}{c} h \\ k-1 \end{array} \right\} = B_h$$

where B_h is the h-th *Bell number* (sequence M1484 in [12]).

4. THE SUCCESSION RULE $(k) \rightsquigarrow (2)(3)\dots(k)(k+1)$

In this section, we examine the following succession rule:

$$\left\{ \begin{array}{l} (2) \\ (k) \rightarrow (2)(3)\dots(k+1). \end{array} \right. \tag{4.1}$$

This rule is such that the relative generating tree is a subtree of the one obtained by the factorial rule and the class described by (4.1) is a deco subclass. Let $\mathcal{P}$ be the class of directed column-convex polyominoes contained in a rectangle R having the following property: the last column of any polyomino of $\mathcal{P}$ touches the upper side of R and each column cannot go over the column on its right (see Fig. 8). $\mathcal{P}$ is called the class of *parallelogram polyominoes* [3–6]. We denote the set of elements of $\mathcal{P}$ whose first column has height $k-1$ by $\mathcal{P}_k$. Let P be a polyomino in $\mathcal{P}_k$. Among the polyominoes of $\theta(P)$, exactly k belong to $\mathcal{P}$ and they have the height of the first column equal to $1, 2, \dots, k$ (Fig. 9). So, the restriction of the operator θ to $\mathcal{P}$ represents a combinatorial interpretation of the succession rule (4.1). In order to enumerate the class objects with respect to the height, we use the generating functions technique. Consider a parallelogram polyomino P.

We denote the height, width and height of its first column (fertility minus one) by $h(P), w(P)$ and $f(P)$, respectively. The generating function of $\mathcal{P}$ according to the mentioned parameters is:

$$F(x,y,t) = \sum_{p\in\mathcal{P}} x^{h(P)} y^{f(P)} t^{w(P)}.$$

In order to determine a functional equation for the generating function we take into account how operator θ changes the parameters height, width and fertility:

if θ adds a cell under the first column, we obtain $P' \in \theta(P)$, such that:

$$h(P') = h(P) + 1,\ w(P') = w(P) \text{ and } f(P') = f(P) + 1;$$

otherwise, if θ adds a column of height j on the left of the first column, we obtain:

$$h(P') = h(P) + 1,\ w(P') = w(P) + 1 \text{ and } f(P') = j$$

with $j = 1, 2, \dots, f(P)$.

Thus,

$$\begin{aligned}\theta(F(x,y,t)) &= F(x,y,t) - xyt \\ &= \sum_{p\in\mathcal{P}} \left(x^{h(P)+1} y^{f(P)+1} t^{w(P)} + \sum_{k=1}^{f(P)} \left(x^{h(P)+1} y^k t^{w(P)+1} \right) \right)\end{aligned}$$

from which we obtain

$$F(x,y,t) = xyt + xyF(x,y,t) + \frac{xyt}{1-y}[F(x,1,t) - F(x,y,t)]$$

and so

$$F(x,y,t)\left[1 - xy + \frac{xyt}{1-y}\right] = xyt + \frac{xyt}{1-y} F(x,1,t).$$

We use the following method ([9], Ex. 2.2.1.4 and Ex. 2.2.1.11) to solve this functional equation: since $1 - xy + \frac{xyt}{1-y} = 0$ implies $xyt + \frac{xyt}{1-y}F(x,1,t) = 0$, we determine $F(x,1,t)$, by substituting the solution y_0 of the first equation in the second one. We choose the solution

$$y_0 = \frac{1 + x - xt - \sqrt{1 - 2x - 2xt + x^2 - 2x^2t + x^2t^2}}{2x}$$

because the conjugate root is not suitable. In fact it gives $F(0,1,t) = \infty$. Therefore:

$$F(x,1,t) = \frac{1 - x - xt - \sqrt{1 - 2x - 2xt + x^2 - 2x^2t + x^2t^2}}{2x} \tag{4.2}$$

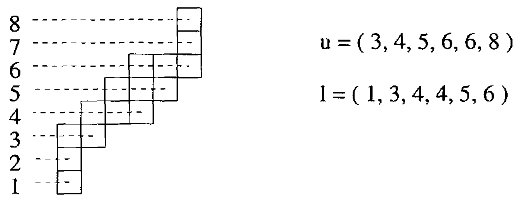

FIGURE 10. A $\mathcal{D}^{(2)}$ parallelogram polyomino.

and by setting $t = 1$, we have:

$$F(x, 1, 1) = \frac{1 - 2x - \sqrt{1 - 4x}}{2x}$$

which is the generating function of the *Catalan numbers*. Consequently, the number p_h of parallelogram polyominoes having height h is the h-th Catalan number:

$$p_h = \frac{1}{h+1}\binom{2h}{h}.$$

Finally, by using the Lagrange inversion formula from (4.2), we obtain an expression for the number $p_{h,w}$ of parallelogram polyominoes having height h and width w:

$$p_{h,w} = \frac{1}{h}\binom{h}{w}\binom{h}{w-1}.$$

These numbers are known as *Narayana numbers* [11] and their relation with Catalan numbers is a classical result [10].

5. THE SUCCESSION RULE $(k) \rightsquigarrow (2)^{k-1}(k+1)$

In this section we are interested in examining the following succession rule:

$$\begin{cases} (2) \\ (k) \to (2)^{k-1}(k+1). \end{cases} \tag{5.1}$$

We denote by $l = (l_1, \dots, l_w)$ and $u = (u_1, \dots, u_w)$ the two vectors whose elements l_i and u_i are the level of the lowest and uppermost cells in the i-th column, respectively (see Fig. 10). Let $\mathcal{D}^{(2)}$ be the class of parallelogram polyominoes satisfying the conditions:

$$u_i \le l_{i+2},\ 1 \le i < w - 1.$$

Thus, each element of $\mathcal{D}^{(2)}$ has no more than two columns of height greater or equal to 2 starting at the same level. If we restrict the operator θ to polyominoes of $\mathcal{D}^{(2)}$, the fertility of any element is $k = l_2 - l_1 + 2$ for $w > 1$, otherwise $k = h + 1$.

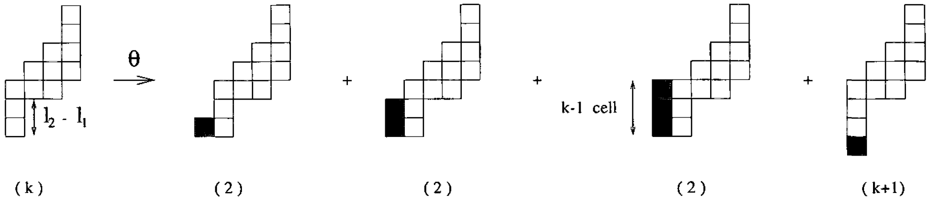

FIGURE 11. The operator θ applied to a parallelogram polyomino of $\mathcal{D}^{(2)}$ having height 10.

Moreover, by applying θ to a polyomino P having fertility k, one polyomino in $\theta(P)$ has fertility $k+1$, while the others have fertility 2 (see Fig. 11). Therefore the rule (5.1) describes the recursive construction of the polyominoes which belong to $\mathcal{D}^{(2)}$. We now go on to determine the generating function of $\mathcal{D}^{(2)}$ according to the parameters $h(P)$, $w(P)$ and $f(P)$ which denote the height, width and fertility of P, respectively.

If θ adds a cell under the first column, we obtain $P' \in \theta(P)$ such that:

$$h(P') = h(P) + 1, \ w(P') = w(P) \text{ and } f(P') = f(P) + 1;$$

otherwise, if θ adds a column of height j on the left of the first column, we obtain:

$$h(P') = h(P) + 1, \ w(P') = w(P) + 1 \text{ and } f(P') = 2,$$

with $j = 1, 2, \ldots, f(P) - 1$.

The generating function is:

$$F(x, y, t) = \sum_{p \in \mathcal{D}^{(2)}} x^{h(P)} y^{f(P)} t^{w(P)}.$$

By applying the operator θ, we obtain:

$$\theta(F(x, y, t)) = F(x, y, t) - xy^2 t = \sum_{P \in D^{(2)}} x^{h(P)+1} y^{f(P)+1} t^{w(P)} + \sum_{P \in D^{(2)}} \left(x^{h(P)+1} y^2 t^{w(P)+1} \right) (f(P) - 1).$$

Therefore, the generating function satisfies the following functional equation:

$$F(x, y, t) = xy^2 t + xyF(x, y, t) + xy^2 t \left[\frac{\partial F(x, y, t)}{\partial y} \right]_{y=1} - xy^2 t F(x, 1, t); \quad (5.2)$$

by deriving with respect to y and setting $y = 1$, we get:

$$\left[\frac{\partial F(x,y,t)}{\partial y}\right]_{y=1}(1-x-2xt) = 2xt + xF(x,1,t)(1-2t). \tag{5.3}$$

By solving the system formed by (5.2), in which $y = 1$, and by (5.3) we can deduce that:

$$F(x,t) = \frac{tx(1-x)}{1-(2+t)x+x^2}\,. \tag{5.4}$$

By setting $t = 1$ in (5.4), we have

$$F(x,1) = \frac{x-x^2}{1-3x+x^2}\,. \tag{5.5}$$

Notice that odd index Fibonacci numbers have the generating function (5.5). So, the number of $\mathcal{D}^{(2)}$ polyominoes having height h is the $(2h-1)$-th Fibonacci number. Let $F(x,y) = \sum_{h\geq 1,\, w\geq 1} a_{h,w}x^h t^w$. We determine the number of $\mathcal{D}^{(2)}$ polyominoes of height h and width w by (5.4). We can extract the coefficient of t^w in formula (5.4) as follows:

$$[t^w]F(x,t) = \frac{x}{1-x}\left[t^{(w-1)}\right]\frac{1}{1-\frac{tx}{(1-x)^2}}\,.$$

We then determine the coefficient of x^h,

$$\left[x^{h-w}\right]\frac{1}{(1-x)^{2w-1}},$$

and we finally obtain:

$$a_{h,w} = \binom{h+w-2}{h-w}.$$

Remark 5.1. *The succession rule:*

$$\begin{cases} (c) \\ (k) \to (c)^{k-1}(k+1) \end{cases} \tag{5.6}$$

generalizes the rule (5.1). $\mathcal{D}^{(c)}$ is the class of polyominoes satisfying the conditions:

$$u_i \leq l_{i+2} + c - 2,\; 1 \leq i < w-1.$$

Each element of $\mathcal{D}^{(c)}$ has no more than two columns of height greater or equal to c starting at the same level. By considering the restriction of θ to the polyominoes of $\mathcal{D}^{(c)}$, the fertility of any element is $k = l_2 - l_1 + c$ for $w > 1$, otherwise $k = h + 1$. If we determine the number of the elements of the class with respect to the height parameter in the case $c = 3$, we find a known sequence (sequence M2847 in [12]) which enumerates the order-consecutive partitions *[8].*

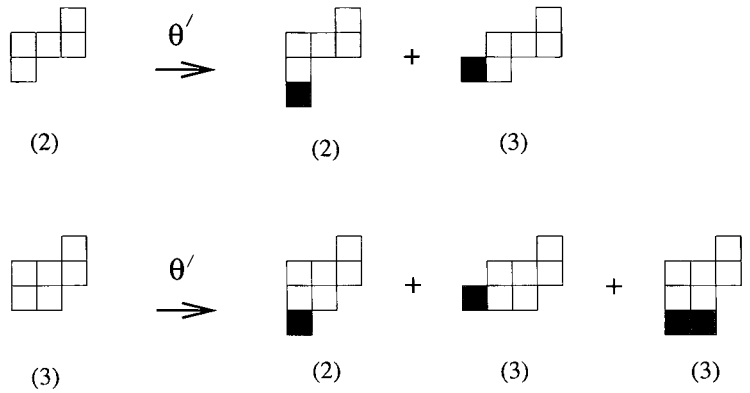

FIGURE 12. A new recursive construction for $\mathcal{D}^{(2)}$.

Remark 5.2. *We point out that the class $\mathcal{D}^{(2)}$ can be also constructed by means of a different operator, say θ'. We divide the class into two subclasses:*

- *the polyominoes whose first two columns start at the same level (they have fertility 3);*
- *the other ones (they have fertility 2).*

If P has fertility 3, θ' adds a cell next to the first column, under the first column, or a row made up of two cells under the first row; if P has fertility 2, θ' adds a cell next to or under the first column (see Fig. 12). This construction is described by the following succession rules.

$$\begin{cases} (2) \\ (2) \rightarrow (2)(3) \\ (3) \rightarrow (2)(3)(3) \end{cases} \tag{5.7}$$

that we call finite *as it does not depend on k. It is not hard to verify that we obtain again the odd index Fibonacci numbers generating function. Moreover we can easily extend this construction to the class $\mathcal{D}^{(c)}$ for which we have the following finite succession rules:*

$$\begin{cases} (c) \\ (c) \rightarrow (c)^{c-1}(c+1) \\ (c+1) \rightarrow (c)^{c-1}(c+1)^2. \end{cases} \tag{5.8}$$

6. CONCLUSIONS

We showed that the factorial succession rule and the succession rules, obtained by making some changes in it, are related to transcendental, algebraic or rational generating functions.

Thus, an interesting question would be whether, given a succession rule, we could immediatly deduce the kind of generating function only on the basis of the labels (without having to do any computation). As a step in this direction, we are interested in dividing the succession rules into classes according to whether their generating functions are transcendental, algebraic or rational. To this regard, as far as we know, all the finite succession rules give rational generating functions.

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