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ASSOCIATIVE CLOSURE AND PERIODICITY OF ω-WORDS (*)

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Abstract. – *We investigate some shuffle-like operations on ω-words and ω-languages. The operations are introduced using a uniform method based on the notion of an ω-trajectory. Our main results concern associativity. An interconnection between associative closure and periodicity will be exhibited. This provides characterizations of periodic and ultimately periodic ω-words. Finally, a remarkable property of the Fibonacci ω-word is proved, i.e., the associative closure of this ω-word properly contains all periodic ω-words.* © Elsevier, Paris

Résumé. – *Nous étudions des opérations similaires à l'opération de mélange sur des mots infinis et sur des ω-langages. Les opérations sont introduites par une méthode uniforme fondée sur la notion d'ω-trajectoire. Nos résultats principaux concernent l'associativité. Une connexion entre la fermeture associative et la périodicité est établie. Elle fournit des caractérisations de mots infinis périodiques et ultimement périodiques. Finalement, nous prouvons une propriété remarquable du mot infini de Fibonacci, à savoir que la fermeture associative de ce mot contient strictement tous les mots infinis périodiques.* © Elsevier, Paris

1. PRELIMINARIES

Parallel composition of words and languages appears as a fundamental operation in parallel computation and in the theory of concurrency. Usually, this operation is modelled by the shuffle operation or restrictions of this operation, such as literal shuffle, insertion, left-merge, or the infiltration product, [6].

We investigate some shuffle-like operations on ω-words and ω-languages. The reader is referred to [11] for an early approach of this problem in connexion to parallel composition of concurrent processes. The shuffle-like

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operations considered below are define using syntactic constraints on the ω-shuffle operation.

The constraints are based on the notion of an *ω-trajectory* and describe the general strategy to switch from one ω-word to another ω-word. Roughly speaking, an ω-trajectory is a brokenline in plane, starting in the origin and continuing parallel with the axis Ox or Oy. The broken line can change its direction only at points with nonnegative integer coordinates. An ω-trajectory defines how to move from an ω-word to another ω-word when carrying out the shuffle operation. Each set T of ω-trajectories defines in a natural way a shuffle operation over T. Given a set T of ω-trajectories the operation of shuffle over T is not necessarily an associative operation. However, for each set T there exists a smallest set of trajectories $\overline{T}$ such that $\overline{T}$ contains T and, moreover, the operation of shuffle over $\overline{T}$ is associative. The set $\overline{T}$ is referred to as the associative closure of T. We show that the associative closure of some very simple finite sets leads to the set of periodic or to the set of ultimately periodic ω-words.

The set of nonnegative integers is denoted by ω. The set of all subsets of a set A is denoted by $\mathcal{P}(A)$.

Let Σ be an alphabet, i.e., a finite nonempty set of elements called *letters*. The free monoid generated by Σ is denoted by Σ^*. Elements in Σ^* are referred to as *words*. The empty word is denoted by λ.

If $w \in \Sigma^*$, then $|w|$ is the length of w. Note that $|\lambda| = 0$. If $a \in \Sigma$ and $w \in \Sigma^*$, then $|w|_a$ denotes the number of occurrences of the letter a in the word w. The *mirror* of a word $w = a_1a_2 \ldots a_n$, where a_i are letters, $1 \leq i \leq n$, is $\tilde{w} = a_n \ldots a_2a_1$ and $\tilde{\lambda} = \lambda$. A word w is a *palindrome* iff $\tilde{w} = w$.

Let Σ be an alphabet. An *ω-word* over Σ is a function $f : \omega \longrightarrow \Sigma$. Usually, the ω-word defined by f is denoted as the infinite sequence $f(0)f(1)f(2)f(3)f(4)\ldots$ An ω-word w is *ultimately periodic* iff $w = \alpha vvvvv\ldots$, where α is a (finite) word, possibly empty, and v is a nonempty word. In this case w is denoted as αv^ω. An ω-word w is *periodic* iff $w = vvv\ldots$ for some nonempty word $v \in \Sigma^*$. In this case w is denoted as v^ω. The set of all ω-words over Σ is denoted by Σ^ω. An *ω-language* is a subset L of Σ^ω. The reader is referred to [12], [14] and [15] for general results on ω-words.

We now recall some operations from formal language theory that simulate the parallel composition of words. The *shuffle* operation, denoted by $\sqcup\!\sqcup$, is defined recursively by:

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) \cup b(au \sqcup\!\sqcup v),$$

and

$$u \sqcup\!\sqcup \lambda = \lambda \sqcup\!\sqcup u = \{u\},$$

where $u,\ v \in \Sigma^*$ and $a,\ b \in \Sigma$.

The shuffle operation is extended in a natural way to languages: the *shuffle* of two languages L_1 and L_2 is:

$$L_1 \sqcup\!\sqcup L_2 = \bigcup_{u \in L_1, v \in L_2} u \sqcup\!\sqcup v.$$

The *literal shuffle*, denoted by $\sqcup\!\sqcup_l$, is defined as:

$$a_1 a_2 \ldots a_n \sqcup\!\sqcup_l b_1 b_2 \ldots b_m = \begin{cases} a_1 b_1 a_2 b_2 \ldots a_n b_n b_{n+1} \ldots b_m, & \text{if } n \leq m, \\ a_1 b_1 a_2 b_2 \ldots a_m b_m a_{m+1} \ldots a_n, & \text{if } m < n, \end{cases}$$

where $a_i, b_j \in \Sigma$.

$$u \sqcup\!\sqcup_l \lambda = \lambda \sqcup\!\sqcup_l u = \{u\},$$

where $u \in \Sigma^*$.

2. ω-TRAJECTORIES

In this section we introduce the notions of ω-trajectory and shuffle on ω-trajectories. The shuffle of two ω-words has a natural geometrical interpretation related to lattice points in the plane (points with nonnegative integer coordinates) and with a certain "walk" in the plane defined by each ω-trajectory.

Let $V = \{r, u\}$ be the set of *versors* in the plane: r stands for the *right* direction, whereas, u stands for the *up* direction.

DEFINITION 2.1: *An ω-trajectory is an element t, $t \in V^\omega$. A set T, $T \subseteq V^\omega$, is called a set of ω-trajectories.* □

Let Σ be an alphabet and let t be an ω-trajectory, $t = t_0 t_1 t_2 \ldots$, where $t_i \in V, i \geq 0$. Let α, β be two ω-word sover Σ, $\alpha = a_0 a_1 a_2 \ldots, \beta = b_0 b_1 b_2 \ldots$, where $a_i, b_j \in \Sigma, i, j \geq 0$.

DEFINITION 2.2: *The shuffle of* α *with* β *on the* ω*-trajectory* t*, denoted* $\alpha \sqcup\!\sqcup_t \beta$*, is defined as follows:*

$\alpha \sqcup\!\sqcup_t \beta = c_0 c_1 c_2 \ldots$*, where, if* $|t_0 t_1 t_2 \ldots t_i|_r = k_1$ *and* $|t_0 t_1 t_2 \ldots t_i|_u = k_2$*, then*

$$\begin{cases} a_{k_1 - 1}, & \text{if } t_i = r, \\ b_{k_2 - 1}, & \text{if } t_i = u. \end{cases}$$

□

If T is a set of ω-trajectories, the *shuffle of* α *with* β *on the set* T *of* ω*-trajectories*, denoted $\alpha \sqcup\!\sqcup_T \beta$, is:

$$\alpha \sqcup\!\sqcup_T \beta = \bigcup_{t \in T} \alpha \sqcup\!\sqcup_t \beta.$$

The above operation is extended to ω-languages over Σ, if$L_1, L_2 \subseteq \Sigma^\omega$, then:

$$L_1 \sqcup\!\sqcup_T L_2 = \bigcup_{\alpha \in L_1, \beta \in L_2} \alpha \sqcup\!\sqcup_T \beta.$$

Notation. If T is V^ω then $\sqcup\!\sqcup_T$ is denoted by $\sqcup\!\sqcup_\omega$.

Example 2.1: Let α and β be the ω-words $\alpha = a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 \ldots$, $\beta = b_0 b_1 b_2 b_3 b_4 \ldots$ and assume that $t = r^2 u^3 r^5 uru \ldots$. The shuffle of α with β on the trajectory t is:

$$\alpha \sqcup\!\sqcup_t \beta = \{a_0 a_1 b_0 b_1 b_2 a_2 a_3 a_4 a_5 a_6 b_3 a_7 b_4 \ldots\}.$$

The result has the following geometrical interpretation (see Fig. 1): the trajectory t defines a broken line (the thinner line in Figure 1) starting in theorigin and continuing one unit right or up, depending on the current letter of t. In our case, first there are two units right, then three units up, then five units right, etc. Assign α on the Ox axis and β on the Oy axis of the plane. The result can be read following the broken line defined by the trajectory t, that is, if being in a lattice point of the trajectory, (the corner of a unit square) and if the trajectory is going right, then one should pick up the corresponding letter from α, otherwise, if the trajectory is going up, then one should add to the result the corresponding letter from β. Hence, the trajectory t defines a broken line in the plane, on which one has "to walk" starting from the origin O. In each lattice point one has to follow one of the versors r or u, according to the definition of t.

Assume now that t' is another trajectory, say: $t' = ur^5u^4r^3 \ldots$. The trajectory t' is depicted in Figure 1 by the bolder broken line.

Observe that:

$$\alpha \sqcup\!\sqcup_{t'} \beta = \{b_0a_0a_1a_2a_3a_4b_1b_2b_3b_4a_5a_6a_7\ldots\}.$$

Consider the set of trajectories $T = \{t, t'\}$. The shuffle of α with β on the set T of trajectories is:

$$\alpha \sqcup\!\sqcup_T \beta = \{a_0a_1b_0b_1b_2a_2a_3a_4a_5a_6b_3a_7b_4\ldots,\\ b_0a_0a_1a_2a_3a_4b_1b_2b_3b_4a_5a_6a_7\ldots\}.$$

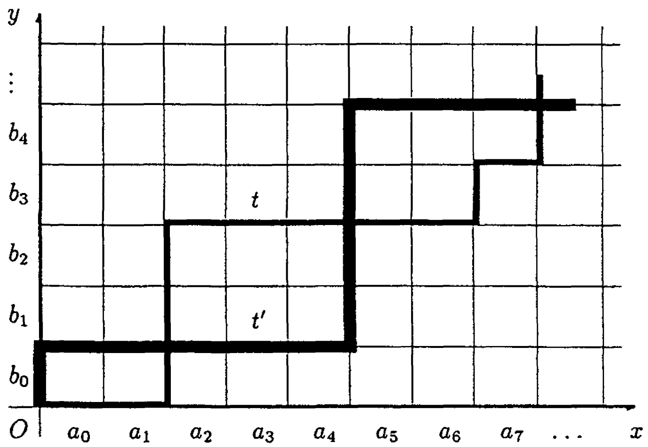

Figure 1.

Remark 2.1: The shuffle on (finite) trajectories of (finite) words is investigated in [9]. In this case a trajectory is an element t, $t \in V^*$.

Let Σ be an alphabet and let t be a trajectory, $t = t_0t_1\ldots t_n$, where $t_i \in V, 1 \le i \le n$. Let α, β be two words over Σ, $\alpha = a_0a_1\ldots a_p, \beta = b_0b_1\ldots b_q$, where $a_i, b_j \in \Sigma, 0 \le i \le p$ and $0 \le j \le q$.

The shuffle of α with β on the trajectory t, denoted $\alpha \sqcup\!\sqcup_t \beta$, is defined as follows:

if $|\alpha| \neq |t|_r$ or $|\beta| \neq |t|_u$, then $\alpha \sqcup\!\sqcup_t \beta = \emptyset$, else

$\alpha \sqcup\!\sqcup_t \beta = c_0c_1c_2 \ldots c_{p+q+2}$, where, if $|t_0t_1 \ldots t_i|_r = k_1$ and $|t_0t_1 \ldots t_i|_u = k_2$, then

$$c_i = \begin{cases} a_{k_1-1}, & \text{if } t_i = r, \\ b_{k_2-1}, & \text{if } t_i = u. \end{cases}$$

Observe that there is an important distinction between the finite case, i.e., the shuffle on trajectories, and the infinite case, i.e., the shuffle on ω-trajectories: sometimes the result of shuffling of two words α and β on a trajectory t can be empty whereas the shuffle of two ω-words over an ω-trajectory is always nonempty and consists of only one ω-word.

3. ASSOCIATIVITY

The main results in this paper deal with associativity. After a few general remarks, we restrict our attention to the set V_+^ω of ω-trajectories t such that both r and u occur infinitely many times in t. (It will become apparent below why this restriction is important.) It turns out that associativity can be viewed as stability under four particular operations, referred to as $\diamondsuit$-operations. This characterization exhibits a surprising interconnection between associativity and periodicity, which in our opinion is of direct importance also for thebasic theory of ω-words.

DEFINITION 3.1: *A set T of ω-trajectories is associative iff the operation $\sqcup\!\sqcup_T$ is associative, i.e.,*

$$(\alpha \sqcup\!\sqcup_T \beta) \sqcup\!\sqcup_T \gamma = \alpha \sqcup\!\sqcup_T (\beta \sqcup\!\sqcup_T \gamma),$$

for all $\alpha, \beta, \gamma \in \Sigma^\omega$. □

The following sets of ω-trajectories are associative:

1. $T = \{r, u\}^\omega$.
2. $T = \{t \in V^\omega \mid |t|_r < \infty\}$.
3. $T = \{\alpha_0\beta_0\alpha_1\beta_1 \ldots \mid \alpha_i \in r^*,\ \beta_i \in u^*$ and, α_i, β_i are of even length, $i \geq 0\}$.

Nonassociative sets of ω-trajectories are for instance:

1. $T = (ru)^\omega$.
2. $T = \{t \in V^\omega \mid t$ is a Sturmian ω-word $\}$.
3. $T = \{w_0w_1w_2 \ldots \mid w_i \in L\}$, where$L = \{r^nu^n \mid n \geq 0\}$.

Observe that for each set of ω-trajectories, T, the operation $\sqcup\!\sqcup_T$ is distributive over union both on the right and on the left side. Moreover, we adjoin to V^ω a unit element with respect toeach $\sqcup\!\sqcup_T$, denoted 1. Note that 1 is not an ω-word. Hence, we obtain the following result:

PROPOSITION 3.1: *If T is an associative set of trajectories, then for any alphabet Σ,*

$$\mathcal{S} = (\mathcal{P}(\Sigma^*), \cup, \sqcup\!\sqcup_T, \emptyset, 1)$$

is a semiring.

Proof: One can easily verify the axioms of a semiring, see [4]or [5]. □

The following proposition provides a characterization of those sets of ω-trajectories that are associative.

DEFINITION 3.2: *Let D be the set $D = \{x, y, z\}$. Define the substitutions σ, τ : $\mathcal{P}(V^\omega) \longrightarrow \mathcal{P}(D^\omega)$, as follows:*

$$\sigma(r) = \{x, y\}, \quad \sigma(u) = \{z\},$$

$$\tau(r) = \{x\}, \quad \tau(u) = \{y, z\}.$$

Consider the morphisms φ and ψ, φ, $\psi : V^\omega \longrightarrow D^\omega$, defined as:

$$\varphi(r) = x, \quad \varphi(u) = y,$$
$$\psi(r) = y, \quad \psi(u) = z.$$

□

PROPOSITION 3.2: *Let T be a set of ω-trajectories. The following conditions are equivalent:*

(i) *T is an associative set of ω-trajectories.*

(ii) $\sigma(T) \cap (\varphi(T) \sqcup\!\sqcup_\omega z^\omega) = \tau(T) \cap (\psi(T) \sqcup\!\sqcup_\omega x^\omega)$.

Proof: (i) $\Rightarrow$ (ii). Assume that T is an associative set of ω-trajectories. Consider w such that $w \in \sigma(T) \cap (\varphi(T) \sqcup\!\sqcup_\omega z^\omega)$. It follows that there exists t_1, $t_1 \in T$, such that $w \in \sigma(t_1)$ and there exists t, $t \in T$, such that $w \in \varphi(t) \sqcup\!\sqcup_\omega z^\omega$. Assume that

$$t_1 = r^{i_0} u^{j_1} r^{i_1} \ldots u^{j_n} r^{i_n} \ldots,$$

for some nonnegative integers i_g, j_h, $0 \le g, 1 \le h$. From the definition of σ we conclude that

$$w \in \{x, y\}^{i_0} z^{j_1} \{x, y\}^{i_1} \ldots z^{j_n} \{x, y\}^{i_n} \ldots$$

Since $w \in \varphi(t) ⧢_\omega z^\omega$, it follows that $t = s_0 s_1 \ldots s_n \ldots$, such that $s_k \in V^*$ and $|s_k| = i_k$ for all k, $0 \leq k$. Therefore,

$$w \in (x^\omega ⧢_t y^\omega) ⧢_{t_1} z^\omega.$$

Because T is associative, there are t' and t'_1 in T such that

$$(x^\omega ⧢_t y^\omega) ⧢_{t_1} z^\omega = x^\omega ⧢_{t'} (y^\omega ⧢_{t'_1} z^\omega).$$

Hence, we obtain that $w \in x^\omega ⧢_{t'} (y^\omega ⧢_{t'_1} z^\omega)$, for some t' and t'_1 in T. Now, it is easy to observe that this simplies that $w \in \tau(T) \cap (\psi(T) ⧢_\omega x^\omega)$. Thus, $\sigma(T) \cap (\varphi(T) ⧢_\omega z^\omega) \subseteq \tau(T) \cap (\psi(T) ⧢_\omega x^\omega)$. The converse inclusion is analogous. Therefore, the equality from (ii) is true.

(ii) $\Rightarrow$ (i). Let Σ be an alphabet and let α, β, γ be ω-words over Σ. Consider an ω-word w, such that $w \in (\alpha ⧢_T \beta) ⧢_T \gamma$. There exist t and t_1 in T such that $w \in (\alpha ⧢_t \beta) ⧢_{t_1} \gamma$. Let v be the ω-word obtained from w by replacing each letter from α by x, each letter from β by y and each letter from γ by z. Observe that v is in $\sigma(t_1)$ and also in $\varphi(t) ⧢_\omega z^\omega$. Therefore, $v \in \sigma(T) \cap (\varphi(T) ⧢_\omega z^\omega)$. By our assumption, it follows that $v \in \tau(T) \cap (\psi(T) ⧢_\omega x^\omega)$. Hence, there are t' and t'_1 in T such that $v \in \tau(t') \cap (\psi(t'_1) ⧢_\omega x^\omega)$. Note that this means that $v \in x^\omega ⧢_{t'} (y^\omega ⧢_{t'_1} z^\omega)$. Hence, it is easy to see that $w \in \alpha ⧢_{t'} (\beta ⧢_{t''} \gamma)$, i.e., $w \in \alpha ⧢_T (\beta ⧢_T \gamma)$. Thus, $(\alpha ⧢_T \beta) ⧢_T \gamma \subseteq \alpha ⧢_T (\beta ⧢_T \gamma)$. The converse inclusion is analogous. Therefore, for all $\alpha, \beta, \gamma \in \Sigma^\omega$,

$$(\alpha ⧢_T \beta) ⧢_T \gamma = \alpha ⧢_T (\beta ⧢_T \gamma).$$

Thus, T is an associative set of ω-trajectories. □

Now we introduce the notion of the associative closure of an arbitrary set of ω-trajectories.

Notation. Let $\mathcal{A}$ be the family of all associative sets of ω-trajectories.

We omit the proof of the following proposition.

Proposition 3.3: *If $(T_\imath)_{\imath \in I}$ is a family of associative sets of ω-trajectories, then,*

$$\bigcap_{\imath \in I} T_\imath,$$

is an associative set of ω-trajectories. □

DEFINITION 3.3: *Let T be an arbitrary set of ω-trajectories. The associative closure of T, denoted $\overline{T}$, is*

$$\overline{T} = \bigcap_{T \subseteq T', T' \in \mathcal{A}} T'. \qquad \square$$

Observe that for every $T \subseteq \{r, u\}^*$, $\overline{T}$ is an associative set of ω-trajectories and, moreover, $\overline{T}$ is the smallest associative set of ω-trajectories that contains T.

Remark 3.1: The function $^-, ^- : \mathcal{P}(V^\omega) \longrightarrow \mathcal{P}(V^\omega)$ defined as above is a closure operator. $\square$

Notation. Let V_+^ω be the set of all ω-trajectories $t \in V^\omega$ such that t contains infinitely many occurrences of both r and u.

Now we give a characterization of an associative set of ω-trajectories from V_+^ω. This is useful in finding an alternative definition of the associative closure of a set of ω-trajectories and also to prove some other properties related to associativity.

However, this characterization is valid only for sets of ω-trajectories from V_+^ω and not for the general case, i.e., not for sets of ω-trajectories from V^ω.

DEFINITION 3.4: *Let W be the alphabet $W = \{x, y, z\}$ and consider the following four morphisms, ρ_i, $1 \le i \le 4$, where $\rho_i : W^\omega \longrightarrow V_+^\omega$, $1 \le i \le 4$, and*

$$\begin{array}{lll} \rho_1(x) = \lambda, & \rho_1(y) = r, & \rho_1(z) = u, \\ \rho_2(x) = r, & \rho_2(y) = u, & \rho_2(z) = u, \\ \rho_3(x) = r, & \rho_3(y) = u, & \rho_3(z) = \lambda, \\ \rho_4(x) = r, & \rho_4(y) = r, & \rho_4(z) = u. \end{array}$$

where λ denotes the empty word. $\square$

Next, we consider four operations on the set V_+^ω of ω-trajectories.

DEFINITION 3.5: *Let $\diamondsuit_i$, $1 \le i \le 4$, be the following operations on V_+^ω:*

$$\diamondsuit_i : V_+^\omega \times V_+^\omega \longrightarrow V_+^\omega, \quad 1 \le i \le 4,$$

defined, for all $t, t' \in V_+^\omega$, by:

1. $\diamondsuit_1(t, t') = \rho_1((x^\omega \sqcup\!\sqcup_t y^\omega) \sqcup\!\sqcup_{t'} z^\omega)$,
2. $\diamondsuit_2(t, t') = \rho_2((x^\omega \sqcup\!\sqcup_t y^\omega) \sqcup\!\sqcup_{t'} z^\omega)$,
3. $\diamondsuit_3(t', t) = \rho_3(x^\omega \sqcup\!\sqcup_{t'} (y^\omega \sqcup\!\sqcup_t z^\omega))$,
4. $\diamondsuit_4(t', t) = \rho_4(x^\omega \sqcup\!\sqcup_{t'} (y^\omega \sqcup\!\sqcup_t z^\omega))$. $\square$

Remark 3.2: Here we like to point out why we restricted our attention to the set V_+^ω and not to the general case V^ω. The operation $\diamondsuit_1$ is defined to produce the ω-trajectory t'_1 (see the above proof). However, if T contains a trajectory t that is not inV_+^ω, then $\diamondsuit_1(t,t)$ is not necessarily in V^ω. For instance, if $t = rur^\omega$, then $\diamondsuit_1(t,t) = ur \notin V^\omega$. Thus the operation $\diamondsuit_1$ is not well defined. A similar phenomenon happens with the operation $\diamondsuit_3$. □

DEFINITION 3.6: *A set $T \subseteq V_+^\omega$ is stable under $\diamondsuit$-operations iff, for all $t_1, t_2 \in T$, it follows that $\diamondsuit_i(t_1,t_2) \in T$, $1 \leq i \leq 4$.* □

PROPOSITION 3.4: *Let T be a set of ω-trajectories, $T \subseteq V_+^\omega$. The following assertions are equivalent:*

(i) T is an associative set of ω-trajectories.

(ii) T is stable under $\diamondsuit$-operations.

Proof: The idea of the proof is that for two ω-trajectories t, t' and for the ω-words x^ω, y^ω and z^ω, the operation $\diamondsuit_1$ applied to t and t' gives the (unique) trajectory t'_1 that occurs in the equality:

$$(x^\omega \sqcup\!\sqcup_t y^\omega) \sqcup\!\sqcup_{t'} z^\omega = x^\omega \sqcup\!\sqcup_{t_1} (y^\omega \sqcup\!\sqcup_{t'_1} z^\omega).$$

The operation $\diamondsuit_2$ gives the (unique) ω-trajectory t_1 that occurrs in the above equality. Analogously, $\diamondsuit_3$ applied to t_1 and t'_1 gives the (unique) trajectory t whereas $\diamondsuit_4(t_1, t'_1) = t'$.

(i) $\Longrightarrow$ (ii) Assume that T_+^ω is an associative set of ω-trajectories. Since T is associative, there are t_1 and t'_1 in T such that

$$(x^\omega \sqcup\!\sqcup_t y^\omega) \sqcup\!\sqcup_{t'} z^\omega = x^\omega \sqcup\!\sqcup_{t_1} (y^\omega \sqcup\!\sqcup_{t'_1} z^\omega).$$

Hence,

$$\begin{aligned}\diamondsuit_1(t,t') &= \rho_1((x^\omega \sqcup\!\sqcup_t y^\omega) \sqcup\!\sqcup_{t'} z^\omega) = \rho_1(x^\omega \sqcup\!\sqcup_{t_1} (y^\omega \sqcup\!\sqcup_{t'_1} z^\omega))\\ &= r^\omega \sqcup\!\sqcup_{t'_1} u^\omega = t'_1 \in T.\end{aligned}$$

Thus T is stable for $\diamondsuit_1$.

Analogously,

$$\begin{aligned}\diamondsuit_2(t,t') &= \rho_2((x^\omega \sqcup\!\sqcup_t y^\omega) \sqcup\!\sqcup_{t'} z^\omega) = \rho_2(x^\omega \sqcup\!\sqcup_{t_1} (y^\omega \sqcup\!\sqcup_{t'_1} z^\omega))\\ &= r^\omega \sqcup\!\sqcup_{t_1} u^\omega = t_1 \in T.\end{aligned}$$

Hence T is stable for $\diamondsuit_2$.

A similar proof shows that T is also stable for $\diamondsuit_3$ and $\diamondsuit_4$.

(ii) $\Longrightarrow$ (i) Now assume that $T \subseteq V_+^\omega$ is a set of ω-trajectories stable under $\diamondsuit_i$, $1 \leq i \leq 4$.

Let Σ be an alphabet and consider $\alpha, \beta, \gamma \in \Sigma^\omega$ and $t, t' \in T$. Note that $\diamondsuit_1(t, t') = t'_1$ and $\diamondsuit_2(t, t') = t_1$, for some t_1, $t'_1 \in T$. Now it is easy to see that

$$(\alpha \sqcup\!\sqcup_t \beta) \sqcup\!\sqcup_{t'} \gamma = \alpha \sqcup\!\sqcup_{t_1} (\beta \sqcup\!\sqcup_{t'_1} \gamma).$$

Thus, we obtain

$$(\alpha \sqcup\!\sqcup_T \beta) \sqcup\!\sqcup_T \gamma \subseteq \alpha \sqcup\!\sqcup_T (\beta \sqcup\!\sqcup_T \gamma).$$

For the converse inclusion, the proof is similar, using the fact that T is stable under $\diamondsuit_3$ and $\diamondsuit_4$. □

Comment. Observe that $\mathcal{D} = (\mathcal{P}(V_+^\omega), (\diamondsuit_i)_{1 \leq i \leq 4})$ is a universal algebra, see [3]. If T is a set of ω-trajectories, then denote by $\check{T}$ the union of all those sets of ω-trajectories that are in the subalgebra generated by T with respect to the algebra $\mathcal{D}$.

PROPOSITION 3.5: *Let $T \subseteq V_+^\omega$ be a set of ω-trajectories.*

(i) $\check{T}$ is anassociative set of ω-trajectories and, moreover,

(ii) $\check{T} = \overline{T}$, i.e., the associative closure of T is exactly the subalgebra generated by T in $\mathcal{D}$.

Proof: (i) $\check{T}$ is stable under the operations $\diamondsuit_i$, $1 \leq i \leq 4$ and thus, by Proposition 3.4, $\check{T}$ is an associative set of ω-trajectories.

(ii) Observe that $T \subseteq \check{T}$ and that $\check{T}$ is associative, hence $\overline{T} \subseteq \check{T}$. For the converse inclusion, let $T' \subseteq V_+^\omega$ be an associative set of ω-trajectories such that $T \subseteq T'$. Note that by Proposition 3.4, T' is stable under the operations $\diamondsuit_i$, $1 \leq i \leq 4$ and thus $\check{T} \subseteq T'$. Therefore$\check{T} \subseteq \overline{T}$. □

4. PERIODICITY AND ASSOCIATIVITY

This section is dedicated to investigate some interrelations between the periodicity property and the associativity of the shuffle on trajectories.

PROPOSITION 4.1: *Let $T \subseteq V_+^\omega$ be a set of ω-trajectories.*

(i) *If each $t \in T$ is a periodic ω-word, each $t' \in \overline{T}$, has the same property, i.e., each ω-trajectory in $\overline{T}$ is a periodic ω-word.*

(ii) *If, additionally, each* $t \in T$ *has a palindrome as its period, then the associative closure of* T, $\overline{T}$, *has the same property.*

(iii) *If* T *is a set of ultimately periodic* ω*-trajectories, then the associative closure of* T, $\overline{T}$, *has the same property, i.e., each* ω*-trajectory in* $\overline{T}$ *is ultimately periodic.*

Proof: (i) Note that the morphisms ρ_i, $1 \leq i \leq 4$, preserve the periodicity. Now consider the operation $\diamondsuit_1$. Let $t_1 = s^\omega$ and $t_2 = s'^\omega$. Define p and q by $p = |s|_r$ and$q = |s|_u$. Observe that $x^\omega = (x^p)^\omega$ and $y^\omega = (y^q)^\omega$. Let v be the unique word x^p ⧢$_s$ y^q (note that this is the shuffle over a finite trajectory, see Remark 2.1). Observe that x^ω ⧢$_{t_1}$ $y^\omega = v^\omega$ is a periodic ω-word for some nonempty word v that contains both r and u ($T \subseteq V_+^\omega$).

Now assume that $i = |s'|_r$, $j = |s'|_u$ and $k = |v|$. Let n be the least common multiple of i, j, k. Assume that $n = ii' = jj' = kk'$ for some positive nonzero integers i', j', k'. Note that

$$(x^\omega ⧢_{t_1} y^\omega) ⧢_{t_2} z^\omega = v^\omega ⧢_{s'^\omega} z^\omega = \left(v^{i'}\right)^\omega ⧢_{s'^{j'\omega}} \left(z^{k'}\right)^\omega = \alpha^\omega,$$

where α is the unique word $v^{i'}$ ⧢$_{s'^{j'}}$ $z^{k'}$.

Hence $\diamondsuit_1(t_1, t_2)$ is a periodic ω-word. Similarly, $\diamondsuit_i(t_1, t_2)$ is a periodic ω-word, $2 \leq i \leq 4$.

(ii) Observe that the morphisms ρ_i, $1 \leq i \leq 4$, are weak codings and hence they preserve the palindromes. The proof now proceeds as above. The resulting periods are palindromes.

(iii) The proof is similar with the proof of (i). □

The above proposition yields:

COROLLARY 4.1: *The following sets of* ω*-trajectories are associative:*

(i) *the set of all periodic* ω*-trajectories from*$V_+{}^\omega$.

(ii) *the set of all periodic* ω*-trajectories from*$V_+{}^\omega$ *that have as their period a palindrome.*

(iii) *the set of all ultimately periodic* ω*-trajectories from*$V_+{}^\omega$. □

DEFINITION 4.1: *Let* sym *be the following mapping,* $sym : V \longrightarrow V$, $sym(r) = u$ *and* $sym(u) = r$. *Also consider the mapping* $\varphi : \{x, y, z\} \longrightarrow \{x, y, z\}$, $\varphi(x) = z$, $\varphi(y) = y$ *and* $\varphi(z) = x$. sym *and* φ *are extended to* ω*-words over* V *and, respectively over* $\{x, y, z\}$. □

Next theorem provides a characterization of those ω-trajectories that are periodic. As such it is also a direct contribution to the study of ω-words, exhibiting an interconnection between periodicity and associativity. The theorem gives also a concrete example of a calculation of the associative closure of a set of ω-trajectories.

THEOREM 4.1: *Let t be an ω-trajectory such that $t \neq r^\omega$ and $t \neq u^\omega$. The following assertions are equivalent:*

(i) *t is a periodic ω-word.*

(ii) *t is in the associative closure of $(ru)^\omega$.*

Consequently, we obtain the following:

COROLLARY 4.2: *An ω-word $t \in V_+^*$ is a periodic ω-word if and only if t can be obtained from the ω-word$(ru)^\omega$ by finitely many applications of operations $\Diamond_i$, $1 \leq i \leq 4$.* □

Proof of Theorem 4.1: (ii) $\Longrightarrow$ (i) It follows from Proposition 3.5, (ii), and Proposition 4.1, (i).

(i) $\Longrightarrow$ (ii) Let A be the associative closure of the ω-trajectory$(ru)^\omega$. Assume that w is a nonempty word from V^*, $w = d_{i_1}^{j_1} d_{i_2}^{j_2} \ldots d_{i_k}^{j_k}$, where $d_{i_p} \in \{r, u\}$, $1 \leq j_p$, for all $1 \leq p \leq k$ and, moreover, $d_{i_q} \neq d_{i_{q+1}}$, for all $1 \leq q < k$. The *degree* of w, denoted $deg(w)$, is by definition k. Note that for each nonempty word w over V, $deg(w)$ is a unique integer greater than 1. Let t be a periodic ω-word over V such that $t \neq r^\omega$ and $t \neq u^\omega$. It follows that $t = w^\omega$ for some nonempy word w. Clearly, $deg(w) \geq 2$.

We prove by induction on $deg(w)$ that $t = w^\omega$ is in A. First we prove two claims:

Claim A: For all $i, j \geq 1$, the ω-trajectories $t = w^\omega$, where $w = r^i u$ or $w = ur^j$, are in A.

Proof of Claim A: Note that $\Diamond_1((ru)^\omega, (ru)^\omega) = (uru)^\omega$. Moreover, $\Diamond_3((uru)^\omega, (ru)^\omega) = (ur)^\omega$. Hence we obtain that $(ur)^\omega \in A$.

Assume now that $w = r^i u$, $i \geq 1$. We show by induction on i that $t = w^\omega \in A$. For $i = 1$, obviously $t \in A$. Assume the statement true for all $w = r^i u$ with $i \leq k$ and consider $w = r^{k+1} u$. If k is an even number, say $k = 2j$, then let t_1, t_2 be the ω-trajectories $t_1 = (ru)^\omega$ and $t_2 = (r^j u)^\omega$. By the inductive hypothesis t_2 is in A. Now observe that:

$$\begin{aligned}\Diamond_4(t_1, t_2) = \rho_4(x^\omega \sqcup\!\sqcup_{t_1} (y^j z)^\omega) &= \rho_4(((xy)^j xz)^\omega)\\ &= (r^{2j+1} u)^\omega = (r^{k+1} u)^\omega.\end{aligned}$$

Consider now the other case, i.e., k is an odd number, say $k = 2j - 1$. Let t_1, t_2 be the ω-trajectories $t_1 = (r^j u)^\omega$ and $t_2 = (ur)^\omega$. By the inductive hypothesis t_1 is in A. Now observe that:

$$\diamondsuit_3(t_1, t_2) = \rho_3(x^\omega ⧢_{t_1} (zy)^\omega) = \rho_3((x^j zx^j xy)^\omega) = (r^{2j}u)^\omega = (r^{k+1}u)^\omega.$$

Therefore $(r^i u)^\omega \in A$ for all $i \geq 1$.

A similar proof shows that $(u^j r)^\omega, (ur^j)^\omega, (ru^j)^\omega \in A$ for all $j \geq 1$.

Claim B: For all $i, j, p, q \geq 1$, the ω-trajectories $t = w^\omega$, where $w = r^i u^j$ or $w = u^p r^q$, are in A.

Proof of Claim B: First, assume that $w = r^i u^j$, $i, j \geq 1$. The proof is by induction on the number $i + j$. Obviously, if $w = ru$, then $t = w^\omega \in A$. The inductive step: let $t_1 = (r^i u^j)^\omega$ be in A. By Claim A it follows that $t_2 = (r^{i+j}u)^\omega \in A$. Observe that:

$$\diamondsuit_2(t_1, t_2) = \rho_2((x^i y^j)^\omega ⧢_{t_2} z^\omega) = \rho_2((x^i y^j z)^\omega) = (r^i u^{j+1})^\omega.$$

Hence Claim B is true also for $w = r^i u^{j+1}$.

Note that the ω-trajectory $t_3 = (ru^{i+j})^\omega$ is also in A, see Claim A. Moreover, $\diamondsuit_4(t_1, t_3) = (r^{i+1}u^j)^\omega$. Therefore, for all words of the form $w = r^i u^j$, $i, j \geq 1, w^\omega \in A$.

A similar proof shows that for all words of the form $w = u^p r^q$, $p, q \geq 1, w^\omega \in A$.

We are now ready to prove the implication (i) $\Longrightarrow$ (ii). Let $t = w^\omega$ be a periodic ω-word such that $t \neq r^\omega$ and $t \neq u^\omega$. The proof is by induction on $k = deg(w)$. The case $k = 2$ follows from Claim B. Assume the implication true forwords w with $deg(w) \leq k$. Let w be a word with $deg(w) = k + 1$, say$w = r^{i_1}u^{i_2}r^{i_3} \ldots r^{i_k}u^q$. Denote $w_1 = r^{i_1}u^{i_2}r^{i_3} \ldots r^{i_k}$ and note that by the inductive hypothesis the ω-trajectory $t_1 = (w_1)^\omega$ is in A. Consider also the ω-trajectory $t_2 = (r^s u^q)^\omega$, where $s = |w_1|$ and note that t_2 is also in A. Observe that:

$$\diamondsuit_2(t_1, t_2) = \rho_2((x^{i_1}y^{i_2} \ldots x^{i_k}) ⧢_{t_2} z^\omega) = \rho_2((x^{i_1}y^{i_2} \ldots x^{i_k}z^q)^\omega) = t.$$

If $w = r^{i_1}u^{i_2}r^{i_3} \ldots u^{i_k}r^p$, then denote $w_1 = r^{i_1}u^{i_2}r^{i_3} \ldots u^{i_k}$ and note that, again by the inductive hypothesis, the ω-trajectory $t_1 = (w_1)^\omega$ is in A. Consider also the ω-trajectory $t_2 = (u^s r^p)^\omega$, where $s = |w_1|$ and note that t_2 is also in A. Moreover, it is easy to see that: $\diamondsuit_4(t_2, t_1) = t$.

The situation when w begins with u is similar. $\square$

Next theorem is similar to Theorem 4.1. It states a characterization of ultimately periodic ω-words in terms of the associative closure of a certain set ofω-trajectories.

THEOREM 4.2: *Let t be an ω-trajectory such that $t \in V_+^\omega$. The following assertions are equivalent:*

(i) *t is a ultimately periodic ω-word.*

(ii) *t is in the associative closure of the ω-trajectory $r(ru)^\omega$.*

Thus, we obtain the following:

COROLLARY 4.3: *Let t be an ω-word, $t \in V_+^*.t$ is a ultimately periodic ω-word if and only if t can be obtained from the ω-word $r(ru)^\omega$ by finitely many applications of operations $\Diamond_i$, $1 \leq i \leq 4$.* □

Proof of Theorem 4.2: We start the proof by considering the notion of *a formal description.* Let T be a set of ω-trajectories. A finite sequence $t_1, t_2, \ldots, t_n$ of ω-trajectories is referred to as a formal description of t_n with respect to T iff for each $1 \leq k \leq n$, either $t_k \in T$, or there exists $1 \leq i \leq 4$, and there are $1 \leq p, q < k$, such that $t_k = \Diamond_i(t_p, t_q)$.

A formal description $t_1, t_2, \ldots, t_n$ of t_n with respect to T is denoted by

$$< t_1, t_2, \ldots, t_n >_T .$$

One can easily verify that

$$\overline{T} = \{t_n \mid \text{ there exists } < t_1, t_2, \ldots, t_n >_T\}.$$

Moreover, if $< t_1, t_2, \ldots, t_n >_T$ and $T \subseteq T'$, then $< t_1, t_2, \ldots, t_n >_{T'}$.

(ii) $\Longrightarrow$ (i) It follows from Proposition 3.5, (ii) and Proposition 4.1, (iii).

(i) $\Longrightarrow$ (ii) We start the proof by proving some claims. Consider the notations $T_0 = \{(ru)^\omega\}$, $T_1 = \{r(ru)^\omega\}$, $T_2 = \{(ru)^\omega, r(ru)^\omega\}$ and $T_3 = \{(ru)^\omega, r(ru)^\omega, u(ru)^\omega\}$.

Claim A: The sets T_1 and T_2 have the same associative closure, i.e., $\overline{T_1} = \overline{T_2}$.

Proof of Claim A: One can easily verify the following two equalities:

$$\Diamond_1(r(ru)^\omega, r(ru)^\omega) = (uru)^\omega$$

and

$$\Diamond_3((uru)^\omega, r(ru)^\omega) = (ur)^\omega.$$

Note that $\overline{sym(T)} = \overline{sym(\overline{T})}$ and using Theorem 4.1 we obtain that $\overline{(ur)^\omega} = \overline{(ru)^\omega}$. Hence $\overline{T_2} \subseteq \overline{T_1}$. The converse inclusion is obviously true. Thus we obtain Claim A.

Claim B: The sets T_2 and T_3 have the same associative closure, i.e., $\overline{T_2} = \overline{T_3}$.

Proof of Claim B: Note that $u(ru)^\omega = (ur)^\omega$ and hence $u(ru)^\omega$ is a periodic ω-trajectory. By Theorem 4.1, $u(ru)^\omega$ is in the associative closure of $(ru)^\omega$. Thus we obtain Claim B.

Claim C: If $< t_1, t_2, \ldots, t_n >_{T_0}$, then

$$< t_1, t_2, \ldots, t_n, rt_1, ut_1, rt_2, ut_2, \ldots, rt_n, ut_n >_{T_2} .$$

Proof of Claim C: We show by induction on k that

$$< t_1, t_2, \ldots, t_n, rt_1, ut_1, rt_2, ut_2, \ldots, rt_k, ut_k >_{T_2} .$$

Assume that $k = 1$. Observe that from the definition of a formal description, it follows that $t_1 = (ru)^\omega$. By definition of T_2 and Claim B we conclude that $< t_1, t_2, \ldots, t_n, rt_1, ut_1 >_{T_2}$. Assume by induction that$< t_1, t_2, \ldots, t_n, rt_1, ut_1, rt_2, ut_2, \ldots, rt_{k-1}, ut_{k-1} >_{T_2}$. If $t_k = (ru)^\omega$, then the conclusion follows as in the case $k = 1$. Now, assume that $t_k = \diamondsuit_i(t_p, t_q)$ for some $1 \leq i \leq 4$, $1 \leq p, q < k$ and we show that the trajectory rt_k has a formal description with respect to T_2. Consider all possible situations:

If $i = 1$, that is $t_k = \diamondsuit_1(t_p, t_q)$, then note that

$$rt_k = \rho_1((x^\omega ⧢_{ut_p} y^\omega) ⧢_{rt_q} z^\omega) = \diamondsuit_1(ut_p, rt_q).$$

Assume that $i = 2$. Hence $t_k = \diamondsuit_2(t_p, t_q)$, and note that

$$rt_k = \rho_2((x^\omega ⧢_{rt_p} y^\omega) ⧢_{rt_q} z^\omega) = \diamondsuit_2(rt_p, rt_q).$$

Consider now that $i = 3$. Thus $t_k = \diamondsuit_3(t_p, t_q)$, and observe that

$$rt_k = \rho_3((x^\omega ⧢_{rt_p} y^\omega) ⧢_{t_q} z^\omega) = \diamondsuit_3(rt_p, t_q).$$

If $i = 4$ and therefore $t_k = \diamondsuit_4(t_p, t_q)$, then note that

$$rt_k = \rho_4((x^\omega ⧢_{rt_p} y^\omega) ⧢_{t_q} z^\omega) = \diamondsuit_4(rt_p, t_q).$$

Finally, we prove how the trajectory ut_k can be obtained. Again, we consider all possible situations.

Assume that $i = 1$ and $t_k = \diamondsuit_1(t_p, t_q)$. Observe that

$$ut_k = \rho_1((x^\omega \sqcup\!\sqcup_{t_p} y^\omega) \sqcup\!\sqcup_{ut_q} z^\omega) = \diamondsuit_1(t_p, ut_q).$$

If $i = 2$, and thus $t_k = \diamondsuit_2(t_p, t_q)$, then note that

$$ut_k = \rho_2((x^\omega \sqcup\!\sqcup_{t_p} y^\omega) \sqcup\!\sqcup_{ut_q} z^\omega) = \diamondsuit_2(t_p, ut_q).$$

Consider that $i = 3$. Hence $t_k = \diamondsuit_3(t_p, t_q)$ and notice that

$$ut_k = \rho_3((x^\omega \sqcup\!\sqcup_{ut_p} y^\omega) \sqcup\!\sqcup_{rt_q} z^\omega) = \diamondsuit_3(ut_p, rt_q).$$

If $i = 4$, that is $t_k = \diamondsuit_4(t_p, t_q)$, then note that

$$ut_k = \rho_4((x^\omega \sqcup\!\sqcup_{ut_p} y^\omega) \sqcup\!\sqcup_{ut_q} z^\omega) = \diamondsuit_4(ut_p, ut_q).$$

Next two claims assert that certain trajectories are in the associative closure of the set$T_2 = \{(ru)^\omega, r(ru)^\omega\}$.

Claim D: If t is a periodic trajectory, $t \in V_+^\omega$, then the trajectories rt and ut are in $\overline{T_2}$.

Proof of Claim D: Let t be a periodic trajectory, $t \in V_+^\omega$. Observe that from Theorem 4.1 and from the definition of the formal description, there exists a formal description of t with respect to $T_0 = \{(ru)^\omega\}$, say $< t_1, t_2, \ldots, t_n, t >_{T_0}$. From Claim B it follows that

$$< t_1, t_2, \ldots, t_n, t, rt_1, ut_1, \ldots, rt_n, ut_n, rt, ut >_{T_2} .$$

Hence, we conclude Claim D.

Claim E: The trajectories $r^2(ru)^\omega$ and $ur(ru)^\omega$ are in $\overline{T_2}$.

Proof of Claim E: Observe that: $r^2(ru)^\omega = \rho_4(x^\omega \sqcup\!\sqcup_{t_1} (y^\omega \sqcup\!\sqcup_{t_2} z^\omega)$, where $t_1 = (ru^3)^\omega$ and $t_2 = r(ru^2)^\omega$. Moreover, $ur(ru)^\omega = \rho_1((x^\omega \sqcup\!\sqcup_{t_3} y^\omega) \sqcup\!\sqcup_{t_4} z^\omega)$, where $t_3 = u(ur)^\omega$ and $t_2 = (ur^2)^\omega$.

Note that as a consequence of Theorem 4.1 and of Claim D, t_i, $1 \leq i \leq 4$ are in $\overline{T_2}$.

From the above Claim E, we deduce that there are the following formal descriptions:

$$< x_1, x_2, \ldots, x_i, r^2(ru)^\omega >_{T_2} \quad \text{and} \quad < y_1, y_2, \ldots, y_j, ur(ru)^\omega >_{T_2} .$$

Using these notations we assert:

Claim F: If $< t_1, t_2, \ldots, t_n >_{T_2}$, then

$$< (ru)^\omega, x_1, ..., x_i, r^2(ru)^\omega, y_1, ..., y_j, ur(ru)^\omega,$$
$$t_1, ..., t_n, rt_1, ut_1, ..., rt_n, ut_n >_{T_2}.$$

Proof of Claim F: We show that for all $1 \leq k \leq n$, the trajectories rt_k and ut_k are in $\overline{T_2}$. Assume $k = 1$ and note that $t_1 = (ru)^\omega$ or $t_1 = r(ru)^\omega$. Thus rt_1 and ut_1 are among the following trajectories: $r(ru)^\omega$, $u(ru)^\omega$, $r^2(ru)^\omega$, $ur(ru)^\omega$. Using Claims B and E we obtain that in each case they are in $\overline{T_2}$. Assume that $k > 1$. If t_k is in T_2, then the claim follows as in case $k = 1$. If t_k is obtained by using an operation $\diamondsuit_i$, $1 \leq i \leq 4$, i.e., $t_k = \diamondsuit_i(t_p, t_q)$, for some $1 \leq i \leq 4$, $1 \leq p, q < k$, then a similar proof as for Claim C shows that rt_k and ut_k are in $\overline{T_2}$.

Now we end the proof of Theorem 4.2.

Let $t = \alpha z^\omega$ be a ultimately periodic ω-word, $t \in V_+^\omega$. Let n be the length of α, i.e., $|\alpha|$. If $n = 0$, then t is a periodic ω word and from Theorem 4.1 we conclude that $t \in \overline{T_2}$. If $n \geq 1$, then using n times Claim F we obtain a formal description of t with respect to T_2. Hence, from the definition of the formal description it follows that $t \in \overline{T_2}$. Since, by Claim A, $\overline{T_2} = \overline{T_1}$, the proof is complete. □

Next theorem shows a remarkable property of the most famous infinite Sturmian word, known as the Fibonacci word. An infinite Sturmian word is an ω-word t over V such that the subword complexity of t is defined by the function $\varphi_t(n) = n + 1$, i.e., for any $n \geq 1$, the number of subwords of t of length n is exactly $n + 1$. See [1], [7] for other equivalent definitions of infinite Sturmian words as well as for a number of properties of these words.

The Fibonacci word f is defined as the limit of the sequence of words $(f_n)_{n \geq 0}$, where:

$$f_0 = r,\ f_1 = u,\ f_{n+1} = f_n f_{n-1},\ n > 0.$$

Note that f is an infinite word which has an initial prefix as follows:

$$f = uruururuuruururuururu\ldots$$

Remark 4.1: Note that for each n, $|f_n|$ is equal with the n-th term of the Fibonacci numerical sequence: $1, 1, 2, 3, 5, 8, 13\ldots$. Moreover, $|f_n|_r = |f_{n-1}|$ and $|f_n|_u = |f_{n-2}|$. □

We recall some properties of the operation of shuffle on finite trajectories, see Remark 2.1. For more details on this topic the reader is referred to [9].

DEFINITION 4.2: *Let W be the alphabet $W = \{x, y, z\}$ and consider the following four morphisms, ρ'_i, $1 \leq i \leq 4$, where $\rho'_i : W \longrightarrow V^*$, $1 \leq i \leq 4$, and*

$$\rho'_1(x) = \lambda, \quad \rho'_1(y) = r, \quad \rho'_1(z) = u,$$
$$\rho'_2(x) = r, \quad \rho'_2(y) = u, \quad \rho'_2(z) = u,$$
$$\rho'_3(x) = r, \quad \rho'_3(y) = u, \quad \rho'_3(z) = \lambda,$$
$$\rho'_4(x) = r, \quad \rho'_4(y) = r, \quad \rho'_4(z) = u.$$

□

DEFINITION 4.3: *Let $\diamondsuit'_i$, $1 \leq i \leq 4$ be the following partial operations on V^*.*

$$\diamondsuit'_i : V^* \times V^* \multimap V^*, \quad 1 \leq i \leq 4,$$

Let t, t' be in V^* and assume that $|t| = n$, $|t|_r = p$, $|t|_u = q$, $|t'| = n'$, $|t'|_r = p'$, $|t'|_u = q'$,

1. *if $n = p'$, then $\diamondsuit'_1(t, t') = \rho_1((x^p \sqcup\!\sqcup_t y^q) \sqcup\!\sqcup_{t'} z^{q'}))$, else, $\diamondsuit'_1(t, t')$ is undefined.*
2. *if $n = p'$, then $\diamondsuit'_2(t, t') = \rho_2((x^p \sqcup\!\sqcup_t y^q) \sqcup\!\sqcup_{t'} z^{q'}))$, else, $\diamondsuit'_2(t, t')$ is undefined.*
3. *if $n = q'$, then $\diamondsuit'_3(t', t) = \rho_3(x^{p'} \sqcup\!\sqcup_{t'} (y^p(\sqcup\!\sqcup_t z^q))$, else, $\diamondsuit'_3(t, t')$ is undefined.*
4. *if $n = q'$, then $\diamondsuit'_4(t', t) = \rho_4(x^{p'} \sqcup\!\sqcup_{t'} (y^p(\sqcup\!\sqcup_t z^q))$, else, $\diamondsuit'_4(t, t')$ is undefined.* □

Note that the morphisms ρ'_i, $1 \leq i \leq 4$, as well as the operations $\diamondsuit'_i$, $1 \leq i \leq 4$, are the versions of the morphisms ρ_i, $1 \leq i \leq 4$, and respectively of the operations $\diamondsuit_i$, $1 \leq i \leq 4$, for the case of finite trajectories. In the sequel ρ'_i is denoted by ρ_i and $\diamondsuit'_i$ is denoted by $\diamondsuit_i$, $1 \leq i \leq 4$. Notice that this simplification does not produce any ambiguity.

Now we are in position to state our result:

THEOREM 4.3: *The associative closure of the Fibonacci word, $\overline{f}$, contains all periodic trajectories and the containment is strict.*

Proof: We start by proving the following equality:

$$\diamondsuit_1(f, \diamondsuit_3(f, f)) = (ru)^\omega. \qquad (I)$$

Let $(e_n)_{n\geq 0}$ be the Fibonacci numerical sequence $1, 1, 2, 3, 5, 8, 13\ldots$. Note that $e_n = |f_n|$ for every $n \geq 0$. Consider the notation $t = \diamondsuit_1(f, \diamondsuit_3(f, f))$. We show that each prefix of t of length $2e_n$ is of the form $(ru)^{e_n}$. The proof is by induction on n.

Let α_n be the value of $\diamondsuit_3(f_{n+1}, f_n)$, i.e.,

$$\alpha_n = \rho_3(x^{e_{n-1}} ⧢_{f_{n+1}} (y^{e_{n-2}} ⧢_{f_n} z^{e_{n-1}})).$$

Therefore, we prove by induction on n that:

$$\diamondsuit_1(f_{n-1}, \alpha_n) = (ru)^{e_{n-2}}. \qquad (II)$$

That is:

$$(x^{e_{n-3}} ⧢_{f_{n-1}} y^{e_{n-2}}) ⧢_{\alpha_n} z^{e_{n-2}} = (ru)^{e_{n-2}}.$$

The base of induction: $n = 3$.

Notice that:

$$\alpha_3 = \rho_3(x^2 ⧢_{uruur} (y ⧢_{uru} z^2)) = \rho_3(x^2 ⧢_{uruur} zyz) = \rho_3(zxyzx) = rur.$$

It follows that:

$$\diamondsuit_1(f_2, \alpha_3) = \rho_1(ur, rur) = \rho_1((x ⧢_{ur} y) ⧢_{rur} z) =$$

$$= \rho_1(yx ⧢_{rur} z) = \rho_1(yzx) = ru = (ru)^{e_1}.$$

Hence, for $n = 3$ the equality (II) is true.

The inductive step: $n \vdash n + 1$

We start by computing the value of α_{n+1}:

$$\alpha_{n+1} = \rho_3(x^{e_n} ⧢_{f_{n+2}} (y^{e_{n-1}} ⧢_{f_{n+1}} z^{e_n})) =$$

$$= \rho_3(x^{e_n} ⧢_{f_{n+2}} (y^{e_{n-2}+e_{n-3}} ⧢_{f_n f_{n-1}} z^{e_{n_1}+e_{n-2}})) =$$

$$= \rho_3(x^{e_n} ⧢_{f_{n+2}} (y^{e_{n-2}} ⧢_{f_n} z^{e_{n_1}}) \cdot (y^{e_{n-3}} ⧢_{f_{n-1}} z^{e_{n_2}})) =$$

$$= \rho_3(x^{e_{n-1}+e_{n-2}} ⧢_{f_{n+1} f_n} (y^{e_{n-2}} ⧢_{f_n} z^{e_{n_1}}) \cdot (y^{e_{n-3}} ⧢_{f_{n-1}} z^{e_{n_2}})) =$$

$$= \rho_3((x^{e_{n-1}} ⧢_{f_{n+1}} (y^{e_{n-2}} ⧢_{f_n} z^{e_{n_1}}))\cdot(x^{e_{n-2}} ⧢_{f_n} (y^{e_{n-3}} ⧢_{f_{n-1}} z^{e_{n_2}})) =$$

$$= \alpha_n \alpha_{n-1}.$$

Therefore, we proved that:

$$\alpha_{n+1} = \alpha_n \alpha_{n-1}.$$

Using the above relation, the inductive hypothesis, the well-known properties of the numerical Fibonacci sequence and the properties of the Fibonacci sequence f, it follows that:

$$\diamondsuit_1(f_n, \alpha_{n+1}) = \rho_1((x^{e_{n-2}} \sqcup\!\sqcup_{f_n} y^{e_{n-1}}) \sqcup\!\sqcup_{\alpha_{n+1}} z^{e_{n-1}}) =$$

$$= \rho_1((x^{e_{n-3}+e_{n-4}} \sqcup\!\sqcup_{f_{n-1}f_{n-2}} y^{e_{n-2}+e_{n-3}}) \sqcup\!\sqcup_{\alpha_{n+1}} z^{e_{n-1}}) =$$

$$= \rho_1((x^{e_{n-3}} \sqcup\!\sqcup_{f_{n-1}} y^{e_{n-2}}) \cdot (x^{e_{n-4}} \sqcup\!\sqcup_{f_{n-2}} y^{e_{n-3}}) \sqcup\!\sqcup_{\alpha_n \alpha_{n-1}} z^{e_{n-2}+e_{n-3}}) =$$

$$= \rho_1(((x^{e_{n-3}} \sqcup\!\sqcup_{f_{n-1}} y^{e_{n-2}}) \sqcup\!\sqcup_{\alpha_n} z^{e_{n-2}}) \cdot ((x^{e_{n-4}} \sqcup\!\sqcup_{f_{n-2}} y^{e_{n-3}}) \sqcup\!\sqcup_{\alpha_{n-1}} z^{e_{n-3}})) =$$

$$\diamondsuit_1(f_{n-1}, \alpha_n)\diamondsuit_1(f_{n-2}, \alpha_{n-1}) = (ru)^{e_{n-2}}(ru)^{e_{n-1}} = (ru)^{e_{n-1}+e_{n-2}} = (ru)^{e_{n-1}}.$$

Hence, equality (II), and consequently equality (I) are true.

Using Theorem 4.1, we deduce that all periodic ω-words from V_+^ω are contained in the associative closure of the Fibonacci sequence, $\overline{f}$.

Since the Fibonacci sequence is not a periodic ω-word, by Proposition 4.1, we conclude that the above containment is strict. □

5. CONCLUSION

The shuffle-like operations considered in this paper provide a new tool for investigating properties of ω-words. Recently, a characterization of ω-words that are ultimately periodic has been obtained in [10]. This characterization is based on a different approach. Interrelations between this characterization and the characterization from the present paper are subject of further research.

Many other problems remain to be investigated. For instance: does exist a proper ultimately periodic ω-word in the associative closure of the Fibonacci ω-word f? What can be said about the associative closure of some other Sturmian ω-words?

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