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Equations on the semidirect product of a finite semilattice by a $J$-trivial monoid of height $k$


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EQUATIONS ON THE SEMIDIRECT PRODUCT OF A FINITE SEMILATTICE BY A $J$-TRIVIAL MONOID OF HEIGHT $k$ (*)

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1. INTRODUCTION

Let $J_k$ denote the $k$th level of Simon’s hierarchy of $J$-trivial monoids. The 1st level $J_1$ is the $M$-variety of finite semilattices. In this paper, we give a complete sequence of equations for the product $J_1 \ast J_k$ generated by all semidirect products of the form $M \ast N$ with $M \in J_1$ and $N \in J_k$. Results of Almeida imply that this sequence of equations is complete for the product $J_{k+1}$ or $J_1 \ast \cdots \ast J_1$ ($k + 1$ times) generated by all semidirect products of $k + 1$ finite semilattices and that $J_1 \ast J_k$ is defined by a finite sequence of equations if and only if $k = 1$. The equality $J_1 \ast J_k = J_{k+1}^1$ implies that a conjecture of Pin concerning tree hierarchies of $M$-varieties is false.

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\( J_1 \ast \ldots \ast J_{1}(k + 1 \text{ times}) \) or \( J_1^{k+1} \), which turns out to be our equational characterization of \( J_1 \ast J_k \). The equality \( J_1 \ast J_k = J_1^{k+1} \) implies that a conjecture of Pin concerning tree hierarchies of M-varieties is false. Almeida [3] implies that \( J_1 \ast J_k \) is defined by a finite sequence of equations if and only if \( k = 1 \). The methods used in this paper were developed by Almeida [1], [2].

1.1 Preliminaries

The reader is referred to the books of Eilenberg [15], Lallement [19] or Pin [20] for terminology not defined in this paper.

Let \( A \) be a finite set called an alphabet, whose elements are called letters. We will denote by \( A^* \) the free monoid over \( A \). The elements of \( A^* \) are the finite sequences of letters called words. The empty word (denoted by \( 1 \)) corresponds to the empty sequence.

Let \( L \) be a subset of \( A^* \) (or a language over \( A \)) and \( \sim \) be an equivalence relation on \( A^* \). We say that \( \sim \) saturates \( L \) if \( L \) is a union of classes modulo \( \sim \) or for every \( u, v \in A^*, u \sim v \) and \( u \in L \) imply \( v \in L \).

The syntactic congruence of \( L \) is the congruence \( \sim_L \) on \( A^* \) defined by \( u \sim_L v \) if and only if for every \( x, y \in A^*, xuy \in L \) if and only if \( xvy \in L \). We can show that \( \sim_L \) is the coarsest congruence saturating \( L \). The syntactic monoid of \( L \) is the quotient monoid \( M(L) = A^*/\sim_L \).

Let \( S \) and \( T \) be semigroups. We say that \( S \) is a quotient of \( T \) if there exists a surjective morphism \( \varphi : T \rightarrow S \) and we say that \( S \) divides \( T \) \( (S \prec T) \) if \( S \) is a quotient of a submonoid of \( T \). The division relation is transitive. The syntactic monoid of a language. \( L \) is the smallest monoid recognizing \( L \), where smallest is taken in the sense of the division relation.

A variety \( V \) is a class of semigroups closed under division and products. By the well-known theorem of Birkhoff such a variety is defined by equations that must hold for all elements of semigroups in \( V \). Thus equations give rise to varieties.

An S-variety is a class of finite semigroups closed under division and finite products and an M-variety is a class of finite monoids closed under division and finite products. Equivalently, a class \( V \) of finite monoids is an M-variety if \( V \) satisfies the following two conditions:

- if \( T \in V \) and \( S \prec T \), then \( S \in V \);
- if \( S, T \in V \), then \( S \times T \in V \).
Eilenberg has shown the existence of a bijection between the \( M \)-varieties and some classes of languages called the \( * \)-varieties of languages.

A class \( \mathcal{V} \) is a \( * \)-variety of languages if

- for every alphabet \( A \), \( A^* \mathcal{V} \) is a set of recognizable languages over \( A \) closed under boolean operations;
- if \( \varphi : A^* \to B^* \) is a free monoid morphism, then \( L \in B^* \mathcal{V} \) implies \( L \varphi^{-1} = \{ u \in A^* | u \varphi \in L \} \) is in \( A^* \mathcal{V} \);
- if \( L \in A^* \mathcal{V} \) and \( a \in A \), then \( a^{-1} L = \{ u \in A^* | au \in L \} \) and \( La^{-1} = \{ u \in A^* | ua \in L \} \) are in \( A^* \mathcal{V} \).

If \( \mathcal{V} \) is an \( M \)-variety and \( A \) is an alphabet, we denote by \( A^* \mathcal{V} \) the set of recognizable languages over \( A \) whose syntactic monoid is in \( \mathcal{V} \). Equivalently, \( A^* \mathcal{V} \) is the set of languages of \( A^* \) recognized by a monoid of \( \mathcal{V} \). If \( \mathcal{V} \) is a \( * \)-variety of languages, we denote by \( \mathcal{V} \) the \( M \)-variety generated by the monoids of the form \( M(L) \) where \( L \in A^* \mathcal{V} \) for some alphabet \( A \).

A result of Simon enables us to describe the \( * \)-variety of languages corresponding to the \( M \)-variety of \( \mathcal{J} \)-trivial monoids denoted by \( \mathcal{J} \).

A word \( a_1 \ldots a_i \in A^* \) is a subword of a word \( u \) of \( A^* \) if there exist words \( u_0, u_1, \ldots, u_i \in A^* \) such that \( u = u_0 a_1 u_1 \ldots a_i u_i \). For each integer \( k \geq 0 \), we define an equivalence relation \( \sim_k \) on \( A^* \) by \( u \sim_k v \) if and only if \( u \) and \( v \) have the same subwords of length less than or equal to \( k \). We can verify that \( \sim_k \) is a congruence on \( A^* \) with finite index. Note that \( u \sim_1 v \) if and only if \( u \) and \( v \) have the same letters. The set of letters that occur in a word \( u \) will be denoted by \( u\alpha \).

A language \( L \) over \( A \) is called piecewise testable if it is a union of classes modulo \( \sim_k \) for some integer \( k \), or equivalently if it is in the boolean algebra generated by all languages of the form \( A^* a_1 A^* \ldots a_i A^* \) where \( i \geq 0 \) and \( a_1, \ldots, a_i \in A \). Simon [24] has proved that a language is piecewise testable if and only if its syntactic monoid is \( \mathcal{J} \)-trivial. For every alphabet \( A \), we will denote by \( A^* \mathcal{J}_k \) the boolean algebra generated by all languages of the form \( A^* a_1 A^* \ldots a_i A^* \), where \( 0 \leq i \leq k \) and \( a_1, \ldots, a_i \in A \). One can show that \( \mathcal{J}_k \) is a \( * \)-variety of languages and we will denote by \( \mathcal{J}_k \) the corresponding \( M \)-variety. The \( M \)-variety \( \mathcal{J} \) is the union of the \( M \)-varieties \( \mathcal{J}_k \).

### 1.2 Product of varieties of semigroups

Let \( S \) and \( T \) be semigroups. To simplify the notation we will represent \( S \) additively (without necessarily supposing that \( S \) is commutative) and \( T \) multiplicatively.

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An action of $T$ on $S$ is a function

$$T \times S \rightarrow S$$

$$(t, s) \mapsto ts$$

satisfying for every $t, t' \in T$ and $s, s' \in S$:

- $t(s + s') = ts + ts'$;
- $t(t's) = (tt')s$.

Given an action of $T$ on $S$, the semidirect product $S \rtimes T$ is the semigroup defined on $S \times T$ by the multiplication

$$(s, t)(s', t') = (s + ts', t't).$$

The multiplication in $S \rtimes T$ is associative. Thus $S \rtimes T$ is a semigroup.

In this paper, we only consider semidirect products $S \rtimes T$ given by actions of $T$ on $S$ that are described by monoid homomorphisms $\varphi : T^1 \rightarrow \text{End } S$ from $T^1$ into the monoid of endomorphisms of $S$. In the terminology adopted by Eilenberg [15], this means that we only consider left unitary actions, that is actions of $T$ on $S$ that satisfy $1s = s$ for every $s \in S$. Here $T^1$ denotes the semigroup $T \cup \{1\}$ obtained from $T$ by adjoining an identity if $T$ does not have one, and $T^1 = T$ otherwise.

If $V$ and $W$ are varieties of semigroups, the product $V \star W$ is the variety generated by all semigroups of the form $S \rtimes T$ with $S \in V$ and $T \in W$. The product of two $S$-varieties (or $M$-varieties) is defined analogously. The operation $\star$ defined on varieties is associative.

There remain many problems to be solved on products of $S$-varieties (or $M$-varieties). The most important of these is the following. Given two decidable $S$-varieties (or $M$-varieties), is the product decidable? A particular case of this problem is well known in the theory of semigroups. Karnofsky and Rhodes [18] have established the decidability of the $M$-varieties $A \star G$ and $G \star A$. Here, $A$ denotes the $M$-variety of aperiodic monoids and $G$ the $M$-variety of groups.

This paper deals in particular with products of the form $J_1^k$. It is known that $\bigcup_{k \geq 0} J_1^k$ is the $M$-variety $R$ of all finite $R$-trivial monoids (Stiffler [25]) and that $J_1^k$ is decidable (Pin [21]).
1.3 Equations on products of varieties of semigroups

Let \( A^+ \) be the free semigroup over a denumerable alphabet \( A \) and let \( u, v \in A^+ \). We say that a semigroup \( S \) satisfies the equation \( u = v \) or the equation \( u = v \) holds in \( S \) (and we write \( S \models u = v \)) if for every morphism \( \varphi : A^+ \to S, u \varphi = v \varphi \). This means that, it we substitute elements of \( S \) for the letters in \( u \) and \( v \), we reach equalities in \( S \). For example, \( S \) is idempotent if it satisfies the equation \( x = x^2 \) and \( S \) is commutative if it satisfies the equation \( xy = yx \). For a sequence \( \mathcal{E} \) of equations and an equation \( u = v \), \( \mathcal{E} \vdash u = v \) (and we say \( u = v \) is deducible from \( \mathcal{E} \)) means that for every semigroup \( S \), if \( S \models \mathcal{E} \), then \( S \models u = v \).

Let \( V(u, v) \) be the class of finite semigroups \( S \) satisfying the equation \( u = v \). It is easy to show that \( V(u, v) \) is an S-variety.

Let \( (u_i, v_i)_{i>0} \) be a sequence of pairs of words of \( A^+ \). Consider the following S-varieties:

\[
W = \bigcap_{i>0} V(u_i, v_i)
\]
\[
W' = \bigcup_{I>0} \bigcap_{i \geq I} V(u_i, v_i).
\]

We say that \( W \) is defined by the equations \( u_i = v_i (i > 0) \). This corresponds to the fact that a finite semigroup is in \( W \) if and only if it satisfies the equations \( u_i = v_i \) for every \( i > 0 \). We say that \( W' \) is ultimately defined by the equations \( u_i = v_i (i > 0) \). This corresponds to the fact that a finite semigroup is in \( W' \) if and only if it satisfies the equations \( u_i = v_i \) for every \( i \) sufficiently large.

The arguments above apply equally well to M-varieties. We only need to replace \( A^+ \) by \( A^* \) throughout.

Eilenberg and Schützenberger [16] have proved the following result. Every nonempty M-variety is ultimately defined by a sequence of equations, or every S-variety containing the trivial semigroup is ultimately defined by a sequence of equations. If \( V \) is the S-variety ultimately defined by the equations \( u_i = v_i, i > 0 \), then the same equations ultimately define the M-variety consisting of all the monoids in \( V \). Also every M-variety generated by a single monoid is defined by a (finite or infinite) sequence of equations.

Equational characterizations of all the M-varieties \( J_k \) are known [23], [5], [6], [10], [11]. In particular,
the M-variety $J_1$ is defined by the equations $x = x^2$ and $xy = yx$, so $J_1$ is the M-variety of idempotent and commutative monoids;

- the M-variety $J_2$ is defined by the equations $xyzx = xyxzx$ and $(xy)^2 = (yx)^2$;

- the M-variety $J_3$ is defined by the equations $xzyxvxwy = xzxyxvxwy$, $ywxvxyzx = ywxvxyzx$ and $(xy)^3 = (yx)^3$.

**Definition 1.1**: Let $k \geq 1$ and let $A = \{x_1, x_2, \ldots\}$ be a denumerable alphabet of variables including $x$ ($x = x_1$).

$E_k$ is the sequence of all equations (over $A$) of the form

$$u_1 \ldots u_i v_1 \ldots v_j = u_1 \ldots u_i x v_1 \ldots v_j$$

where

$$\{x\} \subseteq u_1 \alpha \subseteq \ldots \subseteq u_i \alpha$$

$$\{x\} \subseteq v_1 \alpha \subseteq \ldots \subseteq v_j \alpha$$

and where $i + j = k$.

**Theorem 1.1** [10]: Let $k \geq 1$. The M-variety $J_k$ is defined by $E_k$.

These results lead to the following question. Can the M-varieties $J_k$ be defined by a finite sequence of equations? This question has been answered in [11]. The M-varieties $J_k$ can be defined by a finite sequence of equations if and only if $k = 1$, 2 or 3.

Equations are known for the product of the S-variety of semilattices, groups, and $R$-trivial semigroups by the S-variety of locally trivial semigroups [15]. These results have important applications to language theory [14], [15].

Pin [22] has shown that the M-variety $J_1 \ast J_1$ is defined by the equations $xux = xux^2$ and $xuyvyx = xuyvyx$. A result of Irastorza [17] shows that the M-varieties $J_1 \ast (Z_k)$ are not defined by finite sequences of equations. Here, $(Z_k)$ denotes the M-variety generated by the cyclic group $Z_k$ of order $k$ which is defined by the equations $x^k = 1$ and $xy = yx$. Almeida [3] has shown that $J^k_1$ is defined by a finite sequence of equations if and only if $k = 1$ or 2. Ash [4] has shown that $J_1 \ast G = \Inv$ is defined by the equation $x^\omega y^\omega = y^\omega x^\omega$. The M-variety of groups $G$ is defined by the equation $x^\omega = 1$, and $\Inv$ denotes the M-variety generated by the inverse semigroups.
2. ON A COMPLETE SEQUENCE OF EQUATIONS FOR $J_1 \ast J_k$

In this section, in order to simplify the notation, we will denote also by $J_k$ the S-variety generated by $J_k$. It will be convenient to denote by $J_0$ the S-variety defined by the equation $x = y$. In this section, we work essentially with semigroups.

Our results follow from an approach to the semidirect product that was introduced in Almeida [1].

The free object on the set $X$ in the variety generated by an S-variety (or M-variety) $V$ will be denoted by $F_X V$. We will also write $F_i V$ as an abbreviation for $F_{\{x_1, \ldots, x_i\}} V$. For every $i \geq 1$ and $k \geq 1$, the free object $F_i (J_k)$ can be viewed as a set of representatives of classes modulo $\sim_k$ of words over $\{x_1, \ldots, x_i\}$. This set is finite. For $i \geq 1$ and $k \geq 1$, let $p_i, k : \{x_1, \ldots, x_i\}^+ \to F_i (J_1 \ast J_k)$ be the canonical projection that maps the letter $x_j$ onto the generator $x_j$ of $F_i (J_1 \ast J_k)$, and let $q_i, k : \{x_1, \ldots, x_i\}^+ \to F_i (J_k)$ be the canonical projection that maps the letter $x_j$ onto the generator $x_j$ of $F_i (J_k)$. If $u \in \{x_1, \ldots, x_i\}^+$, then $u q_i, k$ can be viewed as a representative of the class modulo $\sim$ of $u$.

**Definition 2.1:** Let $k \geq 1$ and $u \in \{x_1, \ldots, x_i\}^+$.  
$u \alpha_{i, k}$ is the set of all pairs of the form  
\[(u' q_{i, k}, x) \in (F_i (J_k))^1 \times \{x_1, \ldots, x_i\}\]
where $u = u' x u''$ for some $u', u'' \in \{x_1, \ldots, x_i\}^*$.

In the case of $k = 0$, $(F_i (J_0))^1 = \{1\}$ and so $u \alpha_{i, 0} = \{1\} \times u \alpha$.

The following lemmas will help us give an equational characterization of $J_1 \ast J_k$. Lemma 2.1 provides an algorithm to decide when an equation holds in $J_1 \ast J_k$.

**Lemma 2.1:** Let $k \geq 0$ and $u, v \in \{x_1, \ldots, x_i\}^+$. Then  
\[J_1 \ast J_k \models u = v\]
if and only if $u \alpha_{i, k} = v \alpha_{i, k}$.

**Proof:** For $k = 0$, we have that $J_1 \models u = v$ if and only if $u \alpha = v \alpha$. Since $F_i (J_k)$ is finite for every $i \geq 1$ and $k \geq 1$, a representation of free objects for a semidirect product of S-varieties obtained in [1] implies that $F_i (J_1 \ast J_k)$ is also finite for every $i \geq 1$ and $k \geq 1$. Moreover, there
is an embedding of $F_i (J_1 \star J_k)$ into $F_Y (J_1) \star F_i (J_k)$ that maps $x_j$ into $((1, x_j), x_j)$. Here $Y = (F_i (J_k))^1 \times \{x_1, \ldots, x_i\}$ and the action in the semidirect product of the free objects is given by $x_j (s, x_j^r) = (x_j s, x_j^r)$ for $s \in (F_i (J_k))^1$. The word $x_j \ldots x_j^r$ is mapped into

$$((1, x_j) + (x_j, x_j^2) + \ldots + (x_j \ldots x_j^r, x_j), x_j \ldots x_j^r).$$

Suppose that $J_1 \star J_k \models u = v$, or that $u p_i, k = v p_i, k$. This is equivalent to the two conditions $u \alpha_i, k = v \alpha_i, k$ and $J_k \models u = v$. Observe that $J_k \models u = v$ if and only if $u q_i, k = v q_i, k$. The result follows since $u \alpha_i, k = v \alpha_i, k$ implies $u q_i, k = v q_i, k$.

Let $k \geq 1$. Let $u, v \in \{x_1, \ldots, x_i\}^+$ be such that $u \alpha_i, k = v \alpha_i, k$. Let $x \in u \alpha$ and consider the first occurrence of $x$ in $u$.

Case 1. If $x$ is the last letter occurring for the first time in $u$, then there is a factorization $u = u_1 x u_2$ with $u_1, u_2 \in \{x_1, \ldots, x_i\}^*$, $x \not\in u_1 \alpha$ and $u_2 \alpha \subseteq (u_1 x) \alpha$. In such a case, since $u \alpha_i, k = v \alpha_i, k$, there is also a factorization $v = v_1 x v_2$ with $v_1, v_2 \in \{x_1, \ldots, x_i\}^*$ and $x \not\in v_1 \alpha$.

Case 2. If $x$ is not the last letter occurring for the first time in $u$, then there is a factorization $u = u_1 x u_2 y u_3$ with $u_1, u_2, u_3 \in \{x_1, \ldots, x_i\}^*$, $x \not\in u_1 \alpha$, $u_2 \alpha \subseteq (u_1 x) \alpha$ and $y \not\in (u_1 x u_2) \alpha$. In such a case, since $u \alpha_i, k = v \alpha_i, k$, there is also a factorization $v = v_1 x v_2 y v_3$ with $v_1, v_2, v_3 \in \{x_1, \ldots, x_i\}^*$, $x \not\in v_1 \alpha$ and $y \not\in (v_1 x v_2) \alpha$.

**Lemma 2.2:** In Case 1 and Case 2, $u_2 \alpha_i, k-1 = v_2 \alpha_i, k-1$.

**Proof:** Let $u_2 = u'_2 z u''_2$ with $z \in \{x_1, \ldots, x_i\}$. Consider the pair $(u'_2 q_i, k-1, z)$ in $u_2 \alpha_i, k-1$. The pair $((u_1 x u'_2) q_i, k, z)$ is in $u \alpha_i, k$. Since $u \alpha_i, k = v \alpha_i, k$, there is a factorization $v = v' z v''$ with $(u_1 x u'_2) q_i, k = v' q_i, k$. It follows that the $\sim_k$-class of $u_1 x u'_2$ is equal to the $\sim_k$-class of $v'$ and hence $x \in v' \alpha$ and, in Case 2, $y \not\in v' \alpha$. Therefore, the chosen occurrence of $z$ in $v = v' z v''$ must be in $v_2$. There is then a factorization $v_2 = v'_2 z v''_2$ such that $v' = v_1 x v'_2$. Hence $(u'_2 q_i, k-1, z) = (v'_2 q_i, k-1, z)$ and the pair $(u'_2 q_i, k-1, z)$ is in $v_2 \alpha_i, k-1$. Then inclusion $u_2 \alpha_i, k-1 \subseteq v_2 \alpha_i, k-1$ follows. The reverse inclusion is similar.

**Definition 2.2:** Let $k \geq 1$ and let $A = \{x_1, x_2, x_3, \ldots\}$ be a denumerable alphabet of variables including $x$ and $y$ ($u = x_1$ and $y = x_2$).

$C_k$ is the sequence of all equations (over $A$) of the form

$$u_k \ldots u_1 x = u_k \ldots u_1 x^2.$$

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where

\[ \{x\} \subseteq u_1 \alpha \subseteq \ldots \subseteq u_k \alpha \]

\(D_k\) is the sequence of all equations (over \(A\)) of the form

\[ u_k \ldots u_1 xy = u_k \ldots u_1 yx \]

where

\[ \{x, y\} \subseteq u_1 \alpha \subseteq \ldots \subseteq u_k \alpha. \]

We define \(C_0\) as the sequence consisting of the equation \(x = x^2\) and \(D_0\) the sequence consisting of \(xy = yx\).

Let \(J_k\) denote the variety of all semigroups that satisfy all the equations in \(E_k\). The variety \(J_k\) is locally finite, or every finitely generated semigroup in \(J_k\) is finite. For a class \(C\) of semigroups, we denote by \(C^F\) the class of all finite semigroups of \(C\). The equality \(J_k = (J_k)^F\) holds. By [1], if \(k \geq 1\), then the equality \((J_1 * J_k)^F = J_1 * J_k\) holds and \(J_1 * J_k\) is locally finite. Hence \(J_1 * J_k\) is generated by \(J_1 * J_k\) and so \(F_i(J_1 * J_k)\) is the free object on \(\{x_1, \ldots, x_i\}\) in the variety \(J_1 * J_k\).

**Theorem 2.1:** Let \(k \geq 0\). The variety \(J_1 * J_k\) is defined by \(C_k \cup D_k\).

**Proof:** We first want to show that \(J_1 * J_k \models C_k \cup D_k\). Let \(u, v \in \{x_1, \ldots, x_i\}^+\) be such that \(u = v\) is an equation in \(D_k\) (the case of equations in \(C_k\) is similar). By Lemma 2.1, it suffices to show that \(u \alpha_{i,k} = v \alpha_{i,k}\). Let \(u = u_k \ldots u_1 xy\) and \(v = u_k \ldots u_1 yx\) be such that \(\{x, y\} \subseteq u_1 \alpha \subseteq \ldots \subseteq u_k \alpha\). Note that

\[ ((u_k \ldots u_1) q_{i,k}, x) = ((u_k \ldots u_1 y) q_{i,k}, x) \]

since the words \(u_k \ldots u_1\) and \(u_k \ldots u_1 y\) are \(\sim_k\)-equivalent. Note also that

\[ ((u_k \ldots u_1 x) q_{i,k}, y) = ((u_k \ldots u_1) q_{i,k}, y) \]

The equality \(u \alpha_{i,k} = v \alpha_{i,k}\) follows.

Conversely, we want to show that if \(u, v \in \{x_1, \ldots, x_i\}^+\) are such that \(u \alpha_{i,k} = v \alpha_{i,k}\), then \(C_k \cup D_k \vdash u = v\). So, assume that \(u \alpha_{i,k} = v \alpha_{i,k}\). Let \(x \in u \alpha\) and consider the first occurrence of \(x\) in \(u\) and \(v\). As in Lemma 2.2, we denote by \(u_1\) (respectively \(v_1\)) the longest prefix of \(u\) (respectively \(v\)) in which the letter \(x\) does not occur, and we denote by \(u_2\) (respectively \(v_2\)) the longest segment of \(u\) (respectively \(v\)) following the first occurrence of \(x\) in \(u\) (respectively \(v\)) that does not involve any new letters. By Lemma 2.2, the equality \(u_2 \alpha_{i,k-1} = v_2 \alpha_{i,k-1}\) holds. By the inductive hypothesis on
k, we conclude that the equation \( u_2 = v_2 \) is deducible from \( C_{k-1} \cup D_{k-1} \).
By a result of [3] (Proposition 2.3), since \( C_{k-1} \cup D_{k-1} \vdash u_2 = v_2 \) and \( u_2 \alpha \subseteq (x_1 x) \alpha \), then \( C_k \cup D_k \vdash u_1 xu_2 = u_1 xv_2 \).

Let \( z \in \{x_1, \ldots, x_i\} \). Let \( u' \) (respectively \( v' \)) be the longest prefix of \( u \) (respectively \( v \)) before the first occurrence of \( z \). We show that the equation \( u' = v' \) is deducible from \( C_k \cup D_k \). If \( z \) is the first letter in \( u \) (and so also the first letter in \( v \)), then the equation \( u' = v' \) becomes \( 1 = 1 \). We assume that it is true for the first occurrence of \( z = x \) (as in Lemma 2.2), or \( C_k \cup D_k \vdash u_1 = v_1 \). Here \( u_1 xu_2 = u_1 xv_2 = v_1 xv_2 \) is deducible from \( C_k \cup D_k \). If \( x \) is the last letter occurring for the first time in \( u \) (as in Case 1 of Lemma 2.2), we obtain that the equation \( u = v \) is deducible from \( C_k \cup D_k \). Otherwise, the induction step allows us to proceed until the first occurrence of another letter, say \( z = y \) (as in Case 2 of Lemma 2.2). After every letter of \( u \) has been found, we obtain the deducibility of the equation \( u = v \) from \( C_k \cup D_k \).

Since \( J_1 \ast J_k = (J_1 \ast J_k)^F \), any sequence of equations for \( J_1 \ast J_k \) is also a sequence of equations for \( J_1 \ast J_k \).

**Corollary 2.1:** Let \( k \geq 0 \). The S-variety \( J_1 \ast J_k \) is defined by \( C_k \cup D_k \).

Note that if two words \( u \) and \( v \) form an equation \( u = v \) for \( J_1 \ast J_k \), then \( u \sim_{k+1} v \). Equations for other S-varieties generalizing the S-varieties \( J_k \) have been built from properties of congruences generalizing the congruences \( \sim_k \) (see [7], [8], [9], [12]).

Pin has given the equational characterization of \( J_1 \ast J_1 \) of Theorem 2.2 and Almeida the characterization of \( J_1^k \) of Theorem 2.3.

**Theorem 2.2.** (Pin [22]): The S-variety \( J_1 \ast J_1 \) is defined by \( C_1 \cup D_1 \) or equivalently by the two equations \( xux = xux^2 \) and \( xuyuy = xuyvyx \).

**Theorem 2.3** (Almeida [3]): Let \( k \geq 0 \). The S-variety \( J_1^{k+1} \) is defined by \( C_k \cup D_k \).

From the preceding results, we deduce the following corollary.

**Corollary 2.2:** Let \( k \geq 0 \). The S-varieties \( J_1 \ast J_k \) and \( J_1^{k+1} \) are equal and hence the S-variety \( J_1 \ast J_k \) is decidable.

A result of Almeida [3] implies the following.

**Corollary 2.3:** The S-variety \( J_1 \ast J_k \) is defined by a finite sequence of equations if and only if \( k = 1 \).

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As mentioned at the beginning of this section, we have worked essentially with semigroups in section 2. As explained in [3], since the \( S \)-variety generated by the \( M \)-variety \( J_k \) is monoidal, results such as Theorems 2.2 and 2.3, and Corollaries 2.1, 2.2 and 2.3 can be translated to results on the \( M \)-varieties \( J_1 \ast J_k \) and \( J_1^{k+1} \).

3. ON A CONJECTURE OF PIN

Theorem 3.1 gives a new proof that a conjecture of Pin concerning tree-hierarchies of \( M \)-varieties is false (another proof was given in [13] using different techniques). Let \( M_1, \ldots, M_k \) be finite monoids. The Schützenberger product of \( M_1, \ldots, M_k \), denoted by \( \mathcal{Q}_k (M_1, \ldots, M_k) \), is the submonoid of upper triangular \( k \times k \) matrices with the usual multiplication of matrices, of the form \( x = (x_{ij}) \), \( 1 \leq i, j \leq k \), in which the \( (i, j) \)-entry is a subset of \( M_1 \times \ldots \times M_k \) and all of whose diagonal entries are singletons, that is

1. \( x_{ij} = \emptyset \) if \( i > j \);
2. \( x_{ii} = \{(1, \ldots, 1, m_i, 1, \ldots, 1)\} \) for some \( m_i \in M_i \) (here, \( m_i \) is the \( i \)th component in the \( k \)-tuple);
3. \( x_{ij} \subseteq \{(m_1, \ldots, m_k) \in M_1 \times \ldots \times M_k \mid m_1 = \ldots = m_{i-1} = 1 = m_{j+1} = \ldots = m_k\} \)

(here, 1 is the identity of \( M_1, \ldots, M_k \)).

Condition (2) allows to identify \( x_{ii} \) with an element of \( M_i \) and Condition (3) \( x_{ij} \) with a subset of \( M_i \times \ldots \times M_j \). If \( \bar{m} = (m_i, \ldots, m_j) \in M_i \times \ldots \times M_j \)
and
\( \bar{m}' = (m_i', \ldots, m_j') \in M_i' \times \ldots \times M_j' \),
then \( \bar{m} \bar{m}' = (m_i, \ldots, m_{j-1}, m_j m_i', m_{i'+1}, \ldots, m_j') \) if \( j = j' \), and is undefined otherwise. This multiplication is extended to sets in the usual fashion; addition is given by set union.

We will denote by \( T \) the set of trees on the alphabet \( \{a, \bar{a}\} \). Formally, \( T \) is the set of words in \( \{a, \bar{a}\}^* \) congruent to 1 in the congruence generated by the relation \( a\bar{a} = 1 \). Intuitively, the words in \( T \) are obtained as follows: we draw a tree and starting from the root we code \( a \) for going down and \( \bar{a} \) for going up. For example,
is coded by \texttt{aa\ldots aa} (where each \texttt{a} is coded by \texttt{a}). The number of leaves of a word \(t\) in \(\{a, \bar{a}\}^*\), denoted by \(l(t)\) is by definition the number of occurrences of the factor \(a\bar{a}\) in \(t\). Each tree \(t\) factors uniquely into \(t = at_1 \bar{a}at_2 \bar{a} \ldots at_k \bar{a}\) where \(k \geq 0\) and where the \(t_i\)'s are trees. Let \(t\) be a tree and let \(t = t_1 at_2 \bar{a}t_3\) be a factorization of \(t\). We say that the occurrences of \(a\) and \(\bar{a}\) defined by this factorization are related if \(t_2\) is a tree. Let \(t\) and \(t'\) be two trees. We say that \(t\) is \textit{extracted} from \(t'\) if \(t\) is obtained from \(t'\) by removing in \(t'\) a certain number of related occurrences of \(a\) and \(\bar{a}\). We now give Pin's tree hierarchy construction using Schützenberger's product.

To each tree \(t\) and to each sequence \(V_1, \ldots, V_{l(t)}\) of \(M\)-varieties is associated an \(M\)-variety \(\diamond_t (V_1, \ldots, V_{l(t)})\) defined recursively by:

1. \(\diamond_1 (V) = V\) for every \(M\)-variety \(V\);
2. if \(t = at_1 \bar{a}at_2 \bar{a} \ldots at_k \bar{a}\) with \(k \geq 0\) and \(t_1, \ldots, t_k \in T\),
   \(\diamond_t (V_1, \ldots, V_{l(t)})\) is the \(M\)-variety of monoids that divide some
   \(\diamond_k (M_1, \ldots, M_k)\) with \(M_1 \in \diamond_{t_1} (V_1, \ldots, V_{l(t_1)}), \ldots, M_k \in \diamond_{t_k} (V_{l(t_1)} + \ldots + l(t_{k-1}) + 1, \ldots, V_{l(t_k)} + \ldots + l(t_k))\).

   When \(V_1 = \ldots = V_{l(t)} = V\), we denote simply by \(\diamond_t (V)\) the \(M\)-variety
   \(\diamond_t (V_1, \ldots, V_{l(t)})\). More generally, if \(T\) is a language contained in \(T\), we
   denote by \(\diamond_T (V)\) the smallest \(M\)-variety containing the \(M\)-varieties \(\diamond_t (V)\)
   with \(t \in T\).

Let \(I\) denote the trivial \(M\)-variety. In [21], the following equalities are shown:
\(\diamond_{(a\bar{a})}^+ (I) = J_k\) and \(\diamond_{(a\bar{a})} (I) = J\). Also, it is shown there that if \(V\) is an arbitrary \(M\)-variety, then \(\diamond_{(a\bar{a})}^2 (V, I) = J_1 \ast V\).

Among the many problems concerning these tree hierarchies, is the comparison between the \(M\)-varieties inside a hierarchy. More precisely, the problem consists in comparing the different \(M\)-varieties \(\diamond_t (V)\) (or even \(\diamond_T (V)\)). A partial result and a conjecture on this problem was given in Pin [21]. It was shown that for every \(M\)-variety \(V\), if \(t\) is extracted
from $t'$, then $\diamondsuit_t(V) \subseteq \diamondsuit_{t'}(V)$, and it was conjectured that if $t, t' \in T'$, $\diamondsuit_t(I) \subseteq \diamondsuit_{t'}(I)$ if and only if $t$ is extracted from $t'$. Here, $T'$ denotes the set of trees in which each node is of arity different from 1.

**Theorem 3.1:** The above conjecture is false.

**Proof:** To see this, let $k > 1$ and let $t = a_{k+1}(\bar{a}a\bar{a})^{k+1}$ and $t' = a(a\bar{a})^{k+1} \bar{a}a\bar{a}$. The equalities $\diamondsuit_t(I) = J_1^{k+1}$ and $\diamondsuit_{t'}(I) = \diamondsuit_{(a\bar{a})^2}(J_k, I) = J_1 \ast J_k$ hold. But $J_1 \ast J_k = J_1^{k+1}$ by Corollary 2.2 (M-variety version), and it is easy to verify that the tree $t$ is not extracted from the tree $t'$. □

**REFERENCES**