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ON CONTINUOUS FUNCTIONS COMPUTED BY FINITE AUTOMATA (1)

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Abstract. – *Weighted Finite Automata (WFA) can be used to define functions from [0, 1] into* $\mathbb{R}$. *We give here a method to construct more and more complex WFA computing continuous functions. We give also an example of a continuous function having no derivative at any point, that can be computed with a 4-state WFA.*

1. INTRODUCTION

Finite automata constitute a fundamental and simple method to describe an input-output behaviour, in other words, to compute functions. Computations of such functions are, in any sense, finitarily defined and easy to implement. Typically functions computed are so-called word functions, that is (partial) functions from a set of words over some alphabet into itself or the binary set {yes, no}.

A different, but still finitary, way to use finite automata to compute functions was introduced in [CKarh]. In this approach, finite automata are used to compute ordinary real functions from the unit interval [0, 1] into the set of real numbers. This approach which was motivated by computer graphic, cf. [CKari], or [CD] and [BM] as earlier papers on this topic, and [B] as a related but different approach, is closely related to the theory of

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rational power series, *cf.* [BR] or [SS], the main difference being that the computations are carried out on infinite words.

The automata used in this new approach are ordinary finite nondeterministic automata equipped with the weight function, that is to say, each transition is labelled besides an input symbol also by a real number called its weight. Such automata were called in [CKarh] weighted finite automata, WFA for short, and $\mathbb{R}$-Σ-automata in [E].

A WFA $\mathcal{A}$ computes a real function $\widehat{f_{\mathcal{A}}}$: $[0, 1] \to \mathbb{R}$ as follows. First, the input $x \in [0, 1]$ is identified with the binary infinite word $bin(x)$ in $\{0, 1\}^{\omega}$, its binary representation. Second, the weight associated to $bin(x)$ is computed by $\mathcal{A}$. This is value of $\widehat{f_{\mathcal{A}}}$ at point x. Of course, in order to be well-defined, some convergence considerations are needed.

A particularly interesting class of weighted finite automata, namely that of level automata, was also defined in [CKarh]. For these automata, $\widehat{f_{\mathcal{A}}}$ is always defined. The importance of this class was demonstrated in [CKarh] and [DKLT]. We continue this research here, concentrating on questions when functions computed are continuous.

In Section 3, we establish a method to construct more and more complex WFA (in fact, level automata) computing continuous functions. This method is based on describing a sufficient condition for a level automaton to define a continuous function. As a matter of fact our construction produces exactly the class of strongly continuous level automata, that is, level automata computing continuous functions for any initial distribution. Moreover we prove in Theorem 2 that if a level automaton computes a continuous function for a given initial distribution one can construct a strongly continuous level automaton computing the same function. This gives a straightforward algorithm to decide the continuity of a function computed by a level automaton (cf. [CKarh]).

In Section 4, we apply our above construction to define a 4 states level automaton computing a function which does not have a derivative at any point. This clearly indicates that WFA are powerful to define complicated functions, although as shown in [DKLT] they can compute only relatively few smooth functions, that is functions having all the derivatives. We want to emphasize that only the automaton computing our complicated function is simple, but also the computations to obtain the values (or their approximations) of the function are not difficult – they are essentially as complicated as to compute the values of a polynomial of degree 3!

2. PRELIMINARIES

For a finite alphabet Σ let Σ^* (resp. Σ^ω) be the best set of finite (resp. infinite) words over Σ. For the purposes of this paper we can assume that Σ is binary, say $\Sigma = \{0, 1\}$.

We recall the definition of weighted finite automata from [CKarh].

A WFA (*weighted finite automaton*) is defined as a 5-tuple $\mathcal{A} = (Q, \Sigma, W, I, T)$, where

- Q is a finite set of states,
- Σ is a finite alphabet,
- $W : Q \times \Sigma \times Q \to \mathbb{R}$ is the weight function,
- $I : Q \to \mathbb{R}$ is the initial distribution,
- $T : Q \to \mathbb{R}$ is the final distribution.

A WFA can be represented by matrices: for each letter of a Σ, one defines a $Q \times Q$-matrix W_a over reals, in which $W_a(p, q) = W(p, a, q)$ for all $p, q \in Q$. Moreover, I represented by a row-vector and T by a column-vector. By convention, if not otherwise stated, we assume that $I = (1, 0, \ldots, 0)$ and $T = (0, \ldots, 0, 1)$.

The *distribution* of a word $w = w_1 w_2 \ldots w_n$ on an automaton $\mathcal{A}$ is noted $P_\mathcal{A}(w)$ and is defined by

$$P_\mathcal{A}(w) = I \cdot W_w \qquad \text{with} \quad W_w = W_{w_1} \cdot W_{w_2} \ldots W_{w_n}.$$

A WFA can be used to compute real functions on the interval [0, 1] as follows. Let $w \in \{0, 1\}^\omega$ be $w = w_1 w_2 \ldots w_n \ldots$ with $w_i \in \{0, 1\}$. Then w is interpreted as the real $\hat{w} = \sum_{i=1}^{\infty} w_i 2^{-i}$, and this correspondence comes one-to-one if we assume that $w \notin \Sigma^* 01^\omega$. Now, the WFA $\mathcal{A}$ defines:

$$(1) \qquad f_\mathcal{A}: \quad \Sigma^\omega \to \mathbb{R}, \qquad f_\mathcal{A}(w) = \lim_{n \to \infty} P_\mathcal{A}(w_1 \cdot w_2 \ldots w_n) \cdot T$$

and

$$\widehat{f_\mathcal{A}}: \quad [0, 1] \to \mathbb{R}, \qquad \widehat{f_\mathcal{A}}(x) = f_\mathcal{A}(w),$$

where $\hat{w} = x$ and $w \notin \Sigma^* 01^\omega$.

These definitions assume the existence of the limit (1). In this paper we avoid such considerations either by restricting our family of automata such that the existence of the limit is guaranted or by working under the assumption that the limit exists. A class of WFA guaranteing the existence of the limit (1) was introduced in [CKarh]. These automata were called *level automata* and were defined as WFA satisfying:

(1) The only loops in the underlying automaton of $\mathcal{A}$ are of the form $p \xrightarrow{a} p$,

(2) $0 \leqq W(p, a, p) < 1$ for all $p \in Q$, $a \in \Sigma$ such that there exists $q \in Q$, $q \neq p$, and $b \in \Sigma$ such that $W(p, b, q) \neq 0$, and otherwise $W(p, a, p) = 1$,

(3) $I \in \mathbb{R}_+^n$ and $T \in \mathbb{R}_+^n$, where $n = \text{Card}(Q)$,

(4) The underlying automaton of $\mathcal{A}$ is *reduced*, that is does not have useless states.

Note that we slightly modify the definition given in [CKarh] since here we allow negative weights on connecting transitions.

The *degree* of a state in a level automaton $\mathcal{A}$ is defined as the maximum of lengths of loop-free paths in $\mathcal{A}$ starting from that state, and the degree of $\mathcal{A}$ is the greatest degree of its states.

Clearly one can assume that there is a single state of degree 0, and in the sequel we shall consider only automata having a single state of degree 0.

Finally, a level automaton is called a *line automaton* iff for each $n \in \{0, 1, \ldots, \text{Card}(Q) - 1\}$, there exists exactly one state of degree n.

3. CONTINUITY CONSTRUCTION

In this section we study when a level automaton defines a continuous function.

Clearly level automata with two states are line automata. Hence, at the starting point for our considerations we recall that the continuity of the function defined by such an automaton (shown in figure 1) is characterized by the condition

$$(2) \qquad \alpha + \beta = 1, \qquad \text{or} \qquad (1 - \beta)\gamma = (1 - \alpha)\delta,$$

where α and β are the weights of the loops in the state of degree 1, *cf.* [CKarh]. Let now $\mathcal{A}$ be an arbitrary level automaton and let α and β be fixed non negative real numbers smaller than 1. Denote by Q the state set of $\mathcal{A}$ and by n the cardinality of Q. For each state q in Q let $\mathcal{A}_q$ be the subautomaton of A which constitutes of those states of Q which are accessible from q and of those transitions of $\mathcal{A}$ which connect these states.

We define a family $\mathcal{A}(\alpha, \beta)$ of automata, the elements of which can be viewed as extensions of $\mathcal{A}$, as follows: Each $\mathcal{A}_{ext}$ in $\mathcal{A}(\alpha, \beta)$ contains all

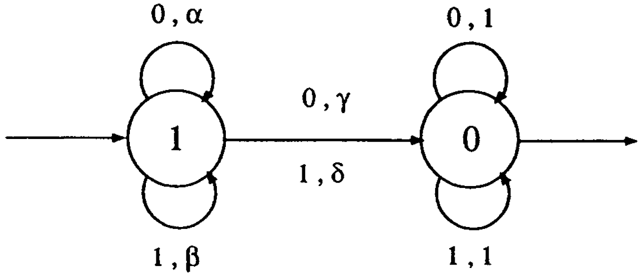

Figure 1. – Line automaton of degree 1.

states and transitions of $\mathcal{A}$, and in addition a new initial state r together with the transitions

$$r \overset{0,\alpha}{\longrightarrow} r, \quad r \overset{1,\beta}{\longrightarrow} r$$

$$r \xrightarrow{i, W(i, q)} q \qquad \text{for} \quad i \in \{0, 1\}, \quad q \in Q.$$

Here $W(i, q)$'s are arbitrary real numbers. It follows that each $\mathcal{A}_{ext}$ in $\mathcal{A}(\alpha, \beta)$ is completely specified by the $2n$ dimensional vector

$$(3) \qquad (W(0, q), W(1, q)) \in \mathbb{R}^{2n}.$$

As we said we consider r the only intial state of $\mathcal{A}_{ext}$, and by convention, we assume that each $\mathcal{A}_{ext}$ is reduced. Consequently, $\mathcal{A}_{ext}$ is of degree $n+1$ iff there exists an input letter j and a state q of $\mathcal{A}$ of degree n such that $W(j, q) \neq 0$. Observe also that our construction is very general: each level automaton can be obtained in this way from the one state level automaton.

With the above terminology, we are going to show

THEOREM 1: *Let $\mathcal{A}$, α and β be fixed as above and assume that for each state q, $\mathcal{A}_q$ defines a continuous function. Then $\mathcal{A}_{ext}$ from $\mathcal{A}(\alpha, \beta)$ represented by (3) defines a continuous function if and only if one of the following conditions holds:*

(i) $\alpha + \beta = 1$ *and* $\forall q \in Q$, $f_{A_q}(0^\omega) = f_{A_q}(1^\omega)$ *(In that case W's are arbitrary).*

(ii) (4) $\sum_{q \in Q} \lambda_q W(0, q) + \sum_{q \in Q} \mu_q W(1, q) = 0.$

for some fixed numbers λ_q and μ_q such that at least one of them is not equal to zero. In that case $(W(0, q), W(1, q))$ belongs to a fixed hyperplane of $\mathbb{R}^{2n}$.

Proof: Consider an arbitrary automaton $\mathcal{A}_{ext}$ in $\mathcal{A}(\alpha, \beta)$ represented by the vector $(W(0, q), W(1, q))$. We first assume that $\widehat{f_{\mathcal{A}_{ext}}}$ is continuous. It is so at the point 1/2, and therefore necessarily

$$f_{\mathcal{A}_{ext}}(01^\omega) = f_{\mathcal{A}_{ext}}(10^\omega) \tag{5}$$

Now the left hand side of (5) can be written as

$$f_{\mathcal{A}_{ext}}(01^\omega) = \alpha \left[\frac{1}{1-\beta} \sum_{q \in Q} W(1, q) f_{\mathcal{A}_q}(1^\omega)\right] + \sum_{q \in Q} W(0, q) f_{\mathcal{A}_q}(1^\omega).$$

Clearly, a similar formula holds for the right hand side of (5). Hence we get (4) with:

$$\lambda_q = f_{\mathcal{A}_q}(1^\omega) - \frac{\beta}{1-\alpha} f_{\mathcal{A}_q}(0^\omega),$$
$$\mu_q = \frac{\alpha}{1-\beta} f_{\mathcal{A}_q}(1^\omega) - f_{\mathcal{A}_q}(0^\omega).$$

Note that the λ_q's and μ_q's can be all equal to zero. This happens if and only if $\alpha + \beta = 1$ and $f_{\mathcal{A}_q}(0^\omega) = f_{\mathcal{A}_q}(1^\omega)$ (case (i)). Then, the vector $(W(0, q), W(1, q))$ can be chosen arbitrary.

Now, we assume that λ_q's and μ_q's are thus fixed. We have to show that the automaton $\mathcal{A}_{ext}$ represented by a vector $(W(0, q), W(1, q))$ satisfying (4) actually defines a continuous function. This is done by considering separately three different cases:

Case 1: Continuity at point 1/2. By the choice of λ_q's and μ_q's, $\mathcal{A}_{ext}$ satisfies (5). Moreover, using the similar arguments as in [CKarh], this implies the continuity at point 1/2. (Observe that we have to modify slightly the considerations of [CKarh], since we allow in a level automaton negative weights in connecting transitions.)

Case 2: Continuity at the point having a finite binary representation, that is two representations $w\,01^\omega$ and $w\,10^\omega$ for some $w \in \Sigma^*$. Now, the continuity at this point is reduced to check the continuity of $\widehat{f_{\mathcal{A}_{ext}(w)}}$ at the point 1/2, where $\mathcal{A}_{ext}(w)$ is the automaton $\mathcal{A}_{ext}$ with the initial distribution $(1, 0, \ldots, 0)\, W_w$. But $\widehat{f_{\mathcal{A}_{ext}(w)}}$ is clearly continuous at the point 1/2, since, by Case 1, $\widehat{f_{\mathcal{A}_{ext}}}$ is so and each $\widehat{f_{\mathcal{A}_q}}$ is continuous (even in the whole interval) by our assumptions.

Case 3: Continuity at the point having only one infinite binary representation. Again, as in [CKarh], this is always true for level automata of our type. □

Theorem 1 deserves several remarks. First, it illustrates very clearly, as was already noted in [CKarh], that a level automaton defines very seldomly a continuous function. This is made more concrete in the following example.

Example 1: Consider the level automaton of figure 2.

Since $\gamma + (1 - \gamma) = 1$, the automaton $\mathcal{A}_1$ defines a continuous function. Since $f_{\mathcal{A}_1}(0^\omega) = 0 \neq f_{\mathcal{A}_1}(1^\omega)$, by Theorem 1, there exists a unique hyperplane in $\mathbb{R}^2$, that is a line going through the origin, such that $\mathcal{A}$ defines a continuous function iff (x, y) belongs to that line. That means that the ratio x/y is unique. In particular, if y is fixed, say $y = 1/2$, then only one value of x makes $\widehat{f_{\mathcal{A}}}$ continuous. For $\gamma = 1/2$, $\alpha = \beta = 1/4$, this value is $1/4$, and $\mathcal{A}$ computes the function $f(x) = x^2$.

The above leads to the following interesting observation. Assume that in $\mathcal{A}$ both x and y making $\widehat{f_{\mathcal{A}}}$ continuous are different from 0. Then the most natural decomposition of $\mathcal{A}$ into two subautomata is by taking $\mathcal{A}'_1$ to be the subautomaton excluding state 1 and $\mathcal{A}'_2$ to be the subautomaton excluding only the transition $2 \overset{1,x}{\to} 0$. Clearly,

$$\widehat{f_{\mathcal{A}}} = \widehat{f_{\mathcal{A}'_1}} + \widehat{f_{\mathcal{A}'_2}}.$$

And although $\widehat{f_{\mathcal{A}}}$ is continuous, both $\widehat{f_{\mathcal{A}'_1}}$ and $\widehat{f_{\mathcal{A}'_2}}$ are not. In particular, if we fix parameters of $\mathcal{A}$ such that it computes the parabola, we obtain a decomposition of the parabola into the sum of two functions both of which are noncontinuous. An interesting point here is, as is easy to see, *cf.* also [DKLT], that the automaton given here for the parabola is the simplest possible, and that the decomposition is the only natural one from the point of view of automata theory.

Our second remark is that Theorem 1 provides a simple systematic method to construct more and more complicated automata computing continuous functions. This method is already illustrated in Example 1, and we use it again in Section 4.

Our third remark is that the conditions that the subautomata $\mathcal{A}_q$ define continuous functions is not necessary, that is a level automaton or even a line automaton can define a continuous function for the standard initial distribution without being obtainable by a recursive application of the construction of Theorem 1. This is seen as follows.

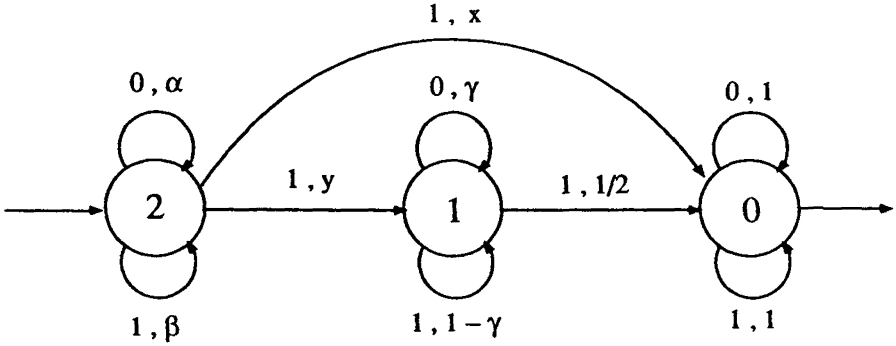

Figure 2. – A level automaton of degree 2.

Example 2: Consider the line automaton $\mathcal{A}$ shown in Figure 3. Now, by the criterion described in (2) the subautomaton $\mathcal{A}_1$ defines a noncontinuous function. It is easy to see that for any $w \in \Sigma^*$, if $P_{\mathcal{A}}(w) = (\alpha, \beta, \gamma, \delta)$ then $\beta = \gamma$. Thus we can delete the state 1, the transitions concerning this state and replace $W_1(2, 0)$ by $W_1(2, 0) + W_1(1, 0)$. This new automaton $\mathcal{A}'$ computes the same function than $\mathcal{A}$ and it can be obtained by the continuity construction. One can verify that the computed function is x^2.

Now we shall prove that the above transformation can always be done.

An automaton is said *strongly continuous* if it computes continuous functions for any initial distribution.

THEOREM 2: *Let $\mathcal{A}$ be a level automaton computing a continuous function for an initial distribution I. If $\mathcal{A}$ is not strongly continuous, one can construct a strongly continuous level automaton having less states than $\mathcal{A}$ and computing the same function.*

Proof: We shall reason by induction on k, the number of states of $\mathcal{A}$. If $k = 1$, $\mathcal{A}$ is clearly strongly continuous. Let $Q = \{q_0, \ldots, q_n\}$ be the set of states of $\mathcal{A}$ such that q_0 is the single state of degree 0 and for $0 \leqq i \leqq j \leqq n$, the degree of q_i is not greater than the degree of q_j. Let us consider $D = \{P_{\mathcal{A}}(w)/w \in \Sigma^*\}$. For the initial distribution $d = P_{\mathcal{A}}(w)$, $\mathcal{A}$ computes a continuous function, namely $\widehat{f_{\mathcal{A}}}(x/(2^{|w|}) + \hat{w})$. The same holds for any initial distribution in $E = \{\lambda_1 \alpha_1 + \cdots + \lambda_p \alpha_p / \lambda_i \in \mathbb{R}, \alpha_i \in D\}$, the linear space generated by D.

If $E = \mathbb{R}^{n+1}$, we are done.

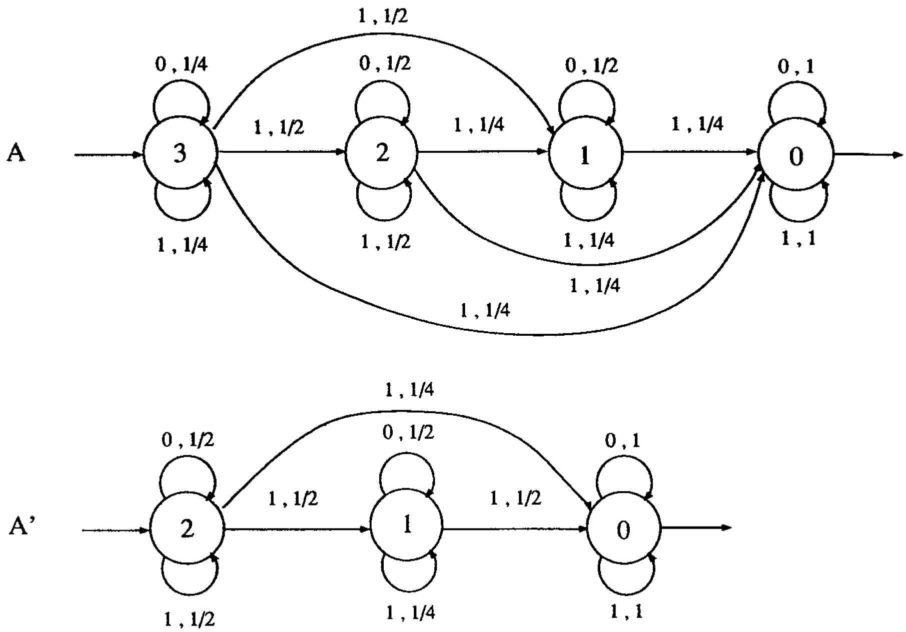

Figure 3. – Line automata $\mathcal{A}$ and $\mathcal{A}'$.

Otherwise, there exists a column-vector $\mu = (\mu_0, \ldots, \mu_n) \neq 0$, such that for any $d \in E$ we have $\mu\, d = 0$.

Let i_0 be the smallest integer such that $\mu_{i_0} \neq 0$. One can assume that $\mu_{i_0} = 1$. Then for any $d = (d_0, \ldots, d_n) \in E$, $d_{i_0} = -(\mu_{i_0+1}\, d_{i_0+1} + \cdots + \mu_n\, d_n)$.

If $i_0 = 0$, we shall show that $f_{\mathcal{A}}$ is the constant function 0. Indeed, since for states of degree >0 the weights of the loops are smaller than 1, $\forall \varepsilon > 0$, $\exists\, n$ such that if $|w| > n$, $d = P_{\mathcal{A}}(w) = (d_0, d_1, \ldots, d_n)$ then $|d_1|, \ldots, |d_n| < \varepsilon$. Hence $|d_0| \leqq \varepsilon\,(\mu_1 + \cdots + \mu_n)$, so that $f_{\mathcal{A}}$ is the zero function.

If $i_0 \neq 0$, we shall delete the state q_{i_0} and the transitions concerning q_{i_0}. Then we have to modify the weights of some connecting transitions in such a way that this automaton $\mathcal{A}'$ computes the same function.

We set $Q' = Q - \{q_{i_0}\}$ and we define the $Q' \times Q'$ matrix W'_a, for $a = 0, 1$, by $W'_a(q_i, q_j) = W_a(q_i, q_j) - \mu_i\, W_a(q_{i_0}, q_j)$. It is easy to check that $\mathcal{A}'$ is a n states level automaton computing $f_{\mathcal{A}}$ and we can finish our construction by the induction hypothesis. □

4. AN EXAMPLE: NOWHERE DERIVABLE FUNCTION

In this section, we shall consider a level automaton $\mathcal{A}(t)$ shown in Figure 4. First, we shall prove that for any value of t and $x(t)$, $\mathcal{A}(t)$ defines a continuous function. Then, for some particular values of t we look at the set of point where $\mathcal{A}(t)$ has a derivative. In particular, if $t = 2/3$, we get a level automaton computing a continuous function having no derivatives at any point of the interval [0, 1].

Claim 1: *$\mathcal{A}(t)$ is strongly continuous.*

Proof: Let $g : \Sigma^\omega \to \mathbb{R}$ be the function computed by the automaton $\mathcal{A}(t)$ for an arbitrary distribution $(\alpha, \beta, \beta', \gamma)$. Then, $g = \alpha f_{\mathcal{A}(t)} + \beta f_{\mathcal{A}_1} + \beta' f_{\mathcal{A}_{1'}} + \gamma$ where $\mathcal{A}_1$ (resp. $\mathcal{A}_{1'}$) is the subautomaton containing only states 1 and 0 (resp. $1'$ and 0). From [CKarh], $\mathcal{A}(t)$ is strongly continuous if and only if $g(10^\omega) = g(01^\omega)$. This condition holds true since $f_{\mathcal{A}_1}(10^\omega) = f_{\mathcal{A}_1}(01^\omega)$, $f_{\mathcal{A}_{1'}}(10^\omega) = f_{\mathcal{A}_{1'}}(01^\omega)$ and $f_{\mathcal{A}(t)}(10^\omega) = f_{\mathcal{A}(t)}(01^\omega)$. Indeed, $\mathcal{A}_1$ and $\mathcal{A}_{1'}$ are strongly continuous and the equality $f_{\mathcal{A}(t)}(10^\omega) = f_{\mathcal{A}(t)}(01^\omega)$ follows from the symmetry of $\mathcal{A}(t)$. □

Although $\mathcal{A}(t)$ is strongly continuous for any value of t, the subautomaton $\mathcal{A}'$ (resp. $\mathcal{A}''$) containing only states 2, 1 and 0 (resp. 2, $1'$ and 0) defines a continuous function only for a particular value of $x(t)$. The condition for $x(t)$ is determined by $f_{\mathcal{A}'(t)}(10^\omega) = f_{\mathcal{A}'(t)}(01^\omega)$ that is by

$$t\left[\frac{1}{1-t}\,x(t) + \frac{1}{1-t}\cdot 1\cdot\frac{4}{3}\right] = x(t),$$

or equivalently,

$$(1-2t)\,x(t) = \frac{4}{3}\,t. \tag{6}$$

Example 3: Let us consider $\mathcal{A}(t)$ for $t = 3/4$.

Now (6) yields $x(t) = -2$.

It is straightforward to compute that for any finite word $w \in \Sigma^*$

$$\widehat{f_{\mathcal{A}(3/4)}}'(\widehat{w\,0^\omega}) \text{ does not exist,}$$

and that

$$\widehat{f_{\mathcal{A}(3/4)}}'(w\,\widehat{(01)}^\omega) = 0.$$

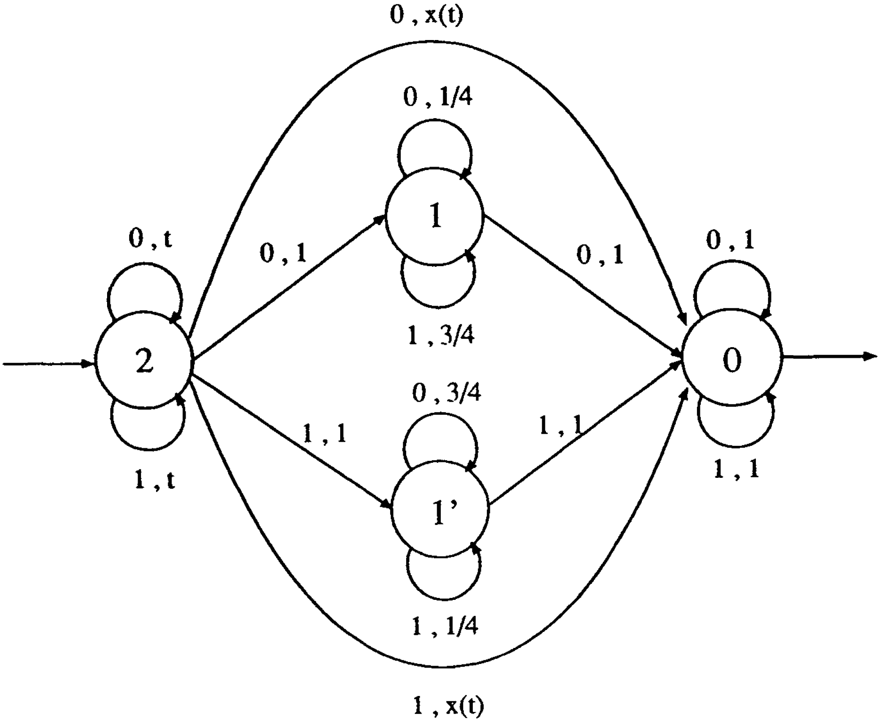

Figure 4. – Level automaton $\mathcal{A}(t)$.

In order to prove these formulas we proceed as follows. First, we note that for any initial distribution $(y, 2y, 2y, z)$, $\mathcal{A}(3/4)$ computes a constant function. Now,

$$\begin{aligned}(\alpha, \beta, \gamma, \delta) W_{0^n} &= \left(\alpha\left(\frac{3}{4}\right)^n, 2\alpha\left(\frac{3}{4}\right)^n - 2\alpha\left(\frac{1}{4}\right)^n \right. \\ &\quad \left. +\beta\left(\frac{1}{4}\right)^n, \gamma\left(\frac{3}{4}\right)^n, z\right) \\ &= (y, 2y, 2y, z) \\ &\quad + \left(0, (\beta - 2\alpha)\left(\frac{1}{4}\right)^n, (\gamma - 2\alpha)\left(\frac{3}{4}\right)^n, 0\right)\end{aligned}$$

for $y = \alpha(3/4)^n$ and for some value of z. Consequently, if for $w \in \Sigma^*$

$$P_{\mathcal{A}(3/4)}(w) = (\alpha, \beta, \gamma, \delta),$$

then

$$f_{\mathcal{A}(3/4)}(w\,0^n\,0^\omega) - f_{\mathcal{A}(3/4)}(w\,0^n\,1^\omega)$$
$$= \left(\left(\frac{1}{4}\right)^n \beta - \left(\frac{1}{4}\right)^n 2\,\alpha\right)\cdot\frac{4}{3} - \left(\left(\frac{3}{4}\right)^n \gamma - \left(\frac{3}{4}\right)^n 2\,\alpha\right)\cdot\frac{4}{3}.$$

Now, it follows directly by induction that $\gamma < 2\,\alpha$, so that the limit

$$\lim_{n\to\infty} \frac{|f_{\mathcal{A}(3/4)}(w\,0^n\,0^\omega) - f_{\mathcal{A}(3/4)}(w\,0^n\,1^\omega)|}{|\widehat{w\,0^n\,0^\omega} - \widehat{w\,0^n\,1^\omega}|}$$
$$= \lim_{n\to\infty} \frac{(4/3)\,(2\,\alpha-\gamma)}{(1/2)^{|w|}}\cdot\frac{(3/4)^n}{(1/2)^n} = \infty,$$

proving that $f'_{\mathcal{A}(3/4)}(\widehat{w\,0^\omega})$ does not exist.

Similarly,

$$(\alpha,\,\beta,\,\gamma,\,\delta)\,W_{(01)^n} = \left(\alpha\left(\frac{3}{4}\right)^{2n},\, 2\,\alpha\left(\frac{3}{4}\right)^{2n}\left(1-\left(\frac{1}{3}\right)^n\right)\right.$$
$$\left. + \beta\left(\frac{3}{16}\right)^n,\, 2\,\alpha\left(\frac{3}{4}\right)^{2n}\left(1-\left(\frac{1}{3}\right)^n\right) + \gamma\left(\frac{3}{16}\right)^n,\, y\right)$$
$$= (y,\,2\,y,\,2\,y,\,z) + (0,\,\beta',\,\gamma',\,0)$$

for some value z and for $y = \alpha\,(3/4)^{2n}$,

$$\beta' = \beta\left(\frac{3}{16}\right)^n - 2\,\alpha\left(\frac{3}{4}\right)^{2n}\left(\frac{1}{3}\right)^n = (\beta - 2\,\alpha)\left(\frac{3}{16}\right)^n$$

and

$$\gamma' = (\gamma - 2\,\alpha)\left(\frac{3}{16}\right)^n.$$

The function computed by $\mathcal{A}\,(3/4)$ for the initial distribution $(0,\,\beta',\,\gamma',\,0)$ is bounded by $4\,|\beta' + \gamma'| \leqq 4\,(4\,\alpha + \beta + \gamma)\,(3/16)^n$. Consequently, if, for $w \in \Sigma^*$, $P_{\mathcal{A}(3/4)}(w) = (\alpha,\,\beta,\,\gamma,\,\delta)$ then for any $w' \in \Sigma^\omega$ we have

$$|f_{\mathcal{A}(3/4)}(w\,(01)^n\,(01)^\omega) - f_{\mathcal{A}(3/4)}(w\,(01)^n\,w')|$$
$$\leqq 2\cdot 4\cdot(4\,\alpha + \beta + \gamma)\left(\frac{3}{16}\right)^n.$$

Since for $w' \in 1\,\Sigma^\omega \cup 00\,\Sigma^\omega$,

$$|\widehat{w\,(01)^n\,(01)^\omega} - \widehat{w\,(01)^n\,w'}| \geqq \frac{1}{12}\cdot\left(\frac{1}{2}\right)^{|w|+2n},$$

we see that

$$\widehat{f_{\mathcal{A}(3/4)}}'\,(w\,(\widehat{01)^{\omega}}) = 0.$$

The graph of the function $\widehat{f_{\mathcal{A}(3/4)}}$ with $x = -2$ is shown in Figure 5.

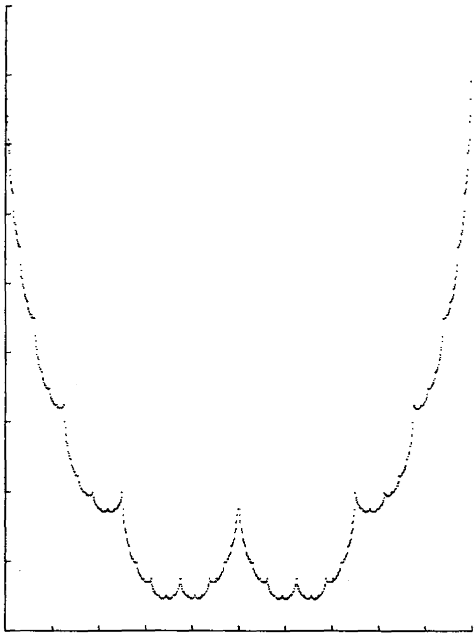

Figure 5. – $t = 3/4$; $x(t) = -2$.

Example 4: For $t = 1/4$ (6) yields $x(t) = 2/3$ and one can draw similar conclusions as in Example 3. Indeed, the graph of $\widehat{f_{\mathcal{A}(1/4)}}$ shown in Figure 6 is in a certain sense dual of that of $\widehat{f_{\mathcal{A}(3/4)}}$.

Example 5: For $t = 2/3$ (6) yields $x(t) = -8/3$.

The graph of the function $\widehat{f_{\mathcal{A}(2/3)}}$ is shown in Figure 7.

After examples 4 and 3 it is a bit surprising. It looks even more complicated than those of the function $\widehat{f_{\mathcal{A}(3/4)}}$ and $\widehat{f_{\mathcal{A}(1/4)}}$, and indeed this is the case. Namely, what we are going to show is that although, by construction $\widehat{f_{\mathcal{A}(2/3)}}$ is continuous on the whole interval [0, 1], it does not have a derivative at any point of this interval. In order to do that let us fixed a point in [0, 1],

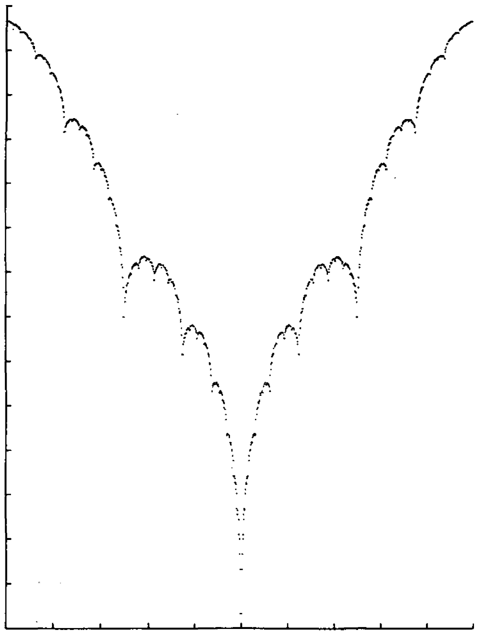

Figure 6. – $t = 1/4$; $x(t) = 2/3$.

and assume that it has a binary representation w. Denote by w_n the prefix of w of the length n. We consider the following infinite words

$$\begin{aligned} w_0(n) &= w_n\, 0^w,\\ w_1(n) &= w_n\, 10^\omega,\\ w_2(n) &= w_n\, 110^\omega,\\ w_3(n) &= w_n\, 1^\omega. \end{aligned}$$

Then, clearly,

$$(7) \qquad |\hat{w} - \widehat{w_i(n)}| \leqq \frac{1}{2^n} \qquad \text{for} \quad i = 0,\ 1,\ 2,\ 3.$$

Consider the distribution given by w_n on $\mathcal{A}(2/3)$, say

$$P_{\mathcal{A}(2/3)}(w_n) = (\alpha_n,\ \beta_n,\ \gamma_n,\ \delta_n).$$

This allows to compute the values $f_{\mathcal{A}(2/3)}(w_i(n))$. Indeed,

$$\begin{aligned} f_{\mathcal{A}(2/3)}(w_0(n)) &= \alpha_n \left[3 \cdot \left(\frac{4}{3} - \frac{8}{3}\right)\right] + \frac{4}{3}\beta_n + \delta_n\\ &= -4\,\alpha_n + \frac{4}{3}\beta_n + \delta_n, \end{aligned}$$

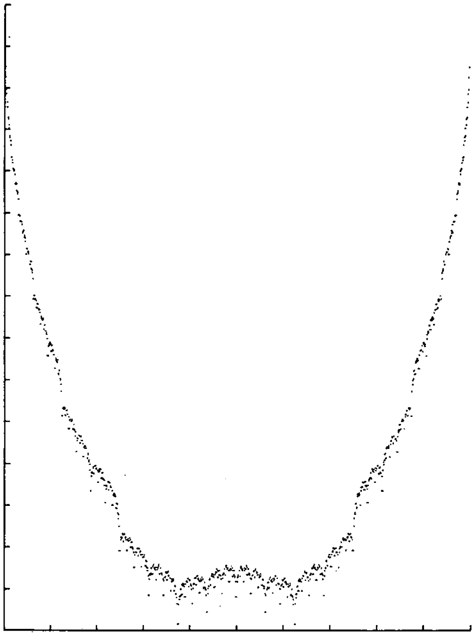

Figure 7. – $t = 2/3$; $x(t) = -8/3$.

and similarly,

$$f_{\mathcal{A}(2/3)}(w_1(n)) = -\frac{16}{3}\alpha_n + \beta_n + \gamma_n + \delta_n,$$
$$f_{\mathcal{A}(2/3)}(w_2(n)) = -\frac{47}{9}\alpha_n + \frac{3}{4}\beta_n + \frac{5}{4}\gamma_n + \delta_n,$$
$$f_{\mathcal{A}(2/3)}(w_3(n)) = -4\alpha_n + \frac{4}{3}\gamma_n + \delta_n.$$

Here the value of α_n is easy to determine:

$$\alpha_n = \left(\frac{2}{3}\right)^n,$$

independantly of w.

Now, we consider the two differences, namely

$$f_{\mathcal{A}(2/3)}(w_0(n)) - f_{\mathcal{A}(2/3)}(w_3(n)) = \frac{4}{3}(\beta_n - \gamma_n),$$

and

$$f_{\mathcal{A}(2/3)}(w_1(n)) - f_{\mathcal{A}(2/3)}(w_2(n)) = -\frac{1}{9}\alpha_n + \frac{1}{4}(\beta_n - \gamma_n).$$

It follows that for each $n \in \mathbb{N}$ the absolute value of at least one of these differences is at least $(1/18)\,\alpha_n$. Therefore for each n ther exist $i_n \in \{0, 1, 2, 3\}$ such that

$$|f_{\mathcal{A}(2/3)}(w) - f_{\mathcal{A}(2/3)}(w_{i_n}(n))| \geqq \frac{1}{36}\,\alpha_n.$$

Since i_n assumes values on a finite set, (8) is actually true for a fixed i_0 and for infinity many values of n. That is there exists an infinite subset I of $\mathbb{N}$ and a value i_0 such that

$$|f_{\mathcal{A}(2/3)}(w) - f_{\mathcal{A}(2/3)}(w_{i_0}(n))| \geqq \frac{1}{36}\left(\frac{2}{3}\right)^n \qquad \text{for} \quad n \in I.$$

Combining this with (7) we obtain

$$\frac{|f_{\mathcal{A}(2/3)}(w) - f_{\mathcal{A}(2/3)}(w_{i_0}(n))|}{|\hat{w} - \widehat{w_{i_0}(n)}|} \geqq \frac{1}{36}\left(\frac{4}{3}\right)^n \qquad \text{for} \quad n \in I.$$

Since I is infinite this proves that $\widehat{f_{\mathcal{A}(2/3)}}$ does not possess a derivative (or even a finite one-sided derivative) at the point presented by w.

We conclude this section with a few remarks. Examples 3-5 provide another evidence of the fact that a small change in the weights of an automaton changes the behaviour of the function it computes drastically. Secondly, we believe that Example 5 has interest of its own. It yields a very simple automata theoretic description to a very wildly behaving continuous function. Indeed the automaton contains only 4 states. This also implies that to compute the values (or their approximations with a given precision) is not more complicated than to compute the values of a polynomial of degree 3. Indeed, as was shown in [DKLT] equally many state automaton is required to compute the values of a cubic polynomial.

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