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*Informatique théorique et applications*, tome 24, nº 6 (1990), p. 521-529

<http://www.numdam.org/item?id=ITA_1990__24_6_521_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol 24, n° 6, 1990, p 521 à 530)

# ON DOT-DEPTH TWO (*)

by F. BLANCHET-SADRI

Communicated by A ARNOLD

Abstract – *For positive integers* $m_1, \ldots, m_k$, *congruences* $\sim_{(m_1, \ldots, m_k)}$ *related to a version of the Ehrenfeucht-Fraissé game are defined which correspond to level k of the Sraubing hierarchy of star-free languages Given any finite alphabet* $A$, *a necessary and sufficient condition is given for the monoids* $A^*/\sim_{(m_1\, m_2\, m_3)}$ *to be of dot-deàth exactly* 2

Résumé – *Étant donnés des entiers positifs* $m_1, \ldots, m_k$, *on définit des congruences* $\sim_{(m_1, \ldots, m_k)}$ *en relation avec une versuin du jeu de Ehrenfeucht-Fraissé, et qui correspondent au niveau k de la hiérarchie de concaténation de Straubing Étant donné un alphabet fini* $A$, *une condition nécessaire et suffisante est donnée pour que les monoides définis par ces congruences soient de dot-depth exactement* 2

## 1. INTRODUCTION

Let $A$ be a given finite alphabet. The regular languages over $A$ are those subsets of $A^*$, the free monoid generated by $A$, constructed from the finite languages over $A$ by the boolean operations, the concatenation product and the star. The star-free languages are those regular languages which can be obtained from the finite languages by the boolean operations and the concatenation product only. According to Schützenberger [15], $L \subseteq A^*$ is star-free if and only if its syntactic monoid $M(L)$ is finite and aperiodic. General references on the star-free languages are McNaughton and Papert [10], Eilenberg [6] or Pin [12].

Natural classifications of the star-free languages are obtained based on the alternative use of the boolean operations and the concatenation product. Let $A^+ = A^* \setminus \{1\}$, where 1 denotes the empty word. Let

(*) Received in January 1989, revised in February 1989

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$A^+ \mathscr{B}_0 = \{ L \subseteq A^+ \mid L \text{ is finite or cofinite} \}$,

$A^+ \mathscr{B}_{k+1} = \{ L \subseteq A^+ \mid L$ is a boolean combination of languages of the form $L_1 \ldots L_n (n \geqq 1)$ with $L_1, \ldots, L_n \in A^+ \mathscr{B}_k \}$.

Only nonempty words over $A$ are considered to define this hierarchy; in particular, the complement operation is applied with respect to $A^+$. The language classes $A^+ \mathscr{B}_0, A^+ \mathscr{B}_1, \ldots$ form the so-called dot-depth hierarchy introduced by Cohen and Brzozowski in [4]. The union of the classes $A^+ \mathscr{B}_0, A^+ \mathscr{B}_1, \ldots$ is the class of star-free languages.

Our attention is directed toward a closely related and more fundamental hierarchy, this one in $A^*$, introduced by Straubing in [18]. Let

$A^* \mathscr{V}_0 = \{ 0, A^* \}$,

$A^* \mathscr{V}_{k+1} = \{ L \subseteq A^* \mid L$ is a boolean combination of languages of the form $L_0 a_1 L_1 a_2 \ldots a_n L_n (n \geqq 0)$ with $L_0, \ldots, L_n \in A^* \mathscr{V}_k$ and $a_1, \ldots, a_n \in A \}$.

$L \subseteq A^*$ is star-free if and only if $L \in A^* \mathscr{V}_k$ for some $k \geqq 0$. The *dot-depth* of $L$ is the smallest such $k$.

Using Eilenberg's correspondence, we have that for each $k \geqq 0$, there is a variety $V_k$ of finite monoids such that for $L \subseteq A^*$, $L \in A^* \mathscr{V}_k$ if and only if $M(L) \in V_k$. An outstanding open problem is whether one can decide if a language has dot-depth $k$, *i. e.*, can we effectively characterize the varieties $V_k$? The variety $V_0$ consists of the trivial monoid alone, $V_1$ of all finite $\mathscr{J}$-trivial monoids [16]. Straubing [19] conjectured an effective characterization, based on the syntactic monoid of the language, for the case $k = 2$. His characterization, formulated in terms of a novel use of categories in semigroup theory recently developed by Tilson [22], is shown to be necessary in general, and sufficient for an alphabet of two elements.

In the framework of semigroup theory, Brzozowski and Knast [1] showed that the dot-depth hierarchy is infinite. Thomas [21] gave a new proof of this result, which shows also that the Straubing hierarchy is infinite, based on a logical characterization of the dot-depth hierarchy that the obtained in [20] (Perrin and Pin gave one for the Straubing hierarchy [11]) and the following version of the Ehrenfeucht-Fraissé game.

First, one regards a word $w \in A^*$ of length $|w|$ as a word model $w = \langle \{1, \ldots, |w|\}, <^w, (Q_a^w)_{a \in A} \rangle$ where the universe $\{1, \ldots, |w|\}$ represents the set of positions of letters in $w$, $<^w$ denotes the $<$-relation in $w$, and $Q_a^w$ are unary relations over $\{1, \ldots, |w|\}$ containing the positions with letter $a$, for each $a \in A$. For a sequence $\bar{m} = (m_1, \ldots, m_k)$ of positive integers, where $k \geqq 0$, the game $\mathscr{G}_{\bar{m}}(u, v)$ is played between two players I and II on the word models $u$ and $v$. A play of the game consists of $k$ moves. In the

$i$-th move, player I chooses, in $u$ or in $v$, a sequence of $m_i$ positions; then player II chooses, in the remaining word, also a sequence of $m_i$ positions. After $k$ moves, by concatenating the sequences chosen from $u$ and $v$, two sequences $p_1 \ldots p_n$ from $u$ and $q_1 \ldots q_n$ from $v$ have been formed where $n=m_1+\ldots+m_k$.

Player II has won the play if

$$p_i <^u p_j \quad \text{if and only if} \quad q_i <^v q_j, \tag{1}$$

and

$$Q^u_a p_i \quad \text{if and only if} \quad Q^v_a q_i, \qquad a \in A \quad \text{for } 1 \leqq i, j \leqq n. \tag{2}$$

If there is a winning strategy for II in the game $\mathscr{G}_{\bar{m}}(u, v)$ to win each play we write $u \sim_{\bar{m}} v$. $\sim_{\bar{m}}$ naturally defines a congruence on $A^*$ which we denote also by $\sim_{\bar{m}}$. The standard Ehrenfeucht-Fraissé game [5] is the special case $\mathscr{G}_{(1, \ldots, 1)}(u, v)$. Thomas [20], [21] and Perrin and Pin [11] imply that $L \in A^* \mathscr{V}_k$ if and only if $L$ is a $\sim_{\bar{m}}$-language for some $\bar{m}=(m_1, \ldots, m_k)$ (or $L$ is a union of classes of the congruence $\sim_{\bar{m}}$). This congruence characterization implies that the problem of deciding whether a language has dot-depth $k$ is equivalent to the problem of effectively characterizing the monoids $M=A^*/\sim$ with $\sim \supseteqq \sim_{\bar{m}}$ for some $\bar{m}=(m_1, \ldots, m_k)$, *i.e.*,

$$V_k=\{A^*/\sim \mid \sim \supseteqq \sim_{\bar{m}} \text{ for some } \bar{m}=(m_1, \ldots, m_k)\}.$$

This paper is concerned with an application of the above congruence characterization. We show that $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2=1$. The proof relies on some properties of the congruences $\sim_{\bar{m}}$ stated in the next section. [2] and [3] include other applications: among them are an answer to a conjecture of Pin [13] concerning tree hierarchies of monoids and also systems of equations satisfied in natural sublevels of level 1 of the Straubing hierarchy. The reader is referred to the books by Eilenberg [6], Lallement [9], Pin [12], Enderton [7] and Fraissé [8] for all the algebraic and logical terms not defined here.

## 2. SOME PROPERTIES OF THE CHARACTERIZING CONGRUENCES

### 2.1. An induction lemma

The following lemma is a basic result (similar to one in [14] regarding $\sim_{(1, \ldots, 1)}$) which allows to resolve games with $k+1$ moves into games with

$k$ moves and thereby allows to perform induction arguments. In what follows, $u^{<p}(u_{>p})$ denotes the subword of $u$ to the left (right) of position $p$ and $u^{<q}_{>p}$ the subword of $u$ between positions $p$ and $q$.

LEMMA 2.1.: *Let* $\bar{m}=(m_1, \ldots, m_k)$. $u \sim_{(m, m_1, \ldots, m_k)} v$ *if and only if*

(1) *for every* $p_1, \ldots, p_m \in u (p_1 \leqq \ldots \leqq p_m)$ *there are* $q_1, \ldots, q_m \in v$ $(q_1 \leqq \ldots \leqq q_m)$ *such that*

(i) $Q^u_a p_i$ *if and only if* $Q^v_a q_i$, $a \in A$ *for* $1 \leqq i \leqq m$,

(ii) $u^{<p_1} \sim_{\bar{m}} v^{<q_1}$,

(iii) $u^{<p_{i+1}}_{>p_i} \sim_{\bar{m}} v^{<q_{i+1}}_{>q_i}$ *for* $1 \leqq i \leqq m-1$,

(iv) $u_{>p_m} \sim_{\bar{m}} v_{>q_m}$ and

(2) *for every* $q_1, \ldots, q_m \in v (q_1 \leqq \ldots \leqq q_m)$ *there are* $p_1, \ldots, p_m \in u$ $(p_1 \leqq \ldots \leqq p_m)$ *such that* (i), (ii), (iii) *and* (iv) *hold.*

## 2.2. A lemma for inclusion

Define

$$\mathcal{N}_{(m_1, \ldots, m_k)} = m_1 + \ldots + m_k + \Sigma_{1 \leqq i_1 < i_2 \leqq k} m_{i_1} m_{i_2} + \ldots + \Sigma_{1 \leqq i_1 < \ldots < i_{k-1} \leqq k} m_{i_1} \ldots m_{i_{k-1}} + m_1 \ldots m_k.$$

One can show that $x^N \sim_{(m_1, \ldots, m_k)} x^{N+1}$ $(N = \mathcal{N}_{(m_1, \ldots, m_k)})$ and that $N$ is the smallest $n$ such that $x^n \sim_{(m_1, \ldots, m_k)} x^{n+1}$ (the proof is similar to the one of a property of $\sim_{(1, \ldots, 1)}$ in [21]). We see that if $u, v \in A^*$ and $u \sim_{(m_1, \ldots, m_k)} v$, then $|u|_a = |v|_a < \mathcal{N}_{(m_1, \ldots, m_k)}$ or $|u|_a, |v|_a \geqq \mathcal{N}_{(m_1, \ldots, m_k)}$ (here $|w|_a$ denotes the number of occurences of the letter a in $w$). The following lemma follows easily from Lemma 2.1 and the above remarks.

LEMME 2.2 :

$$\sim_{(m_1, \ldots, m_k)} \subseteqq \sim_{(\mathcal{N}_{(m_1, \ldots, m_k)})}, \quad \text{and} \quad \sim_{(m_1, \ldots, m_k)} \nsubseteq \sim_{(\mathcal{N}_{(m_1, \ldots, m_k)}+1)}.$$

*If* $k \leqq k'$ *and* $\exists\, 0 = j_0 < \ldots < j_{k-1} < j_k = k'$ *such that* $m_i \leqq \mathcal{N}_{(m'_{j_{i-1}+1}, \ldots, m'_{j_i})}$ *for* $1 \leqq i \leqq k$, *then* $\sim_{(m'_1, \ldots, m'_{k'})} \subseteqq \sim_{(m_1, \ldots, m_k)}$.

## 3. A SEQUENCE OF MONOIDS OF DOT-DEPTH 2

In this section, we show that for positive integers $m_1$, $m_2$ and $m_3$, $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$. The following lemma shows the necessity of the condition.

LEMMA 3.1: *Let* $m_1$ *and* $m_3$ *be positive integers. Then* $A^*/\sim_{(m_1, 2, m_3)}$ *is of dot-depth exactly* 3.

*Proof:* Let $m>0$. Consider $u_m=((xy)^m x (xy)^{2m} y (xy)^m)^m$, $v_m=((xy)^m y (xy)^{2m} x (xy)^m)^m$. A result of Straubing [17] implies that monoids in $V_2$ are 2-mutative and hence satisfy $u_m=v_m$ for all sufficiently large $m$. However, for every $N \geqq \mathcal{N}_{(1,\,2,\,1)}$, $u_N\, x_{(1,\,2,\,1)}\, v_N$. To see this, we illustrate a winning strategy for player I in the game $\mathcal{G}_{(1,\,2,\,1)}(u_N, v_N)$. $(I, i)$ denotes a position chosen by player I in the $i$-th move, $i=1, 2, 3$. Similarly, (II, $i$) denotes a position chosen by player II in the $i$-th move. Player I, in the first move, chooses the

$$u_N = \ldots (xy)^N \underset{\substack{\uparrow\\(\mathrm{II},\,1)}}{x} \overbrace{(xy)(xy)\ldots(xy)(x\underset{\uparrow}{y})}^{2N}\underset{\substack{\uparrow\\(\mathrm{I},\,2)}}{y}\overbrace{(xy)(xy)\ldots(xy)(xy)}^{N}$$

$$v_N = \ldots (xy)^N x \overbrace{(xy)(xy)\ldots(xy)(xy)}^{2N} y \overbrace{(xy)(xy)\ldots(xy)(xy)}^{N}$$

$$\overbrace{(xy)(xy)\ldots(xy)(xy)}^{N} \underset{\substack{\uparrow\\(\mathrm{I},\,1)}}{x} \overbrace{(xy)(xy)\ldots(x\underset{\uparrow}{y})(x\underset{\substack{\uparrow\\(\mathrm{II},\,2)}}{y})\ldots(xy)(xy)}^{N}$$

last $x$ followed immediately by an $x$ in $v_N$. Player II, in the first move, has to choose the last $x$ followed immediately by an $x$ in $u_N$ (if not, player I in the next two moves could win by choosing in the second move the last two consecutive $x$'s in $u_N$). Player I, in the second move, chooses the last two consecutive $y$'s in $u_N$. Player II, in the second move, cannot choose two consecutive $y$'s in $v_N$ to the right of the previously chosen position. Hence he is forced to choose two $y$'s separated by an $x$. Player I, in the third move, selects that $x$. But player II looses since he cannot choose an $x$ between the two consecutive $y$'s chosen in the preceding move by I. The result follows. [ ]

Assume $|u|_a$, $|v|_a>0$. Let $u=u_0\, a u_1 \ldots a u_{|u|_a}$, $v=v_0\, a v_1 \ldots a v_{|v|_a}$. If $Q^u_a p_i$, $Q^v_a q_j$ for $i=1, \ldots, |u|_a$, $j=1, \ldots, |v|_a$, then $u_i=u^{<p_i+1}_{>p_i}$, $i=1, \ldots, |u|_a-1$, $v_j=v^{<q_j+1}_{>q_j}$, $j=1, \ldots, |v|_a-1$. $u_0=u^{<p_1}$, $v_0=v^{<q_1}$, $u_{|u|_a}=u_{>p_{|u|_a}}$, $v_{|v|_a}=v_{>q_{|v|_a}}$.

The next two lemmas will be used in showing that for positive integers $m_1$ and $m_3$, $A^*/\sim_{(m_1,\,1,\,m_3)}$ is of dot-depth exactly 2.

LEMMA 3.2.: *Assume* $u \sim_{(m_1',\, m_2')} v$. *Then*

$$u^{<p}(s-1)\,m_2'+i \sim_{(m_1'-s,\, m_2')} v^{<q}(s-1)\,m_2'+i, \qquad (1)$$

$$u_{>p\,|u|_a+1-(s-1)\,m_2'-i} \sim_{(m_1'-s,\, m_2')} v_{>q\,|v|_a+1-(s-1)\,m_2'-i}$$

$$\textit{for} \quad i=1,\ \ldots,\ m_2' \ \textit{and}\ s=1,\ \ldots,\ m_1'-1. \qquad (2)$$

*Proof:* (1) Let $1 \leqq i \leqq m_2'$ and $1 \leqq s \leqq m_1'-1$. Let $p_1', \ldots, p_{m_1'-s}'$ $(p_1' \leqq \ldots \leqq p_{m_1'-s}')$ be positions in $u^{<p}(s-1)\,m_2'+i$. Consider the following play of the game $\mathscr{G}_{(m_1',\, m_2')}(u, v)$. Player I, in the first move, chooses $p_{m_2'}, p_{2m_2'}, \ldots, p_{(s-1)\,m_2'}, p_{(s-1)\,m_2'+i}, p'^{1}, \ldots, p_{m_1'-s}'$. Hence by the lemma of Induction 2.1, there exist positions $q_1', \ldots, q_{m_1'-s}'(q_1' \leqq \ldots \leqq q_{m_1'-s}')$ in $v^{<q}(s-1)\,m_2'+i$ such that player II, by choosing $q_{m_2'}, q_{2m_2'}, \ldots, q_{(s-1)\,m_2'}$, $q_{(s-1)\,m_2'+i}, q_1', \ldots, q_{m_1'-s}'$ for the corresponding positions, wins this play of the game. It is clear that

(i) $u^{<p_1'} \sim_{(m_2')} v^{<q_1'}$,

(ii) $u_{>p_j'}^{<p_j'+1} \sim_{(m_2')} v_{>q_j'}^{<q_j'+1}$ for $1 \leqq j \leqq m_1'-s-1$,

(iii) $u_{>p_{m_1'-s}'}^{<p_{(s-1)\,m_2'+i}} \sim_{(m_2')} v_{>q_{m_1'-s}'}^{<q_{(s-1)\,m_2'+i}}$.

Note that player II has to choose $q_{m_2'}, q_{2m_2'}, \ldots, q_{(s-1)\,m_2'}, q_{(s-1)\,m_2'+i}$ because there is a number of $a$'s $< m_2'$ between any two consecutive positions among $p_{m_2'}, p_{2m_2'}, \ldots, p_{(s-1)\,m_2'}, p_{(s-1)\,m_2'+i}$.

The proof is similar, when starting with positions in $v^{<q}(s-1)\,m_2'+i$.

For (2), we consider $p_{|u|_a+1-m_2'}, p_{|u|_a+1-2m_2'}, \ldots, p_{|u|_a+1-(s-1)\,m_2'}$, $p_{|u|_a+1-(s-1)\,m_2'-i}, p_1', \ldots, p_{m_1'-s}'$. [ ]

LEMMA 3.3: *Assume* $u \sim_{(m_1',\, m_2')} v$. *Then*

(1) $u_{>p_{(s-1)\,m_2'+i}} \sim_{(m_1'-s,\, m_2')} v_{>q_{(s-1)\,m_2'+i}}$,

(2) $u^{<p}\,|u|_a+1-(s-1)\,m_2'-i \sim_{(m'_1-s,\, m_2')} v^{<q}\,|v|_a+1-(s-1)\,m_2'-i$ *for* $i=1, \ldots, m_2'$ *and* $s=1, \ldots, m_1'-1$.

*Proof:* Similar to Lemma 3.2. [ ]

In the following theorem we talk about positions spelling the first and last occurences of every subword of length $\leqq m$ of a word $w$. We illustrate what we mean by this with the following example. Let $A=\{a, b, c\}$ and

$u = abccccaabbabbacccabababccaaaabbaa\ldots.$

↑ ↑ ↑ ↑ ↑   ↑       ↑ ↑ ↑ ↑   ↑ ↑ ↑ ↑ ↑

$p$

The six arrows on the left point to the positions which spell the first occurences of every subword of length $\leqq 2$ in $u^{<p}$ and the eight arrows on the right (before the one pointing to $p$) to the positions which spell the last occurences of every subword of length $\leqq 2$ in $u^{<p}$.

THEOREM 3.4: *Let $m_1$, $m_2$ and $m_3$ be positive integers. Then $A^*/\sim_{(m_1,\, m_2,\, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$.*

*Proof:* If $A^*/\sim_{(m_1,\, m_2,\, m_3)}$ is of dot-depth exactly 2, then $m_2 < 2$ by

LEMMA 3.1.: *Conversely, for $|A| = r > 1$, we show that for any positive integers $m'_1$, $m'_2$, $\sim_{(m'_1 + (m'_1+1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)} \subseteqq \sim_{(m'_1,\, 1,\, m'_2)}$.*
*To see this, suppose $u \sim_{(m'_1 + (m'_1+1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)} v$. Then there is a winning strategy for player II in the game $\mathscr{G}_{(m'_1 + (m'_1+1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$ to win each play. A winning strategy for player II in the game $\mathscr{G}_{(m'_1,\, 1,\, m'_2)}(u, v)$ to win each play is described as follows. Let $p'_1, \ldots, p'_{m'_1}$ $(p'_1 \leqq \ldots \leqq p'_{m'_1})$ be positions in u chosen by player I in the first move. Player II chooses positions $q'_1, \ldots, q'_{m'_1}$ $(q'_1 \leqq \ldots \leqq q'_{m'_1})$ by considering the following play of the game $\mathscr{G}_{(m'_1 + (m'_1+1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$. In the first move, player I chooses $p'_1, \ldots, p'_{m'_1}$ and the positions which spell the first and last occurences of every subword of length $\leqq m'_2$ in $u^{<p'_1}$, $u^{<p'_2}_{>p'_1}$, $\ldots$, $u^{<p'_{m'_1}}_{>p'_{m'_1-1}}$ and $u_{>p'_{m'_1}}$ for a total of no more than $m'_1 + (m'_1+1)\, 2m'_2\, (r+1)^{m'_2}$ positions (there are $r^{m'_2}$ possible words of length $m'_2$ for a total of no more than $m'_2\, (r+1)^{m'_2}$ positions to spell the first (last) occurences of every subword of length $\leqq m'_2$). More details follow for the special case $u \sim_{(1 + 4m'_2\, (r+1)^{m'_2},\, m'_2)} v$. We have a winning strategy for player II in the game $\mathscr{G}_{(1 + 4m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$ to win each play. Let us describe a winning strategy for player II in the game $\mathscr{G}_{(1,\, 1,\, m'_2)}(u, v)$ to win each play. Let p be a position in u chosen by player I in the first move. Suppose $Q^u_a p$ for some $a \in A$. If p is the* i-th *occurence of a in u $(1 \leqq i \leqq \mathscr{N}_{(1,\, m'_2)} = 2m'_2 + 1)$, then player II chooses the same occurence of a in v, say position q. The fact that $u^{<p} \sim_{(1,\, m'_2)} v^{<q}$ and $u_{>p} \sim_{(1,\, m'_2)} v_{>q}$ follows from Lemmas 3.2 and 3.3 $(\mathscr{N}_{(1,\, m'_2)} \leqq (4m'_2\, (r+1)^{m'_2})\, m'_2)$. If p is the $|u|_a + 1 - i$*-th *occurence of a in u $(1 \leqq i \leqq \mathscr{N}_{(1,\, m'_2)})$, player II chooses the $|v|_a + 1 - i$*-th *occurence of a in v. If p is among $p_{2m'_2+2}, \ldots, p_{|u|_a - 2m'_2 - 1}$, then player II chooses position q, an a, among $q_{2m'_2+2}, \ldots, q_{|v|_a - 2m'_2 - 1}$ by considering the following play of the game $\mathscr{G}_{(1 + 4m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$. In the first move, player I chooses p, the positions which spell the first and last occurences of every subword of length $\leqq m'_2$ in $u^{<p}$ and in $u_{>p}$. Hence there exists a position q in v such that player II, by choosing q, the positions which spell the first and last occurences of every subword of length $\leqq m'_2$* in *$v^{<q}$ and in $v_{>q}$, wins the play of the game. Let us show that $u^{<p} \sim_{(1,\, m'_2)} v^{<q}$ (the proof that $u_{>p} \sim_{(1,\, m'_2)} v_{>q}$ is similar). Let $p'$ be*

*a position in $u^{<p}$ (the proof is similar when starting with a position in $v^{<q}$). Assume $Q^u_{a_i} p'$.*

*Case* 1: $p'$ is among the first $m'_2$ occurences of $a_i$ in $u^{<p}$.

Let $q'$ be the same occurence among the first $m'_2$ occurences of $a_i$ in $v^{<q}$. It is clear that $u^{<p}_{>p'} \sim_{(m'_2)} v^{<q}_{>q'}$ and $u^{<p'} \sim_{(m'_2)} v^{<q'}$.

*Case* 2: $p'$ is among the last $m'_2$ occurences of $a_i$ in $u^{<p}$. Similar to case 1.

*Case* 3: $p'$ is not among the first $m'_2$ nor the last $m'_2$ occurences of $a_i$ in $u^{<p}$.

Let $p''$ and $p'''$ ($p''<p'''$) be the closest positions to $p'$ in $u^{<p'}$ and $u^{<p}_{>p'}$ respectively among the chosen positions by player I. Let $q''$ and $q'''$ ($q''<q'''$) be the corresponding positions chosen by player II.

Since $u^{<p'''}_{>p''} \sim_{(m'_2)} v^{<q'''}_{>q''}$, there is $q'$ in $v^{<q'''}_{>q''}$ such that $Q^v_{a_i} q'$.

Let us show that $u^{<p}_{>p'} \sim_{(m'_2)} v^{<q}_{>q'}$. $u^{<p'} \sim_{(m'_2)} v^{<q'}$ follows similarly.

Let $w = w_1 \ldots w_{|w|}$, $|w| \leqq m'_2$ in $v^{<q}_{>q'}$. The proof is similar when starting with $w$ in $u^{<p}_{>p'}$. If $w \in v^{<q}_{>q'''}$, it is clear that $w \in u^{<p}_{>p'''}$, hence in $u^{<p}_{>p'}$. So let us assume $w \notin v^{<q}_{>q'''}$. Let $p_{w_1}, \ldots, p_{w_{|w|}}$ in $v^{<q}_{>q'}$, at least $p_{w_1}$ being in $v^{<q'''}_{>q'}$, which spell $w_1 \ldots w_{|w|}$. $p_{w_1}, \ldots, p_{w_{|w|}}$ are hence positions which spell an occurence of a subword of length $\leqq m'_2$ in $v^{<q}$. Hence they are smaller than or equal to those positions which spell the last occurence of $w$ in $v^{<q}$ which are in $v^{<q}_{\geqq q'''}$. Hence $w \in u^{<p}_{>p'}$. [ ]

The following corollary gives another result for inclusion (one was Lemma 2.2).

COROLLARY 3.5: *Let $|A| = r$. Then*

$$\sim_{(m'_1 + (m'_1 + 1)\, 2m'_2\, (r+1)^{m'_2},\; m'_2)} \subseteqq \sim_{(m'_1,\; \mathcal{N}(1,\, m'_2))}$$

*Proof*: From Theorem 3.4 and Lemma 2.2. [ ]

## ACKNOWLEDGMENTS

This research was partially supported by F.C.A.R. (Quebec). We would like to thank the referees for their comments and suggestions.

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