Bi-infinitary codes

Informatique théorique et applications, tome 24, no 1 (1990), p. 67-87

<http://www.numdam.org/item?id=ITA_1990__24_1_67_0>
Abstract. — The notion of bi-infinite codes is introduced. For this purpose, the monoid $\mathbb{A}^\omega$ of finite, infinite and bi-infinite words over an alphabet $A$ is defined. A necessary and sufficient condition for a set of words to be a bi-infinite code is formulated. Conditions for a submonoid of $\mathbb{A}^\omega$ to have a minimal generator set are established. Using a specific kind of Thue system, the notion of bi-quasi free sub-monoids is introduced. An "algebraic" characterization of the submonoids generated by bi-infinite codes is obtained. Finally, a "combinatorial" characterization of bi-quasi free submonoids is studied.

Résumé. — On introduit la notion de code biinfini. On définit d’abord le monoïde $\mathbb{A}^\omega$ des mots finis, infinis ou bi-infinis sur un alphabet $A$. On énonce une condition nécessaire et suffisante pour qu’un ensemble de mots soit un code biinfini. On donne également des conditions pour qu’un sous-monoïde de $\mathbb{A}^\omega$ ait un ensemble minimal de générateurs. En utilisant un système de Thue spécifique, on introduit la notion de sous-monoïde bi-quasi libre. Une caractérisation « algébrique » des sous-monoides engendrés par des codes bi-infinis est alors obtenue. Finalement, on étudie une caractérisation « combinatoire » des sous-monoides bi-quasi libres.

INTRODUCTION

There has been a systematic study of codes consisting of finite words, initiated by M. P. Schützenberger [16] and developed by many others taking motivation from information theory (see [11-13]).

Recently, infinitary languages consisting of finite and infinite words have served as an adequate tool for studying behaviours of processes. This is the approach of M. Nivat and A. Arnold [14] in some problems of synchronization which stimulated the study of infinite words including bi-infinite words [15].
Motivated by the theory of codes [1] and the theory of infinitary languages, the notion of infinitary codes has been introduced and examined in [3-10].

This paper is devoted to a study of bi-infinitary codes which are a natural generalization of infinitary codes to bi-infinitary languages i.e., languages of finite, left-infinite, right-infinite and bi-infinite words.

SECTION 1
MONOID \( ^\omega A^\infty \) AND BI-INFINITARY CODES

Let \( A \) be an alphabet. We denote by \( A^* \), the free monoid generated by \( A \). Elements of \( A^* \) are called finite words. The length of a word \( x \) in \( A^* \) is denoted by \( |x| \), the empty word by \( e \) and \( A^+ = A^* - \{e\} \).

We denote by \( A^N \), the set of all right-infinite words, by \( A^-N \), the set of all left-infinite words and by \( A^Z \), the set of all bi-infinite words over \( A \). Every (bi) infinite word \( u \) has a countable length \( |u| = \omega \). For any \( X \subseteq A^* \), we denote by \( X^\omega ({}^\omega X, {}^\omega X^\omega) \), the set of all right-infinite (left-infinite, bi-infinite) words of the form \( x_1 x_2 \ldots (\ldots x_2 x_1, \ldots x_1 x_2 x_3 \ldots) \) for \( x_i \in X \). In particular, if \( x \in A^* \), then \( x^\omega = xxx \ldots, {}^\omega x = \ldots xxx \) and \( {}^\omega x^\omega = \ldots xxx \ldots \). We write \( A^\infty = A^* \cup A^N, {}^\omega A = A^* \cup A^-N \) and \( {}^\omega A^\infty = A^* \cup A^N \cup A^-N \cup A^Z \).

We define a product on elements of \( {}^\omega A^\infty \) as follows:

\[
\alpha \cdot \beta = \begin{cases} 
\alpha, & \text{if } \alpha \in A^N \cup A^Z \\
\alpha \beta, & \text{if } \alpha \in A^* \cup A^-N, \beta \in A^* \cup A^N \\
\beta, & \text{if } \alpha \in A^* \cup A^-N, \beta \in A^-N \cup A^Z.
\end{cases}
\]

It is not difficult to verify that the product is associative and therefore \( {}^\omega A^\infty \) is a monoid. This monoid has \( A^* \), \( A^\infty \) and \( {}^\omega A \) as its submonoids. For simplicity, instead of \( \alpha \cdot \beta \), we write \( \alpha \beta \). For any \( X \subseteq {}^\omega A^\infty \), we denote by \( X^* \), the submonoid of \( {}^\omega A^\infty \) generated by \( X \) and write \( X^+ = X^* - \{e\} \). If \( \alpha \) is a word, instead of \( \{ \alpha \}^* \), we write \( \alpha^* \).

For any \( X \subseteq {}^\omega A^\infty \), we write \( X_{\text{fin}} = X \cap A^* \), \( X_{\text{inf}} = X \cap A^N \), \( X_{-\text{inf}} = X \cap A^-N \), \( X_{\text{biinf}} = X \cap A^Z \), \( X^\infty = X_{\text{fin}} \cup X_{\text{inf}} \), \( {}^\omega X = X_{\text{fin}} \cup X_{-\text{inf}} \), \( \bar{X}^{(0)} = X^{(0)} = \{ e \} \), \( \bar{X}^{(1)} = X^{(1)} = X \),

\[ X^{(n)} = \{ (x_1, x_2, \ldots, x_n)/x_1, x_2, \ldots, x_{n-1} \in X_{\text{fin}}, x_n \in X^\infty \} \quad \text{for} \quad n \geq 2, \]
\( X(n) = \{(x_1, x_2, \ldots, x_n) / x_1 \in ^\omega X, x_2, x_3, \ldots, x_n \in X_{\text{fin}} \} \) for \( n \geq 2 \),

\( X(n) = \{(x_1, x_2, \ldots, x_n) / x_1 \in X_{\text{inf}}, x_2, x_3, \ldots, x_{n-1} \in X_{\text{fin}} \} \) for \( n \geq 2 \)

\( X^{(n)} = X(\pi) \cup X(\pi ') \cup X(\pi ') \) for \( n \geq 2 \),

\( X(\sigma) = \bigcup_{n \geq 0} X(n) \)

\( X(\pi) = \{(x_1, x_2, \ldots, x_n) / (x_1, x_2, \ldots, x_n) \in X(\pi) \} \) for \( n \geq 2 \)

\( X(\pi ') = \{(x_1, x_2, \ldots, x_n) / (x_1, x_2, \ldots, x_n) \in X(\pi ') \} \) for \( n \geq 2 \)

\( X(\pi ') = \{(x_1, x_2, \ldots, x_n) / (x_1, x_2, \ldots, x_n) \in X(\pi ') \} \) for \( n \geq 2 \)

and

\( X(n) = X(\pi) \cup X(\pi ') \cup X(\pi ') \) for \( n \geq 2 \).

We say that a word \( \alpha \in ^\omega A^\omega \) has a factorization on elements of \( X \) if \( \alpha = x_1 x_2 \ldots x_n \) for some \( (x_1, x_2, \ldots, x_n) \in X(n) \).

**Definition 1.1:** A subset \( X \) of \( ^\omega A^\omega \) is called a bi-infinitary code if every word \( \alpha \in ^\omega A^\omega \) has at most one factorization on elements of \( X \). More precisely, \( X \) is a bi-infinitary code if for any \( n, m \geq 1 \) and for any \( (x_1, x_2, \ldots, x_n) \in X^{(n)}, (x'_1, x'_2, \ldots, x'_m) \in X^{(m)} \), the equality \( x_1 x_2 \ldots x_n = x'_1 x'_2 \ldots x'_m \) implies \( n = m \) and \( x_i = x'_i \) \((i = 1, 2, \ldots, n)\).

Unless otherwise stated, from now on code means bi-infinitary code.

**Example 1.1:** If \( A = \{a, b\} \), the subset

\[ X = \{\omega(ab)^\omega, \omega a, b^\omega, ba\} \]

is a code whereas the subset

\[ Y = \{\omega(ab)^\omega, \omega a, b^\omega, ab\} \]

is not a code, since we have,

\[ 2 \omega a \cdot ab \cdot b^\omega = \omega a \cdot ab \cdot b^\omega \]

vol. 24, n° 1, 1990
SECTION 2
A CHARACTERIZATION OF BI-INFINITARY CODES

In this section, we establish a characterization of codes. We first introduce certain concepts and formulate a fundamental formula.

Let $X$ and $Y$ be two subsets of $\infty A^\infty$. Define the sets

$$Y^{-1}X = \{ \alpha \in \infty A^\infty \mid \exists \beta \in Y: \beta \alpha \in X, \}
$$

$$\beta \in Y_{\inf} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon, \}
$$

$$X \cup Y = \{ \alpha \in \infty A^\infty \mid \exists \beta \in Y: \alpha \beta \in X, (\alpha \in A^N \cup A^Z \Rightarrow \beta = \varepsilon), \}
$$

$$\alpha \in \infty A \text{ and } \beta \in Y_{-\inf} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon, \}
$$

We note that if $u, v \in A^{-N}$ and $u \leq v$, then $u^{-1} v$ is a subset of $A^*$. For example, if $u = a^2$ and $v = a^3$, then $u^{-1} v = a^*.$

We associate with every subset $X \subseteq \infty A^\infty$, a sequence of subsets, denoted by $U_n(X)$ or simply by $U_n$, defined recursively by

$$U_1 = X^{-1}X - \{ \varepsilon \}
$$

$$U_{n+1} = X^{-1}U_n \cup U_n^{-1}X, \quad n \geq 1.
$$

**Lemma 2.1:** For any subset $X$ of $\infty A^\infty - \{ \varepsilon \}$, (i) if $n$ is the smallest natural number such that $\varepsilon \in U_n$, then $\forall k \in \{ 1, 2, \ldots, n \}, \exists u \in U_k, \exists i, j \geq 0:

$$u \in A^N \cup A^Z \Rightarrow i = 0
$$

(ii) $\forall n \geq 1, \forall k \in \{ 1, 2, \ldots, n \}$:

$$\exists u \in U_k, \exists i, j \geq 0: u \in A^N \cup A^Z \Rightarrow i + j + k = n,
$$

$$u \in A^N \cup A^Z \Rightarrow i = 0 \Rightarrow \varepsilon \in U_n.
$$

**Proof:** We prove by recurrence on $k$.

(i) Let $n$ be the smallest natural number such that $\varepsilon \in U_n$. If $k = n$, then (2.1) holds obviously with $u = \varepsilon, i = j = 0$. Let $n > k \geq 1$ and suppose the statement is true for $n, n-1, \ldots, k+1$. We prove for $k$. Since the statement...
is true for \( k+1 \), there exist \( v \in U_{k+1} \) and integers \( i', j' \) such that

\[
v(\overline{X}^{(i')} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(j')} \neq \emptyset, \quad i'+j'+k+1 = n,
\]

\( v \in A^{N} \cup A^{Z} \Rightarrow i' = 0 \). Thus we have \( x \in \overline{X}^{(i')} - (A^{-N} \cup A^{Z}) \) and \( y \in \overline{X}^{(j')} \) such that \( vx = y \). The fact that \( v \in U_{k+1} \) gives rise to two cases.

Case (a): \( v \in X^{-1} U_k \). Then, there exists \( z \in X, u \in U_k \) such that

\[
zv = u, \quad (z \in X_{\text{inf}} \cup X_{\text{biinf}} \Rightarrow v = e)
\]

and

\[
(z \in ^{\infty}X \text{ and } v \in A^{-N} \cup A^{Z} \Rightarrow z = e).
\]

If \( v \in A^{N} \), then \( i' = 0, \ x = e, \ z \in ^{\infty}X \) and \( u = zv \). Hence \( u(\overline{X}^{(0)} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(i'+1)} \neq \emptyset \). Thus (2.1) holds with \( i = 0, j = j' + 1 \).

If \( v \in A^{-N} \cup A^{Z} \), then \( z \in ^{\infty}X \) and \( z = e \). Thus \( \varepsilon \in X \) which contradicts the hypothesis that \( X \subseteq ^{\infty}A^{\infty} - \{ e \} \).

If \( v \in A^{*} \) and \( z \in X_{\text{inf}} \cup X_{\text{biinf}} \), then \( v = e \) and \( u = z \). Hence \( u(\overline{X}^{(0)} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(1)} \neq \emptyset \) and therefore (2.1) holds with \( i = 0, j = 1 \).

If \( v \in A^{*} \) and \( z \in ^{\infty}X \), then \( ux = zv \) and so

\[
u(\overline{X}^{(i')} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(j'+1)} \neq \emptyset.
\]

Thus (2.1) holds with \( i = i', j = j' + 1 \).

Case (b): \( v \in U_{k}^{-1} X \). Then, there exist \( u \in U_k \) and \( z \in X \) such that \( uv = z \), \( (u \in A^{N} \cup A^{Z} \Rightarrow v = e) \) and \( (u \in ^{\infty}A, \ v \in A^{-N} \cup A^{Z} \Rightarrow u = e) \).

If \( v \in A^{N} \), then \( i' = 0, \ x = e, \ v = y, \ u \in ^{\infty}A \) and \( uy = z \). Hence \( u(\overline{X}^{(j')} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(1)} \neq \emptyset \). So, (2.1) holds with \( i = j', j = 1 \).

If \( v \in A^{-N} \cup A^{Z} \), then \( u \in ^{\infty}A \) and therefore \( u = e \). Thus \( \varepsilon = u \in U_k \) with \( k < n \), which is contrary to the hypothesis that \( n \) is the smallest natural number such that \( \varepsilon \in U_n \).

If \( v \in A^{*} \) and \( z \in X_{\text{inf}} \cup X_{\text{biinf}} \), then \( v = e, \ u = z \) and \( y = x \). If \( i' = j' = 0 \), then \( k + 1 = n \) and the equality \( u = z \) implies \( u(\overline{X}^{(0)} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(1)} \neq \emptyset \). That is, (2.1) holds with \( i = 0, j = 1 \). Otherwise we have \( k + 1 < n \) and \( v = e \in U_{k+1} \) which gives a contradiction.

If \( v \in A^{*} \) and \( z \in ^{\infty}X \) then \( u \in ^{\infty}A \). The equation \( uy = zv \) gives \( u(\overline{X}^{(i')} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(j'+1)} \neq \emptyset \). Thus (2.1) holds with \( i = j', j = i' + 1 \).

(ii) Suppose there exist \( u \in U_k \) and two integers \( i, j \geq 0 \) such that \( u(\overline{X}^{(i')} - (A^{-N} \cup A^{Z})) \cap \overline{X}^{(0)} \neq \emptyset, \ i + j + k = n, \ u \in A^{N} \cup A^{Z} \Rightarrow i = 0 \). We have to prove that \( \varepsilon \in U_n \). If \( k = n \), then \( i = j = 0 \) and so \( u = e \). Hence \( \varepsilon \in U_n \). Let now
n > k ≥ 1 and suppose the statement is true for n, n − 1, . . . , k + 1. We prove for k. Suppose \( x_1 x_2 \ldots x_i \in \overline{X}^{(i)} - (A^{-N} \cup A^{2}) \) and \( x'_1 x'_2 \ldots x'_j \in \overline{X}^{(j)} \) such that \( u x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j \). We discuss the following cases:

Case (a): Suppose \( u \in A^{N} \cup A^{2} \). Then \( i = 0 \), \( j + k = n \), \( j ≥ 1 \) and \( u = x'_1 x'_2 \ldots x'_j \). Let \( u' = x'_2 x'_3 \ldots x'_j \). Clearly \( u' \in U_{k+1} \) and \( u' (\overline{X}^{(i)} - (A^{-N} \cup A^{2})) \cap \overline{X}^{(j-1)} \neq \Phi \), \( 0 + j - 1 + k + 1 = n \). By recurrence hypothesis \( \varepsilon \in U_{n} \).

Case (b): Suppose \( u \in A^{*} \). If \( j = 0 \), then \( i = 0 \), \( u = \varepsilon \) and \( k = n \). Thus we have \( \varepsilon \in U_{n} \). Let \( j ≥ 1 \). If \( |u| ≥ |x'_1| \), that is, \( u = x'_1 u' \) for some \( u' \), then \( u' \in U_{k+1} \) and

\[
\begin{align*}
u' & x_1 x_2 \ldots x_i = x'_2 x'_3 \ldots x'_j. 
\end{align*}
\]

So \( u' \left( \overline{X}^{(i)} - (A^{-N} \cup A^{2}) \right) \cap \overline{X}^{(j-1)} \neq \Phi \), \( i + j - 1 + k + 1 = n \). By recurrence hypothesis, \( \varepsilon \in U_{n} \). If \( |u| < |x'_1| \), that is, \( x'_1 = uu'' \) for some \( u'' \), then \( u'' \in U_{k+1} \) and \( u'' x'_2 x'_3 \ldots x'_j = x_1 x_2 \ldots x_i \). Hence

\[
\begin{align*}
u'' & \left( \overline{X}^{(j-1)} - (A^{-N} \cup A^{2}) \right) \cap \overline{X}^{(i)} \neq \Phi, \quad j - 1 + i + k + 1 = n.
\end{align*}
\]

This implies \( \varepsilon \in U_{n} \).

Case (c): Suppose \( u \in A^{-N} \). Then \( j ≥ 1 \). If \( j = 1 \), then \( u x_1 x_2 \ldots x_i = x'_1 \) which implies \( x_1 x_2 \ldots x_i \in u^{-1} x'_1 \). Let \( u' = x_1 x_2 \ldots x_i \). We have \( u' \in U_{k+1} \) and

\[
\begin{align*}
u' & \left( \overline{X}^{(i)} - (A^{-N} \cup A^{2}) \right) \cap \overline{X}^{(j-1)} \neq \Phi, \quad 0 + i + k + 1 = n.
\end{align*}
\]

By recurrence hypothesis \( \varepsilon \in U_{n} \). If \( j > 1 \), there are two subcases.

If \( u \) is a left factor of \( x'_1 \), we have

\[
\begin{align*}x_1 x_2 \ldots x_i & = u' x'_2 x'_3 \ldots x'_j 
\end{align*}
\]

with \( u' \in u^{-1} x'_1 \). So, we have \( u' \in U_{k+1} \) and

\[
\begin{align*}
u' & \left( \overline{X}^{(j-1)} - (A^{-N} \cup A^{2}) \right) \cap \overline{X}^{(i)} \neq \Phi, \quad j - 1 + i + k + 1 = n.
\end{align*}
\]

By recurrence hypothesis \( \varepsilon \in U_{n} \).

If \( x'_1 \) is a left factor of \( u \), we have

\[
\begin{align*}x'_2 x'_3 \ldots x'_j = u'' x_1 x_2 \ldots x_i 
\end{align*}
\]

with \( u'' \in (x'_1)^{-1} u \). Then \( u'' \in U_{k+1} \) and

\[
\begin{align*}
u'' & \left( \overline{X}^{(i)} - (A^{-N} \cup A^{2}) \right) \cap \overline{X}^{(j-1)} \neq \Phi, \quad i + j - 1 + k + 1 = n.
\end{align*}
\]

By recurrence hypothesis \( \varepsilon \in U_{n} \). This proves lemma 2.1.
We are now in a position to formulate the main result of this section which is a generalization of the result proved by Do Long Van in [5, 10]. The latter is a generalization of Sardinas-Patterson theorem. This in many cases gives us a procedure to check whether or not a given set is a bi-infinitary code.

**Theorem 2.1:** A subset $X$ of $\bigcup_{n=0}^{\infty} A^n - \{\varepsilon\}$ is a code iff for all $n \geq 1$, $U_n(X)$ does not contain the empty word $\varepsilon$.

**Proof:** Suppose $\varepsilon \notin U_n(X)$, $n \geq 1$. Assume that $X$ is not a code. Then there exists a word $\alpha \in \bigcup_{n=0}^{\infty} A^n$ having two different factorizations on elements of $X$:

$$\alpha = x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$$

where $(x_1, x_2, \ldots, x_i) \in X_i$ and $(x'_1, x'_2, \ldots, x'_j) \in X'_j$.

Case (a): Suppose $\alpha \in A^* \cup A^N$. We may assume that $x_1 \neq x'_1$ and $|x_1| > |x'_1|$. Let $x_1 = x'_1 u$ for some $u \neq \varepsilon$. Clearly $u \in U_1$.

If $x_1 \in X_{\text{fin}}$, then $x'_1 \in X_{\text{fin}}$ and $u \in A^+$. So we have

$$ux_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j, \quad j \geq 2.$$  

Hence

$$u (\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$  

By lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$ which is a contradiction.

If $x_1 \in X_{\text{inf}}$, then $i = 1$, $x'_1 \in X_{\text{fin}}$ and $u \in A^N$. Therefore we have

$$u = x'_2 x'_3 \ldots x'_j, \quad j \geq 2.$$  

This implies

$$u (\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$  

Again by lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$ which is a contradiction.

Case (b): Suppose $\alpha \in A^{-N}$. Clearly $x_1, x'_1 \in X_{\text{inf}}$. Since the case $i = j = 1$ is impossible, we may assume that $i \geq 2$. There are two possibilities.

(i) If $x_1 \neq x'_1$ we can assume that $x_1 = x'_1 u$ with $u \in A^+$ such that

$$ux_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j, \quad j \geq 2.$$  

Then clearly $u \in U_1$ and

$$u (\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$  

Again by lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$. This is a contradiction.

(ii) Suppose $x_1 = x'_1$. Here, if $j = 1$, then $x_1 x_2 \ldots x_i = x'_1$ and so $x_1 = u(x_2 x_3 \ldots x_i)$. Let $x_2 x_3 \ldots x_i = u$. Clearly $u \in x_1^{-1} x'_1 \subseteq U_1$. Hence

$$u (\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i-1)} \neq \Phi.$$  

This implies $\varepsilon \in U_1$ which is a contradiction.
If \( j \geq 2 \), then \( x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j \) with

\[
x_2 x_3 \ldots x_i, x'_2 x'_3 \ldots x'_j \in A^*.
\]

If \( x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j \), we may assume that \( x_2 \neq x'_2 \), and as in case (a), get a contradiction. If \( x_2 x_3 \ldots x_i \neq x'_2 x'_3 \ldots x'_j \), then we have either 

\[
x_1 = x'_1 u \quad \text{and} \quad u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j
\]

or 

\[
x_1 = x'_1 u \quad \text{and} \quad x_2 x_3 \ldots x_i = u x_2 x_3 \ldots x'_j
\]

for some \( u \in A^+ \). By symmetry, we shall discuss one of the two possibilities.

Consider \( x_1 = x'_1 u \) and \( u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j \). Now \( x_1 = x'_1 u \) imply \( x_1 = x'_1 u \) and \( u \in (x'_1)^{-1} x_1 \subseteq U_1 \). Thus 

\[
u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j \quad \text{gives} \quad u (\overline{x}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \overline{x}^{(j-1)} \neq \emptyset \] 

and so \( \varepsilon \in U_{i+j-1} \) which is a contradiction.

Case (c): Suppose \( \varepsilon \in A^Z \). The case \( i = j = 1 \) is impossible. We assume \( j \geq 2 \).

If \( i = 1 \) then \( x_1 = x'_1 x'_2 \ldots x'_j \) and so we have \( u = (x'_1)^{-1} x_1 \in U_1 \) with 

\[
u = x'_2 x'_3 \ldots x'_j.
\]

Hence \( u (\overline{x}^{(0)} - (A^{-N} \cup A^Z)) \cap \overline{x}^{(j-1)} \neq \emptyset \) which gives a contradiction.

If \( i \geq 2 \), then \( x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j \). Now \( x_1, x'_1 \in X_{-\text{inf}} \). There are two possibilities.

(i) If \( x_1 \neq x'_1 \), as in case (a), we obtain a contradiction.

(ii) If \( x_1 = x'_1 \), then we have either

\[
x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j \quad \text{or} \quad x_2 x_3 \ldots x_i \neq x'_2 x'_3 \ldots x'_j.
\]

If \( x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j \), then we can assume \( x_2 \neq x'_2 \) and as in case (a), get a contradiction since \( x_2 x_3 \ldots x_i, x'_2 x'_3 \ldots x'_j \in A^N \). If \( x_2 x_3 \ldots x_i \neq x'_2 x'_3 \ldots x'_j \), we can obtain a contradiction as in the last part of Case b (ii). Thus \( X \) is a code.

We shall prove the converse. Suppose \( X \) is a code. Assume that there are some sets \( U_i(X) \) containing \( \varepsilon \). Let \( U_n(X) \) be one among these, with the smallest index. By lemma 2.1 (i), there exists a word \( u \in U_1 \) with two integers \( i, j \geq 0 \) such that

\[
u (\overline{x}^{(i)} - (A^{-N} \cup A^Z)) \cap \overline{x}^{(j)} \neq \emptyset, \quad i + j + 1 = n,
\]

\( u \in A^N \cup A^Z \Rightarrow i = 0 \). So, we have \( u x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j \) for some \( x_1 x_2 \ldots x_i \in \overline{x}^{(i)} - (A^{-N} \cup A^Z) \) and \( x'_1 x'_2 \ldots x'_j \in \overline{x}^{(j)} \). Since \( u \in U_1 \), there exist words \( x, x' \in X \) with either \( x \neq x' \) and \( x = x' u \) or \( x = x' u \) and \( x = x' u \).

If \( u \in A^+ \), then both \( x, x' \) are either in \( X_{\text{fin}} \) or in \( X_{-\text{inf}} \). Let \( x, x' \in X_{\text{fin}} \). Then we have \( x \neq x' \) and \( x = x' u \). So \( xx_1 x_2 \ldots x_i = x' x'_1 x'_2 \ldots x'_j \) and therefore
$X$ is not a code, a contradiction. Let $x, x' \in X_{\inf}$. If $x \neq x'$ and $x = x' \cdot u$, then as before, we get a contradiction. If $x = x'$ and $x = x' \cdot u$, then $x = x' = u$ and either

$$x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$$
or

$$x_1 x_2 \ldots x_i \neq x'_1 x'_2 \ldots x'_j.$$

If $x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$, then $i = j$ and $x_k = x'_k$ ($k = 1, 2, \ldots, i$) since $X$ is a code. Then the equation $ux_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$ implies $u = \varepsilon$, a contradiction. If $x_1 x_2 \ldots x_i \neq x'_1 x'_2 \ldots x'_j$, then the equation $x' x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$ shows that $X$ is not a code, a contradiction.

If $u \in A^N$, then $i = 0$ and either $x \in X_{\inf}$, $x' \in X_{\fin}$ or $x \in X_{\blinf}$, $x' \in X_{\inf}$. In both cases, we have $x = x' x'_1 \ldots x'_j$ which shows $X$ is not a code, a contradiction.

If $u \in A^2$, then $i = 0$, $x \in X_{\blinf}$ and $x' = \varepsilon$. Since $x' \in X \subseteq \infty A^\infty - \{\varepsilon\}$, this case is not possible.

If $u \in A^{-N}$, then $x = u$ and $x' = \varepsilon$. As before, this case is also not possible. Thus $\varepsilon \notin U_n (X)$, $\forall n \geq 1$.

**Example 2:** (i) Let $X = \{a, b\}^*$, $U_1 (X) = \{a^+\}$, $U_2 (X) = \{b\}$, $U_3 (X) = \{b^a\}$ and $U_4 (X) = \{\varepsilon\}$. So $X$ is not a code.

(ii) Let $X = \{a, b\}^\infty$, $U_1 (X) = \{a^+\}$, $U_2 (X) = \emptyset$. So, $X$ is a code.

**SECTION 3**

**MINIMAL GENERATOR SET OF A SUBMONOID OF $\infty A^\infty$.**

We recall that a generator set $X$ of a monoid $M$ is minimal if $X$ is contained in any generator set of $M$. Such a set, if it exists, is unique and called the base of $M$, denoted as $\text{BASE} (M)$. Every submonoid of $A^*$ has a minimal generator set whereas there are submonoids of $\infty A^\infty$ which have no minimal generator sets. We illustrate this in the following example.

**Example 3.1:** Let $A = \{a, b\}$ and let $M$ be the submonoid of $\infty A^\infty$ given by $M = \{\alpha \in \infty A^\infty \mid |\alpha|_a = |\alpha|_b\}$ where $|\alpha|_a$ stands for the number of occurrences of $a$ in $\alpha$. This monoid has no minimal generator set.

**Definition 3.1:** Let $M$ be a submonoid of $\infty A^\infty$ and $u$, $v$, two elements of $M_{\inf}$. We say that $u$ precedes $v$, denoted by $u < v$, if there exists $f \in M_{\fin} - \varepsilon$ such that $u = fv$. An element $u \in M_{\inf}$ is called stable if $\forall v \in M_{\inf}$: $(u < v) \Rightarrow (u = v)$. The set of all stable elements of $M_{\inf}$ is denoted by $\text{STAB} (M_{\inf})$. 

vol. 24, n° 1, 1990
Let $x, y$ be two elements of $M_{\infty}$. Here also we say that $x$ precedes $y$, denoted by $x < y$ if there exists $g \in M_{\text{fin}} - e$ such that $x = yg$. As before, $x \in M_{\infty}$ is called stable if $\forall y \in M_{\infty}: (x < y) \Rightarrow (x = y)$. The set of all stable elements of $M_{\infty}$ is denoted by $\text{STAB}(M_{\infty})$.

We say that a submonoid $M$ satisfies the stability condition if every unstable element of $M_{\infty}$ (resp. $M_{\text{inf}}$) precedes a stable element of $M_{\infty}$ (resp. $M_{\text{inf}}$). We introduce the following two sets:

$$\text{BASE}(M_{\text{fin}}) = (M_{\text{fin}} - e) - (M_{\text{fin}} - e)^2$$

$$\text{UNFAC}(M_{\text{biinf}}) = M_{\text{biinf}} - (M_{\text{inf}} M_{\text{fin}} M_{\text{inf}}).$$

**Theorem 3.1**: A submonoid $M$ of $A^\infty$ has a minimal generator set iff $M$ satisfies the stability condition and in that case, the minimal generator set of $M$ is

$$X = \text{BASE}(M)$$

$$= \text{BASE}(M_{\text{fin}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}).$$

**Proof**: Assume $X$ satisfies the stability condition. Let

$$X_{\text{fin}} = \text{BASE}(M_{\text{fin}}), \quad X_{\text{inf}} = \text{STAB}(M_{\text{inf}}), \quad X_{\text{inf}} = \text{STAB}(M_{\text{inf}}),$$

$$X_{\text{biinf}} = \text{UNFAC}(M_{\text{biinf}}) \quad \text{and} \quad X = X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{inf}} \cup X_{\text{biinf}}.$$

Since

$$X^*_{\text{fin}} = M_{\text{fin}}, \quad M_{\text{inf}} = \text{STAB}(M_{\text{inf}}) \cup (M_{\text{fin}} - e) \text{STAB}(M_{\text{inf}})$$

$$= M_{\text{fin}} \text{STAB}(M_{\text{inf}}) = X^*_{\text{fin}} X_{\text{inf}}.$$

Similarly,

$$M_{\text{inf}} = X_{\text{inf}} X^*_{\text{fin}} \quad \text{and} \quad M_{\text{biinf}} = \text{UNFAC}(M_{\text{biinf}})$$

$$\cup M_{\text{inf}} M_{\text{fin}} M_{\text{inf}} = X_{\text{biinf}} \cup X_{\text{inf}} X^*_{\text{fin}} X_{\text{inf}}.$$

Therefore,

$$M = M_{\text{fin}} \cup M_{\text{inf}} \cup M_{\text{inf}} \cup M_{\text{biinf}}$$

$$= X^*_{\text{fin}} \cup X^*_{\text{fin}} X_{\text{inf}} \cup X_{\text{inf}} X^*_{\text{fin}} \cup X_{\text{biinf}}$$

$$\cup X_{\text{inf}} X^*_{\text{fin}} X_{\text{inf}} = X^*.$$

Thus $X$ is a generator set of $M$. We shall prove that $X$ is minimal. Let $Y$ be an arbitrary generator set of $M$. We can assume that $e \notin Y$. It is enough if
we prove that

\[ X_{\text{fin}} \subseteq Y_{\text{fin}}, \quad X_{\text{inf}} \subseteq Y_{\text{inf}}, \]

\[ X_{-\text{inf}} \subseteq Y_{-\text{inf}} \quad \text{and} \quad X_{\text{biinf}} \subseteq Y_{\text{biinf}}. \]

As \( Y^*_{\text{fin}} = M_{\text{fin}} \) and \( X_{\text{fin}} \) is the minimal generator set of \( M_{\text{fin}} \), we have \( X_{\text{fin}} \subseteq Y_{\text{fin}} \). Let \( u \in X_{\text{inf}} \). Then \( u = y_1 y_2 \ldots y_n \) for some \((y_1, y_2, \ldots, y_n) \in Y^{(n)}\), \( n \geq 1 \). If \( n = 1 \), then \( u = y_n \in Y_{\text{inf}} \). If \( n > 1 \), we have \( u = y_n \) with \( f = y_1 y_2 \ldots y_{n-1} \in M_{\text{fin}} \), i.e., \( u < y_n \). Since \( u \) is stable \( u = y_n \in Y_{\text{inf}} \). Thus \( X_{\text{inf}} \subseteq Y_{\text{inf}} \). Similarly we can show that \( X_{-\text{inf}} \subseteq Y_{-\text{inf}} \). Let \( u \in X_{\text{biinf}} \). Then \( u = w_1 w_2 \ldots w_n \) for some \((w_1, w_2, \ldots, w_n) \in Y^{(n)}\), \( n \geq 1 \). If \( n = 1 \), \( u = w_1 \) where \( w_1 \in Y_{\text{biinf}} \). If \( n \geq 2 \), \( u = w_1 w_2 \ldots w_n \) is an element of \( M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}} \) since \( w_1 \in Y_{-\text{inf}} \) \( Y_{\text{fin}} = M_{\text{fin}} \), \( w_n \in Y_{\text{fin}} Y_{\text{inf}} = M_{\text{inf}} \) and \( w_2 \ldots w_{n-1} \in Y^*_{\text{fin}} \). This contradicts the choice of \( u \) since \( u \in \text{UNFAC}(M_{\text{biinf}}) \). Hence \( u \in Y_{\text{biinf}} \) and so \( X_{\text{biinf}} \subseteq Y_{\text{biinf}} \).

We prove the converse part now. Let \( Y \) be a minimal generator set of \( M \). Suppose \( M \) does not satisfy the stability condition. Then, there exists an unstable element of \( M_{\text{inf}} \) (resp. \( M_{-\text{inf}} \)), which does not precede any stable element of \( M_{\text{inf}} \) (resp. \( M_{-\text{inf}} \)). Let \( u \) be an unstable element of \( M_{\text{inf}} \) and \( v \) any element of \( M_{\text{inf}} \) such that \( v \neq u \) and \( u < v \). If \( u \in Y_{\text{inf}} \), then since \( Y^*_{\text{inf}} = M_{\text{fin}} \), the set \( Y' = (Y - \{u\}) \cup \{v\} \) is a generator set of \( M \). Since \( Y' \) does not contain \( Y \), we get a contradiction to the minimality of \( Y \). If \( u \notin Y_{\text{inf}} \), then \( u = y_1 y_2 \ldots y_n \) for some \((y_1, y_2, \ldots, y_n) \in Y^{(n)}\), \( n > 1 \). Therefore \( u < y_n \). By hypothesis, \( y_n \) is unstable. Therefore there exists \( w \in M_{\text{inf}} \) such that \( w \neq y_n \) and \( y_n < w \). Thus, the set \( Y'' = (Y - \{y_n\}) \cup \{w\} \) is a generator set of \( M \). Since \( Y'' \) does not contain \( Y \), we have a contradiction. Hence \( M \) satisfies the stability condition.

**Example 3.2:** Let \( A = \{a, b\} \). Let \( M \) be the submonoid of \( \omega A^\omega \) given by

\[ M = \{ a^m (ab)^n \} \cup A^* \cup \omega b A^* \cup A^* \omega \cup \omega b A^* \omega. \]

Every element of \( M_{\text{inf}} \) precedes the unique stable element \( a^\omega \). Every element of \( M_{-\text{inf}} \) precedes the unique stable element \( \omega b \). \( M \) satisfies the stability condition. By theorem 3.1, \( M \) has a minimal generator set which is \( A \cup \{a^\omega, \omega b, \omega a (ab)^\omega\} \).

**Definition 3.2:** Let \( M \) be a submonoid of \( \omega A^\omega \). Any increasing sequence \( u_1 < u_2 < \ldots \) of elements of \( M_{\text{inf}} \) or \( M_{-\text{inf}} \) is called a chain. An infinite chain is called stationary if there exists \( n \geq 1 \), such that \( u_m = u_n \), for all \( m \geq n \). We say that \( M \) satisfies the stationary chain condition if every infinite chain of \( M_{\text{inf}} \) as well as \( M_{-\text{inf}} \) is stationary.
We note that stationary chain condition implies the stability condition but
the converse is not true.

**Definition 3.3:** A submonoid \( M \) of \( \omega A^\omega \) is freeable if \( M^{-1} M \cap MM^{-1} \subseteq M \).

The next theorem explains the existence of the minimal generator set for a
freeable monoid \( M \).

**Theorem 3.2:** For any freeable submonoid \( M \), the following conditions are
equivalent.

(i) \( M \) has a minimal generator set.

(ii) \( M \) satisfies the stationary chain condition.

(iii) \( M \) satisfies the stability condition.

Proof is similar to that of theorem 2.4 of Chapter II in [10] and is therefore
omitted. The main difference is to consider infinite chains of elements of
\( M_{-\inf} \).

**Definition 3.4:** Let \( M \) be a submonoid of \( \omega A^\omega \). An element \( u \) of
\( M_{-\inf} \) (resp. \( M_{-\inf} \)) is maximal if there is no element \( v \) of \( M_{\inf} \) (resp. \( M_{-\inf} \))
such that \( u < v \). The set of all maximal elements of \( M_{\inf} \) (resp. \( M_{-\inf} \)) is
denoted by \( \text{MAX}(M_{\inf}) \) [resp. \( \text{MAX}(M_{-\inf}) \)]. It is evident that
\( \text{MAX}(M_{\inf}) \subseteq \text{STAB}(M_{\inf}) \) and \( \text{MAX}(M_{-\inf}) \subseteq \text{STAB}(M_{-\inf}) \). We say that
\( M \) satisfies the maximality condition if every non maximal element of
\( M_{\inf} \) (resp. \( M_{-\inf} \)) precedes a maximal element of \( M_{\inf} \) (resp. \( M_{-\inf} \)). Clearly,
maximality condition implies stability condition but not the converse.

**Definition 3.5:** Any subset \( X \) of \( \omega A^\omega \) is called distinguished if
\( X_{\inf} \cap X_{-\inf}^+ = \emptyset \), \( X_{-\inf} \cap X_{-\inf} X_{\inf}^+ = \emptyset \) and \( X_{\inf} \cap X_{-\inf} X_{\inf}^+ = \emptyset \).

The following theorem gives the connection between maximality condition
and the distinguished minimal generator set of a monoid \( M \).

**Theorem 3.3:** For any submonoid \( M \), the following conditions are equivalent.

(i) \( M \) has a distinguished minimal generator set which is
\[
\text{BASE}(M_{\inf}) \cup \text{MAX}(M_{\inf}) \cup \text{MAX}(M_{-\inf}) \cup \text{UNFAC}(M_{\inf}) \\
= (M - \varepsilon) - [(M_{\inf} - \varepsilon)^2 \cup (M_{\inf} - \varepsilon) M_{\inf} M_{-\inf} (M_{\inf} - \varepsilon)]
\]

(ii) \( M \) has a distinguished generator set

(iii) \( M \) satisfies the maximality condition
Proof: It is clear that (i) implies (ii). We show that (ii) implies (iii). Let \( Y \) be a distinguished generator set of \( M \). Since \( Y \) is a generator set, it is easy to see that every element of \( M_{\inf} - Y_{\inf} \) (resp. \( M_{-\inf} - Y_{-\inf} \)) precedes an element of \( Y_{\inf} \) (resp. \( Y_{-\inf} \)) and so it is enough to prove that

\[
Y_{\inf} \subseteq \text{MAX}(M_{\inf}) \quad \text{[resp. } Y_{-\inf} \subseteq \text{MAX}(M_{-\inf})] \text{.}
\]

We shall prove that \( Y_{\inf} \subseteq \text{MAX}(M_{\inf}) \). Suppose this is not true. Then, there exists \( y \in Y_{\inf} \) which is not maximal. So, for some \( v \in M_{\inf} \), we have \( y < v \). Let \( y = gv \) where \( g \in M_{\inf} - \varepsilon \) and \( v = y_1 y_2 \ldots y_n \) for some \( (y_1, y_2, \ldots, y_n) \in Y^{(n)}, \ n \geq 1 \). Since \( gy_1 y_2 \ldots y_{n-1} \in Y_{\inf}^* \), we have \( y \in Y_{\inf} \cap Y_{\inf}^* \cap Y_{\inf} \). This is a contradiction since \( Y \) is distinguished. Hence (ii) implies (iii).

We now prove (iii) \( \Rightarrow \) (i). Let \( M \) satisfy the maximality condition. This means \( M \) satisfies the stability condition. By theorem 3.1, \( M \) has a minimal generator set \( X \), namely,

\[
X = \text{BASE}(M_{\inf}) \cup \text{STAB}(M_{\inf}) \cup \text{STAB}(M_{-\inf}) \cup \text{UNFAC}(M_{\biinf}) .
\]

Since a non maximal stable element cannot precede a maximal element,

\[
\text{STAB}(M_{\inf}) = \text{MAX}(M_{\inf}) = M_{\inf} - (M_{\inf} - \varepsilon) M_{\inf}
\]

and

\[
\text{STAB}(M_{-\inf}) = \text{MAX}(M_{-\inf}) = M_{-\inf} - M_{-\inf} (M_{\inf} - \varepsilon).
\]

Since

\[
\text{UNFAC}(M_{\biinf}) = M_{\biinf} - (M_{\inf} M_{\inf} M_{\inf})
\]

and

\[
\text{BASE}(M_{\inf}) = (M_{\inf} - \varepsilon) - (M_{\inf} - \varepsilon)^2,
\]

we have

\[
X = (M - \varepsilon) - [(M_{\inf} - \varepsilon)^2 \cup (M_{\inf} - \varepsilon) M_{\inf} \cup M_{-\inf} (M_{\inf} - \varepsilon) \cup M_{-\inf} M_{\inf} M_{\inf}].
\]

Since \( X = X_{\inf} \cup X_{\inf} \cup X_{-\inf} \cup X_{\biinf} \), and \( X_{\inf} = \text{MAX}(M_{\inf}) \), \( X_{-\inf} = \text{MAX}(M_{-\inf}) \) and \( X_{\biinf} = \text{UNFAC}(M_{\biinf}) \). Thus \( X_{\inf} \cap X_{\inf} \cap X_{\inf} \cap X_{\inf} = \Phi \) and

\[
X_{\biinf} \cap X_{-\inf} X_{\biinf} X_{\inf} = \Phi.
\]

vol. 24, n° 1, 1990
Hence $X$ is distinguished.

SECTION 4
SUBMONOID GENERATED BY CODES AND A THUE SYSTEM

In this section we introduce a bi-quasi free monoid whose underlying set is the set of all normal forms with respect to a specific Church-Rosser Thue system. We establish a characterisation of codes in terms of morphisms of monoids. We show the relation between bi-quasi free monoids, minimal generator sets and codes.

Let $B$ be any finite alphabet. Let $R$ be a binary relation on $B^*$. Elements of $R$ are written as equations, i.e., $R = \{(u=v) \mid u, v \in B^*\}$. Let $T(B) = \langle B; R \rangle$. We call $T(B)$ as a Thue system associated with $B$. We say $(u=v)$ is in $T(B)$ iff $(u=v)$ is in $R$.

Define the relation $\equiv_{T(B)}$ on elements of $B^*$ as follows: For any $(u=v)$ in $T(B)$ and any $x, y \in B^*$, we write $xuv = xuv$. The reflexive transitive closure of the symmetric relation $\equiv_{T(B)}$ is denoted as $\equiv_{T(B)}$. Clearly $\equiv_{T(B)}$ is a congruence relation on $B^*$. If $x \equiv_{T(B)} y$, for any $x, y \in B^*$, we say that $x$ is congruent to $y$. The congruence class of $x$ is denoted by $[x]$.

If $(u=v)$ is in $T(B)$, we write $u \rightarrow_{T(B)} v$ if the length of $u$ is greater than the length of $v$. $\rightarrow_{T(B)}$ is the reflexive, transitive closure of the relation $\rightarrow_{T(B)}$.
Now, \( \equiv_T(B) \) is a congruence relation on \( B^* \). Consider the quotient monoid \( B^*/\equiv_T(B) \) and denote this by \( B[*] \). It is easy to see that \( T(B) \) is Church-Rosser. Hence every congruence class has a unique normal form. It is interesting to note that the set of all normal forms of elements of \( B^* \) is

\[
B^*_1 \cup B^*_2 B_2 \cup \bar{B}_2 B^*_1 \cup \bar{B}_2 B^*_1 B_2 \cup B_3.
\]

By a mild abuse of language, we write

\[
B[*] = B^*_1 \cup B^*_2 B_2 \cup \bar{B}_2 B^*_1 \cup \bar{B}_2 B^*_1 B_2 \cup B_3.
\]

Define a product on \( B[*] \) as follows: For \( x, y \in B[*] \),

\[
x . y = \begin{cases} 
xy & \text{if } x \in B^*_1, \ y \in B^*_2 B_2 \cup B^*_1 \\
& \text{or} \\
x & \text{if } x \in \bar{B}_1 B^*_1, \ y \in B^*_1 \\
y & \text{if } x \in B^*_1 \cup \bar{B}_2 B^*_1, \\
& y \in \bar{B}_2 B^*_1 \cup B_3 \cup \bar{B}_2 B^*_1 B_2.
\end{cases}
\]

Clearly \( B[*] \) is a monoid which we shall call as a bi-quasi free monoid generated by \( B \).

**LEMMA 4.1:** If \( \varphi : B[*] \to \infty A^{\infty} \) is an injective morphism and \( \varphi(B) = X \), then \( \varphi(B_1) = X_{\text{fin}}, \varphi(B_2) = X_{\text{inf}}, \varphi(B_2) = X_{\text{inf}} \) and \( \varphi(B_3) = X_{\text{biinf}} \).

**Proof:** We first show that \( \varphi(B_1) \subseteq X_{\text{fin}} \). Suppose it is not true. Then there exists \( b \in B_1 \) such that

\[
\varphi(b) \in X_{\text{inf}} \cup X_{\text{inf}} \cup X_{\text{biinf}}.
\]

If \( \varphi(b) \in X_{\text{inf}} \cup X_{\text{biinf}} \), for \( b' \in B \),

\[
\varphi(bb') = \varphi(b) \varphi(b') = \varphi(b).
\]

Since \( \varphi \) is injective, \( bb' = b \) which is impossible. If \( \varphi(b) \in X_{\text{inf}} \), for \( b' \in \bar{B}_2 \cup B_3, bb' = b' \). So, \( \varphi(b) \varphi(b') = \varphi(b') \) which is impossible since \( \varphi(b) \neq \varepsilon \). Hence \( \varphi(B_1) \subseteq X_{\text{fin}} \).

To prove that \( \varphi(B_2) \subseteq X_{\text{inf}} \), we suppose that it is not true. Then there exists \( b \in B_2 \) such that \( \varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{biinf}} \). For \( b' \in B \), \( bb' = b \). So, \( \varphi(b) \varphi(b') = \varphi(b) \) and this is not possible since \( \varphi(b') \) need not be \( \varepsilon \). Hence \( \varphi(B_2) \subseteq X_{\text{inf}} \).

vol. 24, n° 1, 1990
We now show that $\varphi(B_2) \subseteq X_{\text{inf}}$. If it were not so, there would exist $b \in B_2$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{biinf}}$. Now, for $b' \in B_2 \cup B_3$, $bb' = b'$ and so $\varphi(b) \varphi(b') = \varphi(b')$. This is not possible since $\varphi(b) \neq \varepsilon$.

Finally, in order to prove that $\varphi(B_3) \subseteq X_{\text{biinf}}$, assume that it is not true. Then there exists $b \in B_3$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{inf}}$. For $b' \in B$, $bb' = b$. Therefore $\varphi(b) \varphi(b') = \varphi(b)$ which is not possible since $\varphi(b') \neq \varepsilon$.

Since $\varphi(B) = X$, we have, $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$, $\varphi(B_2) = X_{\text{inf}}$ and $\varphi(B_3) = X_{\text{biinf}}$. This proves the lemma.

Given a quadruple alphabet $B = (B_1, B_2, B_2, B_3)$ we denote $B^{(1)} = B$ and

$$
B^{(n)} = \{ (b_1, b_2, \ldots, b_n) | b_1, b_2, \ldots, b_{2n} \subset B_1, b_n \in B_2 \cup B_2, \quad \text{or} \quad b_1 \in B_1 \cup B_2, \quad b_2, b_3, \ldots, b_n \in B_1 \}
$$

**Lemma 4.2:** (i) If a subset $X$ of $\mathbb{A}^\infty$ is a code, then every morphism $\varphi : B^* \rightarrow \mathbb{A}^\infty$ which induces a bijection from $B$ onto $X$ with $\varphi(B_1) \subseteq X_{\text{fin}}$, $\varphi(B_2) \subseteq X_{\text{inf}}$ and $\varphi(B_2) \subseteq X_{\text{inf}}$ is injective.

(ii) If $\varphi : B^* \rightarrow \mathbb{A}^\infty$ is an injective morphism, then $X = \varphi(B)$ is a code.

Proof is on lines close to that of lemma 1.3 of Chapter III in [10] and is therefore omitted.

We now give a necessary and sufficient condition for a subset of $\mathbb{A}^\infty$ to be a code.

**Theorem 4.1:** A subset $X$ of $\mathbb{A}^\infty$ is a code iff there exists a bi-quasi free monoid $B^*$ and an injective morphism $\varphi : B^* \rightarrow \mathbb{A}^\infty$ such that $\varphi(B) = X$.

Proof: Let $X$ be a code. Let $B = (B_1, B_2, B_3, B_3)$ be a quadruple alphabet chosen so that $B_1$, $B_2$, $B_2$ and $B_3$ are in one to one correspondence with $X_{\text{fin}}$, $X_{\text{inf}}$, $X_{\text{inf}}$ and $X_{\text{biinf}}$ respectively. This correspondence shows the existence of an isomorphism

$$
\varphi : B^* \rightarrow X^* \quad \text{with} \quad \varphi(B_1) = X_{\text{fin}}, \quad \varphi(B_2) = X_{\text{inf}}, \quad \varphi(B_3) = X_{\text{biinf}}.
$$

By lemma 4.2, the theorem holds.

**Definition 4.1:** A submonoid $M$ of $\mathbb{A}^\infty$ is said to be bi-quasi free if it is isomorphic to a bi-quasi free monoid $B^*$.
The following theorem exhibits that the class of submonoids generated by
codes coincides with the class of biquasi free submonoids.

**Theorem 4.2:** (i) Every biquasi free submonoid $M$ has a minimal generator
set $X$ which is a code.

(ii) If $X$ is a code, then $X^*$ is a biquasi free submonoid having $X$ as its
minimal generator set.

Proof: (i) Suppose $M$ is a biquasi free submonoid. Then there is an
isomorphism $\varphi : B^{[*]} \to M$ from a biquasi free monoid onto $M$. By theorem
4.1, $X = \varphi(B)$ is a code. By lemma 4.2, $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$,
$\varphi(B_0) = X_{\text{biinf}}$, and $\varphi(B_3) = X_{\text{biiinf}}$. We have

$$M = \varphi(B^{[*]}) = \varphi(\mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_2 \cup \mathbb{B}_3)$$

$$= \{\varphi(B_1)^* \cup \varphi(B_2)^* \cup \varphi(B_2)^* \cup \varphi(B_3)^* \}$$

$$\cup \varphi(\mathbb{B}_0)^* \cup \varphi(\mathbb{B}_0)^* \cup \varphi(\mathbb{B}_0)^*$$

$$= X_{\text{fin}}^* \cup X_{\text{fin}}^* \cup X_{\text{inf}}^* \cup X_{\text{inf}}^* X_{\text{inf}}^* \cup X_{\text{inf}}^* X_{\text{biiinf}} = X^*.$$ 

Hence $X$ generates $M$. To prove the minimality of $X$, let $Y$ be any generator
set of $M$ and $x \in X$. Then $x = y_1 y_2 \ldots y_n$ for some $(y_1, y_2, \ldots, y_n) \in Y^{(n)}$,
n $\geq 0$. Since $x \neq e$, $n \geq 1$. Since $X$ is a code, $n = 1$ and so $x = y_1$. Hence $X \subseteq Y$.
Thus $X$ is minimal.

(ii) Suppose $X$ is a code. By theorem 4.1, there exists a biquasi free
monoid $B^{[*]}$ and an injective morphism $\varphi : B^{[*]} \to A^\infty$ such that $\varphi(B) = X$.
Now $\varphi$ is indeed an isomorphism from $B^{[*]}$ onto $\varphi(B^{[*]}) = X^*$. Thus $X^*$ is a
biquasi free submonoid. By the similar argument as in (i), $X$ is a minimal
generator set of $X^*$.

**Section 5**

**A combinatorial characterization of bi-quasi free submonoids**

**Lemma 5.1:** Every bi-quasi free submonoid is freeable.

Proof: Let $M$ be a biquasi free submonoid with the minimal generator
set $X$. By theorem 4.2, $X$ is a code. Let $\alpha \in M^{-1} M \cap M M^{-1}$. Since
$\alpha \in M^{-1} M$, there exists $\beta \in M$ such that $\beta \alpha \in M$, ($\beta \in M_{\text{inf}} \cup M_{\text{biiinf}} \Rightarrow \alpha = e$)
and ($\beta \in A^\infty$ and $\alpha \in A^{\infty} \cup A^2 \Rightarrow \beta = e$). Since $\alpha \in MM^{-1}$, there exists $\gamma \in M$
such that $\alpha \gamma \in M$,

$$\alpha \in A^\infty \text{ and } \gamma \in M_{\text{inf}} \cup M_{\text{biiinf}} \Rightarrow \alpha = e$$
Let
\[ \beta = x_1 x_2 \ldots x_k \quad \text{with} \quad (x_1, x_2, \ldots, x_k) \in X^k, \]
\[ \alpha \psi = x_{k+1} \ldots x_n \quad \text{with} \quad (x_{k+1}, x_{k+2}, \ldots, x_n) \in X^{n-k}, \]
\[ \beta \alpha = x'_1 x'_2 \ldots x'_l \quad \text{with} \quad (x'_1, x'_2, \ldots, x'_l) \in X^l, \]
\[ \psi = x'_{l+1} x'_{l+2} \ldots x'_m \quad \text{with} \quad (x'_{l+1}, \ldots, x'_m) \in X^{m-l}. \]

If \( \beta \in M_{\inf} \cup M_{\biinf} \), then \( \alpha = \epsilon \in M \). If \( \beta \in \infty M \) and \( \alpha \in A^{-N} \cup A^Z \), then \( \beta = \epsilon \).

Therefore \( \beta \alpha \in M \) implies \( \alpha \in M \). If \( \alpha \in A^N \), then we have \( \psi = \epsilon \) and so \( \alpha \psi \in M \) implies \( \alpha \in M \). When \( \alpha \in \infty A \) and \( \psi \in M_{\inf} \cup M_{\biinf} \), then \( \alpha = \epsilon \in M \). We have to consider the only case when \( \beta \in \infty M, \alpha \in A^* \) and \( \psi \in M^\infty \). Since \( \beta (\alpha \psi) = (\beta \alpha) \psi \), we get

\[ x_1 x_2 \ldots x_k x_{k+1} \ldots x_n = x'_1 x'_2 \ldots x'_l x'_{l+1} \ldots x'_m. \]

Since \( X \) is a code, \( n = m \) and \( x_i = x'_i, i = 1, 2, \ldots, n \). Since \( \beta \) is a left factor of \( \beta \alpha \), we have \( l \geq k \). Therefore

\[ \beta \alpha = x'_1 x'_2 \ldots x'_l = x_1 x_2 \ldots x_k x_{k+1} \ldots x_l = \beta x_{k+1} \ldots x_l. \]

This implies \( \alpha = x_{k+1} \ldots x_l \in M \). Thus \( M^{-1} M \cap MM^{-1} \subseteq M \) and so \( M \) is freeable.

**Definition 5.1:** We say that a submonoid \( M \) satisfies finite chain condition if all the chains in \( M_{\inf} \) and \( M_{\biinf} \) are finite.

The finite chain condition implies the maximality condition.

**Lemma 5.2:** Every bi-quasi free submonoid satisfies the finite chain condition.

**Proof:** This is similar to that of proposition 3.3 of Chapter III in [10] and is omitted. The difference is to consider infinite chains in \( M_{\biinf} \).

**Theorem 5.1:** For any submonoid \( M \), the following conditions are equivalent.

(i) \( M \) is bi-quasi free i.e., generated by a code.

(ii) \( M \) is freeable and satisfies the finite chain condition.

(iii) \( M \) is freeable and satisfies the maximality condition

(iv) \( M \) is freeable and has a distinguished (minimal) generator set.

**Proof:** It is clear that (iii) \( \iff \) (iv) by theorem 3.3. (i) \( \Rightarrow \) (ii) is by lemmas 5.1 and 5.2. (ii) \( \Rightarrow \) (iii) is evident. We have to show that (iii) \( \Rightarrow \) (i).
Suppose $M$ is freeable and satisfies the maximality condition. By theorem 3.3, $M$ has a distinguished minimal generator set $X$ which is $\text{BASE}(M_{\text{fin}}) \cup \text{MAX}(M_{\text{inf}}) \cup \text{MAX}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}})$.

By theorem 4.2, it is enough if we prove that $X$ is a code. Suppose $X$ is not a code. Then there exists a word $\alpha$ such that it has two different factorizations on elements of $X$. I.e.,

$$\alpha = x_1 x_2 \ldots x_n = x_1' x_2' \ldots x_m'$$

where $n, m \geq 1$, $(x_1, x_2, \ldots, x_n) \in X^{(n)}$ and $(x_1', x_2', \ldots, x_m') \in X^{(m)}$. Clearly either $n$, or, $m$ should be greater than 1. Let $m > 1$.

**Case (a):** Suppose $\alpha \in A^*$. We may assume that $x_1 \neq x_1'$. Let $|x_1| > |x_1'|$. Then there exists a word $f \neq \varepsilon$ such that $x_1 = x_1' f$ and $f x_2 x_3 \ldots x_n = x_2' x_3' \ldots x_m'$. From the freeability of $M$, it follows that $f \in M_{\text{fin}} - \varepsilon$. This contradicts the hypothesis that $x_1 \in \text{BASE}(M_{\text{fin}})$.

**Case (b):** Suppose $\alpha \in A^N$. Then $x_n, x_m' \in \text{MAX}(M_{\text{inf}})$. If $n = 1$, then $x_1 = x_2 x_3 \ldots x_m$ and so $x_1 < x_m'$ which is a contradiction to the maximality of $x_n$. Suppose $x \geq 2$. If

$$|x_1 x_2 \ldots x_{n-1}| = |x_1' x_2' \ldots x_{m-1}'|,$$

then

$$x_1 x_2 \ldots x_{n-1} = x_1' x_2' \ldots x_{m-1}' \in A^*.$$  

As in case (a), we get a contradiction. If not, we assume that $|x_1 x_2 \ldots x_{n-1}| > |x_1' x_2' \ldots x_{m-1}'|$. This implies that there exists $f \neq \varepsilon$ with

$$x_1 x_2 \ldots x_{n-1} = x_1' x_2' \ldots x_{m-1}' f \quad \text{and} \quad f x_n = x_m'.$$

Again, by freeability of $M$, $f \in M_{\text{fin}} - \varepsilon$ and so we have $x_m' < x_n$ which contradicts the maximality of $x_m'$.

**Case (c):** Suppose $\alpha \in A^{-N}$. Then $x_1, x_1' \in \text{MAX}(M_{-\text{inf}})$. We can discuss as in case (b) and obtain a contradiction.

**Case (d):** Suppose $\alpha \in A^2$. If $n = 1$, then $x_1 = x_1' x_2' \ldots x_m'$ which is a contradiction since $X$ is a distinguished minimal generator set. Suppose $n \geq 2$. Then $x_1, x_1' \in \text{MAX}(M_{-\text{inf}})$. There are two possibilities.

(i) If $x_1 \neq x_1'$, we assume $x_1 = x_1' f$ and so we have

$$f x_2 x_3 \ldots x_n = x_2' x_3' \ldots x_m', \quad m \geq 2.$$
This implies $f \in M_{\text{fin}} - \varepsilon$ since $M$ is freeable. Hence $x_1 < x'_1$ which contradicts the maximality of $x_1$.

(ii) Suppose $x_1 = x'_1$. We have either

$$x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$$

or

$$x_2 x_3 \ldots x_n \neq x'_2 x'_3 \ldots x'_m.$$ 

If $x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$, then we assume $x_2 \neq x'_2$ and proceed as in case (b) and get a contradiction. If $x_2 x_3 \ldots x_n \neq x'_2 x'_3 \ldots x'_m$ since $\alpha$ has two factorizations, we have either $x_1 = x_1 f$ and

$$f x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$$

or $x'_1 = x_1 f$

and

$$x_2 x_3 \ldots x_n = f x'_2 x'_3 \ldots x'_m.$$ 

Since the two cases are similar, it is enough to consider any one of the possibilities, say $x_1 = x'_1 f$ and $f x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$. Clearly $f \in M_{\text{fin}} - \varepsilon$ as $M$ is freeable and hence $x_1 < x'_1$ which contradicts the maximality of $x_1$.

REFERENCES