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GENERALIZED CANCELLATION-AND-PERMUTATION PROPERTIES, REGULAR LANGUAGES AND SUPPORTS OF RATIONAL SERIES (*)

by Gérard JACOB [1]

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Abstract. – *In this paper, we prove two results.*

The first one is a characterization of regular languages by generalized stability property under cancellation-and-permutation of factors. This result includes as particular cases the characterization of regular languages, of Ehrenfeucht and al. *By the "block cancellation property", as well as the characterization of regular languages in the class of periodic languages by the "transposition property", or by the "σ-property" given by Restivo and Reutenauer.*

Secondly, solving an open question of Restivo and Reutenauer, we prove that supports of rational power series satisfy very strong stability property by any finite set of cancellations.

Résumé. – *Dans cet article, nous prouvons deux résultats.*

Le premier est une caractérisation des languages réguliers par des propriétés généralisées de stabilité par les effacements-avec-permutations de facteurs. Ce résultat contient comme cas particuliers la caractérisation des languages réguliers par la « block cancellation property » due à Ehrenfeucht et all., aussi bien que la caractérisation des languages réguliers dans la classe des languages périodiques par la « transposition property » ou par la « σ-property » donnée par Restivo et Reutenauer.

Le second résultat répond à une question ouverte de Restivo et Reutenauer : nous prouvons que les supports des séries formelles rationnelles satisfont une très forte propriété de stabilité pour tout ensemble fini d'effacements.

INTRODUCTION

Regular languages satisfy some very strong sstability properties with respect to cancellation, or pumping, or permutation of factors in a word.

Indeed, let $\mathscr{A}$ be a finite deterministic automaton that recognizes a regular language L. Then, for each k integer, it is clear that any "long enough"

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transition in $\mathscr{A}$ passes $(k+1)$ times through the same state. Thus this implies the "pumping lemma" for regular sets. More generally, this implies that for any word $w = x u_1 u_2 \ldots u_n y$ (where the u_i are nonempty words, and n some great enough integer), we can obtain an other factorization $w = x' v_1 v_2 \ldots v_k y'$ where each v_j is a concatenation of consecutive factors u_i, and operates a loop on the state $q_0 * x'$ obtained from initial state q_0 by the transition labelled by the word x'). Therefore we obtain that any transformation θ of w by cancellation, or pumping, or permutation of the factors v_j gives a transformed word $w \circ \theta$ that satisfies the two symetric conditions:

$$w \in L \quad \Rightarrow \quad w \circ \theta \in L$$

and

$$w \in CL \quad \Rightarrow \quad w \circ \theta \in CL.$$

Converse results attempt to determine which kind of stability by cancellation, or pumping, or permutation implies regularity. The first converse result has been proved by Ehenfeucht, Parikh and Rozenberg [2] stating that the "block cancellation property", i. e. stability by cancellation of a single block, implies regularity. The main argument of the proof is a well known theorem of Ramsey [14] (*see* also [5] theorem 1.7.1). A similar result of Restivo and Reutenauer [18] state that stability by transposition of two consecutive blocks, and more generally by a single permutation, implies also regularity, but only for the *periodic* languages, i. e. for languages having a torsion syntactic monoïd. The proof uses conjointly theorem of Ramsey, and a theorem of Shirshov [21, 11].

In this paper, we obtain a general theorem including as particular cases the previous results, in the following way. First we replace a single cancellation or permutation by a "transformation scheme", i. e. any finite set of transformations. Secondly, instead of the use of one cancellation *or* permutation, we permit any transformation by cancellation *and* permutation, in other words any non increasing transformation.

For a transformation scheme without permutation (that is strictly decreasing) we obtain a characterization of regularity that generalizes the block cancellation property [2].

For a transformation scheme eventually with permutations (but not increasing!) we obtain *for periodic languages* a characterization of regularity that generalizes the transposition property, or the σ-property [18].

Our arguments are, essentially, the same as that of the previous authors: Ramsey's theorem and Shirshov's theorem. It must be noted also that our characterizations are *not symetric*, and we require a strong property on L and a weak property on CL (or the converse). But they become symetric again in case of transformation scheme reduced to a single transformation.

Finally, we present – as an illustration – a theorem concerning supports of rational power series on a commutative field. We prove that these languages satisfy the same strong stability property by *cancellation* as regular languages. As a particular case, we solve an open question of Restivo and Reutenauer [18]. Consequently, using our main theorem, we obtain: in order to prove that some support L of rational series is a regular language, it would suffice to prove for CL a very weak stability property by cancellation.

1. FACTORISATIONS, RAMSEY, SHIRSHOV

Let A be a fixed finite alphabet. Denote by A^* the free monoïd over A. The elements of A^*, also called words over A, are the finite sequences of letters, i. e. of elements A. We note $|w|$ the lenght of a word w (i. e. of the sequence defining w). The empty word (i. e. the empty sequence) is noted ε, we note A^+ the set of all the nonempty words over A.

DEFINITION 1.1: We call *n-factorization* ($n \in \mathbb{N}$) over A any finite sequence $F=(u, f_1, f_2, \ldots, f_n, v)$ of words of A^*, such that each f_i is not empty.

Then we set

$$\operatorname{word}(F)=u f_1 f_2 \ldots f_n v.$$

Finally, we note Φ_n the set of all n-factorizations.

DEFINITION 1.2: Soit $F=(u, f_1, f_2, \ldots, f_n, v) \in \Phi_n$.

A k-factorization G is said to be *compatible with* F if and only if there is a sequence of integers

$$1 \leqq s_1 < s_2 < \ldots < s_{k+1} \leqq n+1$$

that satisfies:

$$u'=u f_1 \ldots f_{s_1-1}, \qquad v'=f_{s_{k+1}} \ldots f_{n-1} f_n v$$

and

$$g_j=f_{s_j} f_{s_j+1} \ldots f_{s_{j+1}-1} \quad \text{for any } j \in \{1, 2, \ldots, k\}.$$

Finally, we note $\Phi_k(F)$ the set of all k-factorizations compatible with F. (see *Fig.* 1).

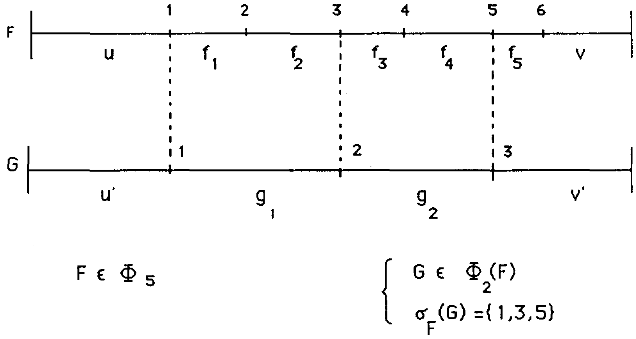


Figure 1. — A factorization *G* compatible with *F*.

(We adopt the evident generalizations in case of $1=s_1$ or $s_{k+1}=n+1$: see example below.)

Observe that $G\in\Phi_k(F)$ implies:

$$\text{word }(G)=\text{word }(F).$$

So, with the notations of the last definition, $G\in\Phi_k(F)$ if and only if:

$$\text{word }(F)=\text{word }(G);$$
u is a prefix of u';
v is a suffix of v';

for any $j\in\{1, 2, \ldots, k\}$, g_j is a concatenation of consecutive factors f_i of F. Thus, $G\in\Phi_k(F)$ if and only if $w(G)=w(F)$, and F is a refinement of G.

To any word w of length n

$$w=x_{i_1}x_{i_2}\ldots x_{i_n}$$

there is exactly one factorization $l_w\in\Phi_n$ such that

$$w=\text{word }(l_w).$$

We call it the *litteral factorization* of w:

$$l_w=(\varepsilon, x_{i_1}, x_{i_2}, \ldots, x_{i_n}, \varepsilon).$$

Now, let $F\in\Phi_n$, and $k\leq n$ a positive integer.

Let X be a finite set. We note $X[k+1]$ the set of all subsets of X whose cardinality is $k+1$.

Now, according with notations of definition 1.2, we note $\sigma_F(G)=\{s_1, s_2, \ldots, s_{k+1}\} \subset X_F$ the sequence of $k+1$ integers s_j that characterizes $G \in \Phi_n(F)$.

LEMMA 1.1: *Let $F \in \Phi_n$. The map σ_F is a bijection*

$$\sigma_F:\quad \Phi_k(F) \to X_F[n+1].$$

Furthermore (transitivity of factorizations), if H is any m-factorization compatible with F, and if $G \in \Phi_k(F)$ for some integer $k \leqq m$, we have

$$G \in \Phi_k(H) \qquad \textit{iff} \quad \sigma_F(G) \subset \sigma_F(H).$$

Consequently, σ_F induces a bijection of $\Phi_k(H)$ onto the set $\sigma_F(H)[k+1]$.

Proof: Straightforward from the definitions (see Fig. 2).

These remarks allow to translate a combinatorial theorem of Ramsey in the terminology of factorizations.

THEOREM 1 (of Ramsey [14], equivalent translation in factorization's terminology):

For any natural integers k and N, one can compute an integer Ram(k, N) *that satisfies:*

For any Ram(k, N)*-factorization F, and for any partition of $\Phi_k(F)$ in two subsets B and C, there is a factorization $G \in \Phi_N(F)$ such that*

either $\Phi_k(G) \subset B$;

or $\Phi_k(G) \subset C$.

We recall now also, concerning factorizations, a combinatorial theorem of Shirshov [21]. *See* also Lothaire [11].

DEFINITION 1.3: We call p**-power of a word** $h \in A^+$ any p-factorization of the form

$$H=(x, \underbrace{h, h, \ldots, h}_{p \text{ times}}, y).$$

Otherwise, let $<$ be a total order on A. We call **lexicographical-by-length order** the order $<$ over A^* defined as follows:

$w<w'$ if and only if

- either $|w|<|w'|$;
- or $|w|=|w'|$ and w is smaller than w' for the lexicographical order.

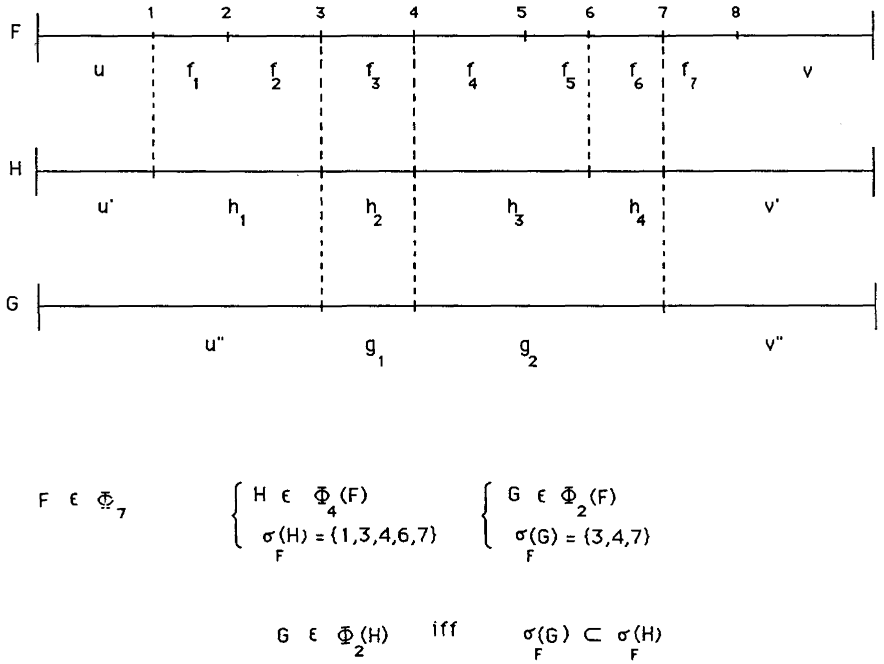


Figure 2. — "Transitivity of factorizations".

DEFINITION 1.4: We call k-**division** any factorization

$$H=(x, h_1, h_2, \ldots, h_k, y)$$

such that for any permutation σ of $\{1, 2, \ldots, k\}$, we have:

$$x\, h_{\sigma(1)}\, h_{\sigma(2)} \ldots h_{\sigma(k)}\, y < x\, h_1\, h_2 \ldots h_k\, y$$

[that will be written in the following section: word $(H \circ \sigma) <$ word (H)].

THEOREM 2 (Shirshov 1957): *Let d, p, $N \in \mathbb{N} - \{0\}$. One can compute an integer* Sh (d, p, N) *that satisfies:*

For any totally ordered alphabet A, of cardinality d, and any word $w \in A^*$ of length $|w| = \mathrm{Sh}(d, p, N)$ *we have:*

either $w =$ *word* (H) *for a p-power H of a word h, with* $1 \leq |h| < N$,

or $w =$ *word* (G) *for an N-division G.*

2. TRANSFORMATION SCHEMES

Definition and notations

We call k-**transformation** a couple (q, θ) where $q \in \mathbb{N}$ and θ is an injective application not equal to the identity map:

$$\theta: \quad \{1, 2, \ldots, q\} \rightarrow \{1, 2, \ldots, k\}.$$

We call k-**transformation-scheme** any finite set Θ of k-transformations.

If (q, θ) is a k-transformation and $G=(x, g_1, g_2, \ldots g_k, y)$ any k-factorization, we set

$$G \circ \theta=(x, g_{\theta(1)}, g_{\theta(2)}, \ldots, g_{\theta(q)}, y)$$

$$\operatorname{word}(G \circ \theta)=x\, g_{\theta(1)} g_{\theta(2)} \ldots g_{\theta(q)} y.$$

Example 1 (Cancellation of one factor): Θ is the unique map $\theta: \{\ \} \rightarrow \{1\}$

$$g=(u, g_1, v), \qquad g \circ \theta=(u, v)$$

$$\operatorname{word}(G)=u\, g_1\, v, \qquad \operatorname{word}(G \circ \theta)=u\, v.$$

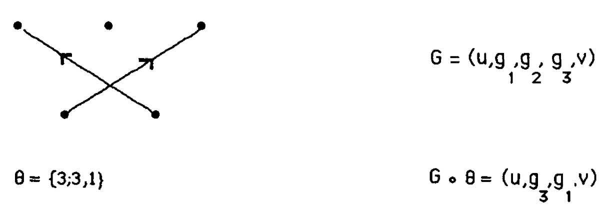
Example 2 (Cancellation of one factor and permutation of the two others):

$$\theta: \quad \{1, 2\} \rightarrow \{1, 2, 3\} \text{ is defined by } \theta(1)=3 \text{ and } \theta(2)=1$$

$$G=(u, g_1, g_2, g_3, v), \qquad G \circ \theta=(u, g_3, g_1, v)$$

$$\operatorname{word}(G)=u\, g_1 g_2 g_3\, v, \qquad \operatorname{word}(G \circ \theta)=u\, g_3\, g_1\, v.$$

(*see* Fig. 3).



where, for $\theta: \{1,2,\ldots,q\} \longrightarrow \{1,2,\ldots,k\}$, we adopt the notation:

$$\theta = \{k;\ \theta(1), \theta(2), \ldots, \theta(q)\}$$

Figure 3. — The 3-transformation θ (example 2).

Example 3 (The set of all k-cancellations): In the last part of this paper, we shall use the k-transformation scheme C_k where C_k is the set of all strictly increasing map (not surjective) into $\{1, 2, \ldots, k\}$.

Then C_k describes all effective cancellations of factors of the k-factorization. For example, if $k=2$, then $C_2=\{\theta_1, \theta_2, \theta_3\}$ such that

$$\text{if } G=(u, g_1, g_2, v)$$
$$G\circ\theta_1=(u, g_1, v)$$
$$G\circ\theta_2=(u, g_2, v)$$
$$G\circ\theta_3=(u, v).$$

Observe that $C_{k+1} \cup \{\mathrm{Id}_{k+1}\}$ is the disjoint union of the two following subsets

$$\{\theta\in C_{k+1} \cup (\mathrm{Id}_{k+1}\} \mid k+1\in \mathrm{Im}\,\theta\}$$

and

$$\{\theta\in C_{k+1} \mid k+1\notin \mathrm{Im}\,\theta\}$$

and there is a bijection of each one with $C_k \cup \{\mathrm{Id}_k\}$.

Remark: Let $G\in\Theta_k$ and let θ be some k-transformation.
(a) If θ is not surjective, then we have

$$|\mathrm{word}(G\circ\theta)| < |\mathrm{word}(G)|.$$

(b) We suppose A totally ordered, and suppose that G is compatible with a p-division D. Then for any θ eventually surjective (but not equal to the identity), we have:

$$\mathrm{word}(G\circ\theta)<\mathrm{word}(G)$$

Notations 2.2: Let Θ be some k-transformation scheme. For any $L\subset A^*$, let $E_\Theta(L)$ and $U_\Theta(L)$ be the two subsets of Φ_k defined as follows:

$$G\in E_\Theta(L) \iff \exists\,\theta\in\Theta,\quad \mathrm{word}(G\circ\theta)\in L.$$
$$G\in U_\Theta(L) \iff \forall\,\theta\in\Theta,\quad \mathrm{word}(G\circ\theta)\in L.$$

It is clear that these two sets coïncide if Θ is reduced to a single transformation.

Let L be a language

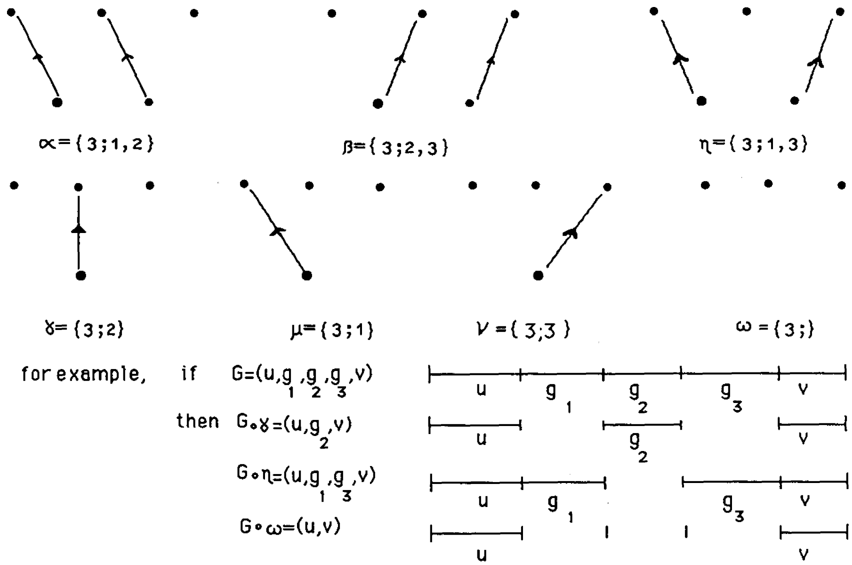


Figure 4. — The 3-transformation scheme C_3 (example 3).

Lemma 2.1: *We have two partitions of* Φ *defined by*

$$\Phi_k = E_\Theta(L) \sqcup U_\Theta(CL)$$

and

$$\Phi_k = U_\Theta(L) \sqcup E_\Theta(CL).$$

These two partitions will be use later in order to apply the combinatorial Ramsey Theorem.

Definition 2.2: Let Θ be some k-transformation scheme, and let $N \in \mathbb{N} - \{0\}$. We say that a language L is *strictly* Θ*-transformable at the order* N is any N-factorization F satisfies:

$$\Phi_k(F) \subset E_\Theta(CL) \quad \Rightarrow \quad \text{word}(F) \in CL.$$

Definition 2.3: Let Θ be some k-transformation scheme, we say that a language L is *weakly* Θ*-transformable at the order* N if any N-transformation F satisfies:

$$\Phi_k(F) \subset U_\Theta(CL) \quad \Rightarrow \quad \text{word}(F) \in CL.$$

These two definitions are given in another equivalent form by Ehrenfeucht *and al.*, and Restivo and Reutenauer, as it can be showed by the two following lemmas.

Thus, let L be a language, N a natural integer, and Θ a k-transformation scheme.

LEMMA 2.2 (strictly transformable factorizations): *For an N-factorization F, the following statements are equivalent*

(i) $\Phi_k(F) \subset E_\Theta(CL) \Rightarrow word(F) \in CL$;

(ii) $word(F) \in L \Rightarrow \exists\, G \in \Phi_k(F),\ G \in U_\Theta(L)$;

(iii) $\exists\, G \in \Phi_k(F),\ word(F) \in L \Rightarrow G \in U_\Theta(L)$.

LEMMA 2.3 (Weakly transformable factorization): *For an N-factorization F, the following statements are equivalent*

(i) $\Phi_k(F) \subset U_\Theta(CL) \Rightarrow word(F) \in CL$;

(ii) $word\ (F) \in L \Rightarrow \exists\, G \in \Phi_k(F),\ G \in E_\Theta(L)$;

(iii) $\exists\, G \in \Phi_k(F),\ word\ (F) \in L \Rightarrow G \in E_\Theta(L)$.

Proof: By the lemma 2.1, these two lemmas are only logical equivalences.

THEOREM 3 (The main result): *Let Θ be a k-transformation scheme, and $L \subset A^*$. We suppose satisfied one of the two hypothesis:*

– **either** Θ *does not contain any surjection;*

– **or** L *is a periodic language* (see *below*).

Then the following two statements are equivalent:

(i) *L is a regular language.*

(ii) *There exists an integer $N \in \mathbb{N} - \{0\}$ such that L is strictly k-transformable at the order N, and such that CL is weakly k-transformable at the order N.*

Remark: We can permute L and CL in the theorem, because a language L is regular (resp. periodic) if and only if CL is regular (resp. periodic).

Particular cases of theorem 3: (1) If Θ is the cancellation of one factor (example 1), we obtain the characterization of regular languages by the "block cancellation property" of Ehrenfeucht *and al.*, [2].

(2) If $\Theta = \{\sigma\}$ where σ is a permutation of $\{1, 2, \ldots, k\}$, we obtain the characterization of the regularity of periodic languages by the "σ-property" of Restivo and Reutenauer [18].

Supports of rational series will give examples where Θ is the set C_k of all the k-cancellations (*Fig.* 4) (*see* example 3). Of course, for $k \geqq 2$, C_k is not reduced to a single transformation. We shall prove that for any $k \in \mathbb{N}$, the supports of rational series are strictly C_k-transformable.

3. PROOF OF THE MAIN THEOREM

(A) *Proof of the direct part* ((*i*) $\Rightarrow$ (*ii*))

Let k, $N \in \mathbb{N} - \{0\}$, and Θ a k-transformation scheme, and let L be a regular language, recognized by a finite deterministic automaton, with d states.

Let us say that an N-factorization $F = (x, f_1, f_2, \ldots, f_N, y)$ crosses k times the same state if there is a state q_1, and k integers j_i, with $1 \leqq j_1 \leqq j_2 \leqq \ldots \leqq j_k \leqq N$ such that for any $i \in \{1, 2, \ldots, \mathrm{k}\}$ we have:

$$q_0 * f_1 f_2 \ldots f_{j_i} = q_1$$

(where q_0 is the initial state, and $*$ denote the right action of the words on the states). Clearly, we have:

LEMMA 3.1: *If L is recognized by some deterministic automaton with d states, then any k.d-factorization crosses k + 1 times the same state.*

Consequently, if F is a $k.d$-factorization, the lemma gives some k-factorization $G \in \Phi_k(F)$, each factor of which being the transition word of a loop on the same state. Hence, for any map

$$\theta : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, k\}$$

(not necessarily injective), one has:

$$\mathrm{word}(F) \in L \quad \Leftrightarrow \quad \mathrm{word}(G \circ \theta) \in L$$

and that achieves the proof of the direct part.

(B) *Proof of the converse* ((ii) $\Rightarrow$ (i))

First case (Θ does not contain any surjection).

Let Θ be a k-transformation scheme.

Denote by $L_N(\Theta)$ the set of languages $L \subset A^*$ that are strictly Θ-transformable at the order N, and whose the complementary is weakly Θ-transformable at the order N.

Recall that the residual of $L \subset A^*$ by a word $u \in A^*$ is the language

$$L \circ u = \{ w \in A^* \mid uw \in L \}$$

clearly, we have

$$C(L \circ u) = (CL) \circ u.$$

Recall also the "Nerode's criterium" [13]: a language is regular if and only if it has only a finite set of distinct residuals.

Thus the theorem will be proved, in this case, by the following two lemmas.

LEMMA 3.2: *$L_N(\Theta)$ is closed by residuals.*

Proof: For any m-factorization $F=(x, f_1, f_2, \ldots, f_m, y)$, we set

$$uF=(ux, f_1, f_2, \ldots, f_m, y).$$

We have clearly:

$$\text{word}(F)\in L\circ u \quad\Leftrightarrow\quad u\,\text{word}(F)\in L \quad\Leftrightarrow\quad \text{word}(uF)\in L.$$

Furthermore, a k-factorization G is in $\Phi_k(F)$ if and only if uG is in $\Phi_k(uF)$.

Now, if we suppose for any $F\in\Phi_N$

$$\Phi_k(F)\subset E_\Theta(CL) \quad\Rightarrow\quad \text{word}(F)\in CL$$

we deduce in particular, for any $F\in\Phi_N$, and $u\in A^*$

$$\Phi_k(uF)\subset E_\Theta(CL) \quad\Rightarrow\quad \text{word}(uF)\in CL$$

and finally:

$$\Phi_k(F)\subset E_\Theta(C(L\circ u)) \quad\Rightarrow\quad \text{word}(F)\in C(L\circ u).$$

Hence, L strictly Θ-transformable at the order N implies $L\circ u$ strictly Θ-transformable at the order N.

The same argument agrees in the weakly Θ-transformable case, and that proves the lemma.

LEMMA 3.3: *Let $N\in\mathbb{N}-\{0\}$, and Θ a k-transformation scheme. If Θ does not contain any surjection, then $L_N(\Theta)$ is finite.*

Proof: Let $L\in L_N(\Theta)$.

Let $\rho=\text{Ram}(k, N)$, and let $w_1\in A^*$ such that $|w_1|=m\geqq\rho$.

Let $A^{<m}=\{w\in A^* \mid |w|<m\}$, and let

$$E=(\varepsilon, l_1, l_2, \ldots, l_m, \varepsilon)$$

the litteral factorization of w_1.

(*a*) If $G\in\Phi_k(E)$, recall that we have

$$G\in U_\Theta(L) \qquad \text{iff} \quad \forall\,\theta\in\Theta, \quad \text{word}(G\circ\Theta)\in L.$$

Now, because Θ does not certain any surjection we have:

$$\forall \theta \in \Theta, \quad |\text{word}(G \circ \theta)| < |\text{word}(G)| = m.$$

Hence the set $L \cap A^{<m}$ suffices to determine if G belongs to $U_\Theta(L)$, or not.

Thus the set $L \cap A^{<m}$ completely determines the partition of $\Phi_k(E)$ induced by:

$$\Phi_k = U_\Theta(L) \sqcup E_\Theta(CL).$$

(*b*) Otherwise, since $m \geqq \rho = \text{Ram}(k, N)$, and by the Ramsey's theorem, there is an N-factorization F compatible with E, such that

$$\text{either } \Phi_k(F) \subset U_\Theta(L)$$
$$\text{or } \Phi_k(F) \subset E_\Theta(CL).$$

In the second case, since L is strictly Θ-transformable at the order N, we have:

$$w_1 = \text{word}(F) \in CL.$$

In the first case, since CL is weakly Θ-transformable at the order N, we obtain

$$w_1 = \text{word}(F) \in L.$$

Thus, we see that the set $L \cap A^{<m}$ completely determines whether or not $w_1 \in A^m$ belongs to L.

(*c*) Finally, the set $L \cap A^{<m}$ completely determines $L \cap A^m$, if $m > \rho$. By induction, we obtain that $L \cap A^{<\rho}$ completely determines any language $L \subset L_N(\Theta)$. Thus we have, if $d = \text{Card } A$:

$$\text{Card}(L_N(\Theta)) \leqq \text{Card}(\mathcal{P}(A^{<\rho})) = 2^s,$$

with

$$s = (d^{\rho-1})/(d-1).$$

That achieves the proof of the theorem, in the case "without surjection".

(*c*) *Second case* (*Θ contains at least one surjection, but L is a periodic language*)

Recall first that a language is called *periodic* if for any $v \in A^*$, the image of v in the syntactic monoid of L is a periodic element. In other words, L is periodic if and only if

$\forall\, v \in A^*, \exists\, p_v, q_v \in \mathbb{N};$

$\forall\, x, y \in A^*;$

$x v^{p_v + q_v} y \in L \Leftrightarrow x v^{q_v} y \in L.$

We would like to replace the families of integers p_v and q_v by a single (p, q) independent of v. That is generally not possible, however:

DEFINITION: Let $R, p, q \in \mathbb{N} - \{0\}$. A language L will be called *R-periodic of type* (p, q) if and only if

$\forall\, v \in A^{<R};$

$\forall\, x, y \in A^*;$

$x v^{p+q} y \in L \Leftrightarrow x v^q y \in L$

and we note $\mathrm{Per}_R(p, q)$ the set of languages $L \subset A^*$ that are R-periodic of type (p, q).

LEMMA 3.4 (Restivo and Reutenauer): *For any* $R \in \mathbb{N} - \{0\}$ *and any periodic language* P, *one can compute two integers* $p = p(R, P)$ *and* $q = q(R, P)$ *such that*

$$P \in \mathrm{Per}_R(p, q).$$

Proof: Let p be a common multiple of all the p_v such that $|v| < R$, and let q any integer greater than all the q_v such that $|v| < R$. Clearly, p and q satisfy the lemma.

Now, as in the first case, the theorem will be prove if we establish the two following lemmas:

LEMMA 3.5: *For any* $R, p, q \in \mathbb{N}$, *the set* $\mathrm{Per}_R(p, q)$ *is closed by residuals, and by complementation.*

Proof: The stability by complementation is a direct consequence of the definition of $\mathrm{Per}_R(p.q)$.

Furthermore, suppose $L \in \mathrm{Per}_R(p, q)$. Thus for any words $u, x, y \in A^*$ and $v \in A^{<R}$, we have:

$$u x v^{p+q} y \in L \quad \Leftrightarrow \quad u x v^q y \in L$$

and equivalently:

$$x v^{p+q} y \in L \circ u \quad \Leftrightarrow \quad x v^q y \in L \circ u$$

and consequently

$$L \circ u \in \mathrm{Per}_R(p, q).$$

LEMMA 3.6: *Let $N \in \mathbb{N} - \{0\}$ and let Θ be a k-transformation scheme. Then for any R, p, $q \in \mathbb{N} - \{0\}$, the set $L_N(\Theta) \cap \mathrm{Per}_R(p, q)$ is finite (restricted to a fixed alphabet A).*

Proof: For any word $w_1 \in A^*$, we set:

$A^{<w_1} = \{ w \in A^* \mid w < w_1 \}$.

Let $d = \mathrm{Card}$ (A). We set:

$$\rho = \mathrm{Ram}(k, N), \qquad \text{and} \qquad \sigma = \mathrm{Sh}(d, p+q, \rho).$$

Let $L \subset A^*$ be any language belonging to

$$L_N(\Theta) \cap \mathrm{Per}_\rho(p, q).$$

We shall prove that for any word $w_1 \in A^*$ such that $|w_1| \geqq \sigma$, the subset

$$L \cap A^{<w_1}$$

completely determines if w_1 belongs to L or not.

Indeed:

By the Shirshov's theorem, and since $|w_1| > \sigma = \mathrm{Sh}(d, p+q, \rho)$, we have:

either $w_1 = \mathrm{word}(H)$ for some $(p+q)-$power H of a word h such that $1 \leqq |h| < \rho$,

or $w_1 = \mathrm{word}$ (E) for some ρ-division E.

In the *first case*, since $|h| < \rho$ and $L \in \mathrm{Per}_\rho(p, q)$, we have:

$$w_1 = xh^{p+q} y \in L \quad \Leftrightarrow \quad xh^q y \in L.$$

Now, $|xh^q y| < |w_1|$, and consequently $xh^q y < w_1$. That implies, of course, that $L \cap A^{<W_1}$ completely determines whether w_1 belongs to L or not.

In the *last case*, we proceed as in the proof of lemma 3.3:

(*a*) Recall that for any $G \in \Phi_k(E)$

$$G \in U_\Theta(L) \qquad \text{iff} \quad \forall \theta \in \Theta, \quad \mathrm{word}(G \circ \theta) \in L.$$

Now, for any $G \in \Phi_k(E)$, because E is a ρ-division, we have:

$$\forall \theta \in \Theta, \quad \mathrm{word}(G \circ \theta) < \mathrm{word}(G) = w_1$$

(recall that the identity map does not belong to Θ). Hence the subset $L \cap A^{<w_1}$ completely determines whether G belongs to $U_\Theta(L)$, or not.

Thus the set $L \cap A^{<w_1}$ completely determines the partition of $\Phi_k(E)$ induced by:

$$\Phi_k = U_\Theta(L) \sqcup E_\Theta(CL).$$

(*b*) Otherwise, by the Ramsey's theorem, and because $E \in \Phi_\rho$ and $\rho = \mathrm{Ram}(k, N)$, there exists an N-factorization $F \in \Phi_N(E)$ such that:

either $\Phi_k(F) \subset U_\Theta(L)$;

or $\Phi_k(F) \subset E_\Theta(CL)$.

Because L is strictly Θ-transformable, and CL is weakly Θ-transformable, at the order N, we deduce in the first case $w_1 \in L$, and in the second case $w_1 \in CL$.

Thus in any case, $L \cap A^{<w_1}$ completely determines whether w_1 belongs to L, or not.

(*c*) Let u_0 be the smallest word of length σ for the lexicographical-by-length ordering. By induction on that order, we deduce that $L \cap A^{<u_0} = L \cap A^{<\sigma}$ completely determines any language L on A that belongs to $L_N(\Theta) \cap \mathrm{Per}_\rho(p, q)$. An finally:

$$\begin{aligned} &\mathrm{Card}(\mathscr{P}(A^*) \cap L_N(\Theta) \cap \mathrm{Per}_\rho(p, q)) \\ &\leqq \mathrm{Card}(\mathscr{P}(A^{<\sigma})) = 2^t \\ &\text{with } t = (d^\sigma - 1)/(d - 1). \end{aligned}$$

That achieves the proof of the theorem in the case of periodic languages.

4. SUPPORTS AND CANCELLATIONS: THE MAIN THEOREM

(A) *Rational power series*

For definitions and general developments on rational power series, *see* for example [1].

DEFINITION 4.1: We call *formal power series* (in noncommuting variables) on A with coefficients in a field K any map $S : A^* \to K$, and we denote it by a formal sum:

$$S = \sum_{w \in A^*} \langle S \mid w \rangle w$$

We call *support of S* the language

$$\mathrm{supp}(S)=\{w\in A^*\,|<S\,|\,w>=0\}.$$

In all the sequel, we identify any vector $V\in K^N$ with the linear application:

$$V:\quad K\to K^N$$
$$1\mapsto V$$

In particular, any $k\in K$ will be identified with the linear endomorphism of K, "product by k".

DEFINITION 4.2: A power series S will be called *recognizable*, or *rational*, if there are:

some finite dimensional vector space $E=K^d$ (d interger) and

– a linear map $\gamma: K\to E$ (i. e. an element of E);

– a monoid homomorphism $\mu: A^*\to \mathrm{End}(E)$;

– a linear form $\lambda: A\to K$,

such that for any $w\in A^*$, *we have:*

$$\langle S\,|\,w\rangle=\lambda\circ\mu w\circ\gamma.$$

The quadruplet $(E, \lambda, \mu, \gamma)$ will be called a *linear representation* of the series S.

Remark: The sign $\circ$ can be interpreted as a matricial product, λ (resp. γ) as an arrow matrix (resp. column matrix) and each $\mu\, w$ as a square matrix.)

DEFINITION 4.3: Following Restivo and Reutenauer [17], we shall call *support* any language that is support of some rational power series.

It is well known that *any regular language is a support.* Indeed, since any rational language is non-ambiguously rational, its characteristic power series $\chi L=\sum_{w\in L} 1.\, w$ is a rational series.

But *the converse is false: for* example, it is well known that

$$D_1^*=\{w\in\{a, b\}^*\,|\,\mathrm{length}_a(w)=\mathrm{length}_b(w)\}$$

is not a regular language nor its complement:

$$T=\{w\in\{a, b\}^*\,|\,\mathrm{length}_a(w)\neq\mathrm{length}_b(w)\}.$$

In spite of that, T is the support of the following rational power series

$$S = \sum_{w \in \{a, b\}^*} [\text{length}_a(w) - \text{length}_b(w)]\, w.$$

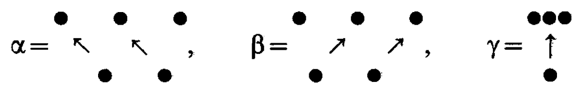
Indeed, S can be defined by the linear representation of the series S:

$$\lambda = (1 \quad 0)$$

$$\mu a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mu b = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Restivo and Reutenauer have proved that each support has "the weak cancellation property". In other words, each support is $\{\theta\}$-transformable, where θ is the unique application of the empty set into $\{1\}$ (*see* example 1), and they have deduced that if L and CL are supports, then L is a regular language (particular case of our theorem 3).

They have proved also (in the their lemma 4.2) that any support is strictly Γ-transformable, where Γ is the following 3-transformation scheme:



that is, in case $G = (f, g_1, g_2, g_3, h)$:

word $[g \circ \alpha] = f g_1 g_2 h$;

word $[g \circ \beta] = f g_2 g_3 h$;

word $[g \circ \gamma] = f g_2 h$.

And they ask whether there is a more general result (p. 258, remark).

Now we state our main theorem on supports.

THEOREM 4: *For any $k \in \mathbb{N} - \{0\}$, and any support L, one can compute an integer m such that L is strictly k-cancellable at the order m.*

COROLLARY: *Let L be a support. In order L to be regular, it suffices that there are two integers k and m such that CL is weakly k-cancellable at the order m.*

Proof: This corollary is a direct consequence of our Theorem 3.

Thus, in a certain sense, supports are very close to regular languages, because a very weak condition on their complementary implies regularity.

The proof of theorem 4 requires a combinatorial result on sequences of vectors in a finite dimensional vector space, that we present in the following section.

(*B*) *sequences of vectors*

If $\mathcal{V}=(V_1, V_2, \ldots, V_m)$ is a sequence of vectors, we denote by span ($\mathcal{V}$) the vector space spanned by the vectors $V_j (j=1, 2, \ldots, m)$.

DEFINITION 4.4: A sequence $\mathcal{V}=(V_1, V_2, \ldots, V_m)$ of nonnul vectors of K^d (d integer) will be called a *festoon of size q* if it is the concatenation of q subsequences $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_q$ that satisfy:

$$\text{span}(\mathcal{V}_1)=\text{span}(\mathcal{V}_2)=\ldots=\text{span}(\mathcal{V}_q)=\text{span}(\mathcal{V}).$$

In other words, for $i=1, 2, \ldots, q$ there is a sequence

$$\mathcal{V}_i=(V_{j_i+1}, V_{j_i+2}, \ldots, V_{j_{i+1}})$$

with

$$0 \leqq j_1 < j_2 < \ldots < j_q < j_{q+1}=m$$

such that for any $i \in \{1, 2, \ldots, q\}$

$$\text{span}(\mathcal{V}_i)=\text{span}(\mathcal{V}).$$

The integer dim(span($\mathcal{V}$)) is called *dimension of the festoon.* Each subsequence ($\mathcal{V}_i$) is called *a mesh of the festoon.*

THEOREM 5: *For any* $q, d \in \mathbb{N}-\{0\}$ *and any sequence* $\mathcal{W}$ *of* q^d *nonnul vectors of* K^d, *there exists a subsequence of consecutive vectors of* $\mathcal{W}$ *which is a festoon of size* q.

Proof: We could obtain this theorem as an application of Ramsey's theorem, or of the "théorème du rang constant" [7]. We give here a direct and economical proof.

The theorem is trivially true for $d=1$.

Suppose now the theorem already proved for $d \leqq d_0$, and let $\mathcal{W}$ be a sequence of q^{d_0+1} vectors of K^{d_0+1}. We distinguish two cases.

(*a*) Suppose that there is a subsequence $\mathcal{V}$ of q^{d_0} consecutive vectors of $\mathcal{W}$ such that dim (span ($\mathcal{V}$)) $=d_0$. Then, by induction hypothesis, we can find in $\mathcal{V}$ the required feston.

(*b*) In other case, any sequence of q^{d_0} consecutive vectors of $\mathcal{W}$ generates span($\mathcal{W}$), because it is the only (d_0+1)-dimensional subspace of span($\mathcal{W}$).

Hence, if $s=q^{d_0}$, the q following sequences:

$\mathcal{W}_1=(V_1, V_2, \ldots, V_s)$;

$\mathcal{W}_2=(V_{s+1}, V_{s+2}, \ldots, V_{2s})$;

$\mathscr{W}_q = (V_{(q-1)s+1}, V_{(q-1)s+2}, \ldots, V_{qs})$.
satisfy:

Span $(\mathscr{W}_1) =$ span $(\mathscr{W}_2) = \ldots =$ Span $(\mathscr{W}_s) =$ span $(\mathscr{W})$, and consequently, the $\mathscr{W}_i$ are the meshes of a festoon of dimension $d_0 + 1$ and of size q.

Finally, the theorem 5 is proved by induction on d_0.

5. SUPPORTS ARE STRICTLY k-CANCELLABLE: THE PROOF

(*a*) Let f, g, f', g' be four linear applications such that $f \circ g$ and $f' \circ g'$ are defined.

Then $(f \circ g) \otimes (f' \circ g') = (f \otimes f') \circ (g \otimes g')$.

Recall also that the tensor product of two elements of the field K is nothing else as their product in K.

Thus, if V and $W \in E = K^d$ and if f and g are two linear forms on K^d, then we have

$$(f \otimes g) \circ (V \otimes W) \neq 0$$

if and only if

$$f \circ V \neq 0 \qquad \text{and} \qquad g \circ W \neq 0.$$

Thus the tensor product allows to "simultaneously control" two inequalities.

(*b*) Now, let $(E \cong K^d, \lambda, \mu, \gamma)$ be a linear representation of some rational series S, and let k be any positive integer. In order to prove that supp (S) is strictly k-cancellable at some order m, we shall have to "simultaneously control" 2^{k-1} inequalities, and thus to compute in the 2^{k-1}-th tensor power of E.

Thus we set

$$\hat{E} = \underbrace{E \otimes E \otimes \ldots \otimes E}_{2^{k-1} \text{ times}}$$

hence E is a vector space of dimension

$$\dim(\hat{E}) = \tau, \qquad \tau = d^{2^{k-1}}$$

Now let $M = (k+1)\tau$. We shall prove that the support of S is strictly k-cancellable at the order M, that is [using lemma 2.1, (iii)]: for any $F \in \Phi_M$ such that word $(F) \in$ supp (S) we shall prove the following hypothesis.

(H) one can find some $G \in \Phi_k(F)$ such that

$$G \in U_{C_k}(\text{supp } S)$$

(*c*) In fact, we shall prove a stronger result. For any $F=(x, u_1, u_2, \ldots, u_M, y) \in \Phi_M$ such that word $(F) \in \text{supp}(S)$ we set

$$s_j = u_j u_{j+1} \ldots u_M y$$
$$S_j = \mu(u_j u_{j+1} \ldots u_M y) \circ \gamma$$
$$V_j = \underbrace{S_j \otimes S_j \otimes \ldots \otimes S_j}_{2^{k-1} \text{ times}}$$

Then $\mathscr{V} = (V_1, V_2, \ldots, V_M)$ is a sequence of $M=(k+1)^\tau$ vectors in the space $\hat{E}$ of dimension τ. Thus let (F) be a festoon of size $k+1$,

$$F = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_{k+1})$$

that can be found in $\mathscr{V}$ by theorem 4.

In fact we shall prove the following hypothesis, for $q=1, 2, \ldots, k+1$.

(H_q) *One can find some factorization $G_q \in \Phi_q(F)$, defined by a sequence of integers*

$$0 \leqq j_1 < j_2 \ldots < j_{q+1} \leqq M$$

that satisfies

(i) $G_q \in U_{C_q}(\text{supp } S)$.

(ii) $V_{j_i} \in \mathscr{W}_i$ *for* $i=1, 2, \ldots, q+1$

(*end of* H_q) (*Fig.* 5).

Clearly (H_k) implies (H), with $G=G_k$.

Clearly also, (H_0) is trivially true. Indeed, C_0 being the empty set (because $\mathrm{Id}_{\{\ \}}$ is not a transformation), so (H_0) only asserts the existence of some $V_{j_1} \in \mathscr{W}_1$.

(*d*) Now, we suppose (H_q) satisfied for $q<k$, and we prove (H_{q+1}).

Introduce first some notations. The factorizations G is of the form

$$G_q = (f, g_1, g_2, \ldots, g_q, s_{j_{q+1}}).$$

We set

$$H_q = (f, g_1, g_2, \ldots, g_q, \varepsilon)$$

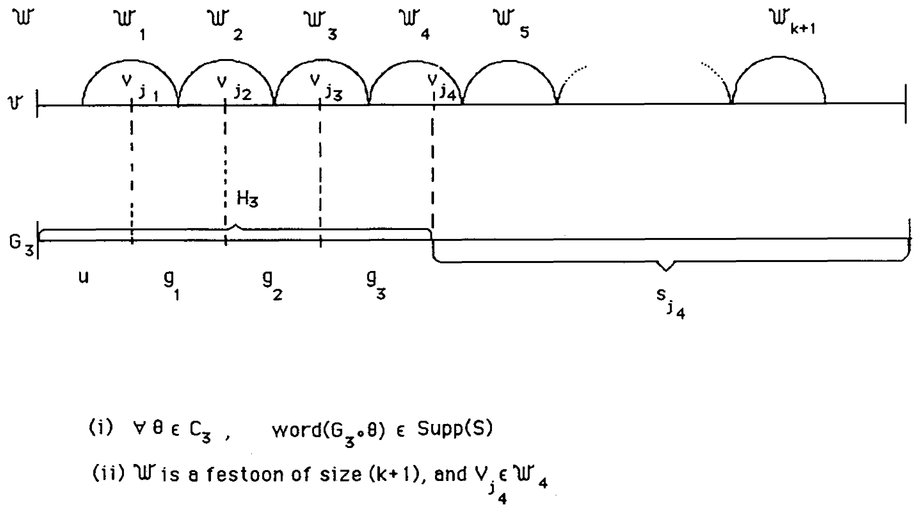


Figure 5. — **(H_3)-hypothesis.**

and so we have, for any $\theta \in C_q \cup \{\mathrm{Id}_q\}$

$$\text{word}\,(G_q \circ \theta) = \text{word}\,(H_q \circ \theta)\,.\, s_{j_{q+1}}$$

[where word $(G_q \circ \mathrm{Id}) = $ word (G_q)].

But word $(G_q \circ \theta)$ belongs to supp L if and only if

$$\langle S \,|\, \text{word}\,(G_q \circ \theta) \rangle \neq 0$$

in other words, if and only if

$$\lambda \circ \mu(\text{word}\,(G_q \circ \theta)) \circ \gamma \neq 0$$

$$\lambda \circ \mu\,(\text{word}\,(H_q \circ \theta)) \circ \mu\,(s_{j_{q+1}}) \circ \gamma \neq 0$$

and that can be write

$$P_\theta \circ S_{j_{q+1}} \neq 0$$

where we have set:

$$P_\theta = \lambda \circ \mu\,[\text{word}\,(H_q \circ \theta)].$$

Hence we have, by (H_q) and because word $(G_q) \in \operatorname{supp} S$:

(E_q) $\forall \theta \in C_q \cup \{\mathrm{Id}_q\}$, $P_\theta \circ S_{j_{q+1}} \neq 0$.

We have Card $(C_q \cup \{\mathrm{Id}_q\}) = 2^q \leqq 2^{k-1}$. Thus we can "simultaneously control" these set of 2^q equations as follows. We set

$$\hat{P} = \Big(\bigotimes_{\Theta \in C_q \cup \{\mathrm{Id}_q\}} P_\theta \Big) \otimes \big(P_{\mathrm{Id}}^{\otimes (2^{k-1} - 2^q)} \big)$$

Recall that $V_{j_{q+1}} = \underbrace{S_{j_{q+1}} \otimes S_{j_{q+1}} \otimes \ldots \otimes S_{j_{q+1}}}_{2^{k-1} \text{ times}}$

Thus the 2^q equations (E_q) are equivalent to

$$\hat{P} \circ V_{j_{q+1}} \neq 0.$$

(*e*) Now, $V_{j_{q+1}} \in W_{q+1}$, and because $q+1 < k+1$, there is a mesh $\mathscr{W}_{q+2}$ in the festoon F, and we have:

$$V_{j_{q+1}} \in \operatorname{span}(\mathscr{W}_{q+2}).$$

Thus we have successively, for some $\alpha_s \in k$,

$$V_{j_{q+1}} = \sum_{V_s \in \mathscr{W}_{q+2}} \alpha_s V_s$$

$$\hat{P} \circ V_{j_{q+1}} = \sum_{V_s \in \mathscr{W}_{q+2}} \alpha_s (\hat{P} \circ V_s) \neq 0.$$

Consequently, we can find $V_{j_{q+2}} \in \mathscr{W}_{q+2}$ such that

$$\hat{P} \circ V_{j_{q+2}} \neq 0$$

and that is equivalent to

$$(E'_q) \qquad \forall \theta \in C_q \cup \{\mathrm{Id}_q\}, \quad P_\theta \circ S_{j_{q+2}} \neq 0$$

and in other words:

$$\lambda \circ \mu [\mathrm{word}(H_q \circ \theta)] \circ \mu(s_{j_{q+2}}) \circ \gamma \neq 0$$

and finally we obtain

$$\forall \Theta \in C_q \cup \{\mathrm{Id}_q\}$$
$$\mathrm{word}\,(H_q \circ \theta) \,.\, s_{j_{q+2}} \in \operatorname{supp} L.$$

(see *Fig.* 6).

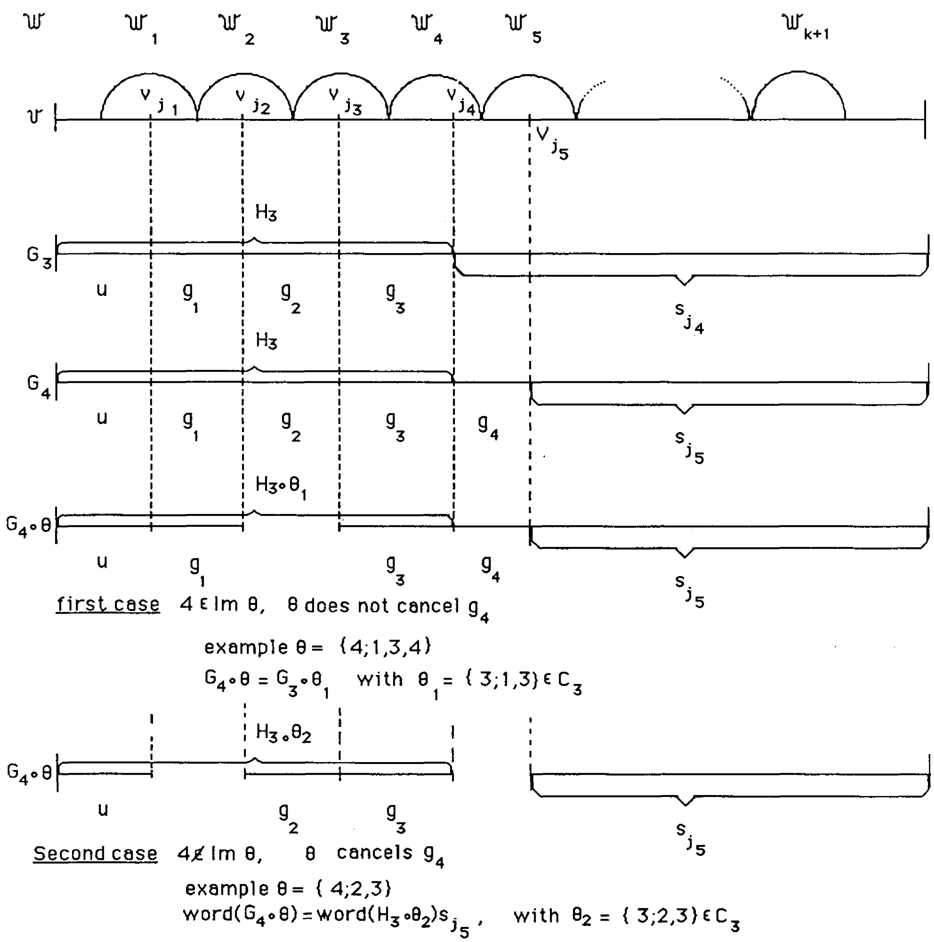


Figure 6. — hypothesis (H_3) implies hypothesis (H_4).

(*f*) Now we can define $G_{q+1} \in \Phi_{q+1}(F)$ by:

$$G_{q+1} = (f, g_1, g_2, \ldots, g_{q+1}, s_{j_{q+2}})$$

and we claim that

$$\forall \theta \in C_{q+1}, \quad \text{word}\,(G_{q+1} \circ \theta) \in \text{supp}\, L.$$

Indeed, if $\theta \in C_{q+1}$ (*see* example) *either* $q+1$ belongs to Im θ, and then

$$\begin{aligned}\text{word }(G_{q+1}\circ\theta) &= \text{word }(G_q\circ\theta_1)\\ &= \text{word }(H_q\circ\theta_1)\,g_{q+1}\,s_{j_{q+2}}\\ &\text{for some } \theta_1\in C_q\end{aligned}$$

and word $(G_q\circ\theta_1)$ belongs to supp L by hypothesis (H_q)

or $q+1$ does not belong to Im θ, and then

$$\begin{aligned}\text{word }(G_{q+1}\circ\theta) &= \text{word }(H_q\circ\theta_2)\,s_{j_{q+2}}\\ &\text{for some } \theta_2\in C_q\cup\{\text{Id}_q\}\end{aligned}$$

and word $(G_{q+\theta})$ belongs to supp L by (E'_q).

Thus we have proved (H_{q+1}), and that achieves the proof of the theorem 5.

CONCLUSION

We recall only the corollary:

Let L be a support. In order L to be regular, it suffices that CL is weakly k-cancellable at the order m, for some k, m strictly positive integers.

Perhaps that will be a useful tool in order to study the following conjecture of Restivo and Reutenauer [18], p. 26 of "rational separability of disjoint supports". If L_1 and L_2 are two disjoint supports of rational power series, there exists a regular language K containing L_1 and not intersecting L_2.

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