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LOWER BOUNDS ON THE COMPLEXITY OF REAL-TIME BRANCHING PROGRAMS (*)

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Abstract. – A $(2m)^{n/2^4}$ lower bound is given for the real-time decision graph complexity of the Dyck language D_m^* . Furthermore, a $2^{n/48}$ lower bound for the real-time branching program complexity of an encoding of the Dyck language D_m^* is proved. Previously known similar lower bounds are $2^{c \cdot n}$, $c \approx 10^{-13}$, for one-time-only branching programs (a less powerful model), and $2^{\Omega(\sqrt{n})}$ for real-time branching programs.

Résumé. – Dans cet article, nous montrons que le nombre de nœuds d'un arbre de décision en temps réel pour le langage de Dyck D_m^* est borné inférieurement par $(2m)^{n/2^4}$. On donne également une borne inférieure en $2^{n/48}$ pour la complexité des programmes temps réel pour un codage du langage de Dyck D_m^* . Les bornes précédemment connues étaient en $2^{c \cdot n}$ avec $c \approx 10^{-13}$ pour les programmes à un seul branchement (un modèle moins puissant) et en $2^{\Omega(\sqrt{n})}$ pour les programmes à branchements temps réel.

1. INTRODUCTION

A Σ -decision graph T , for Σ a finite alphabet, is a directed acyclic graph with the following properties:

- it has exactly one source, i. e. vertex with indegree 0;
- every vertex has outdegree 0 or $|\Sigma|$;
- sinks, i. e. vertices with outdegree 0, are labelled 0 or 1;

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– branching vertices, i. e. vertices with outdegree $|\Sigma|$ are labelled i , for some $1 \leq i \leq n$, and the $|\Sigma|$ outgoing edges are labelled with the elements of Σ , where each $\sigma \in \Sigma$ occurs exactly once.

To every word $w_1 w_2 \dots w_n = w \in \Sigma^n$ there corresponds a unique path p_w leading from the source to a sink (at a branching vertex labelled i it chooses the edge labelled $w_i \in \Sigma$). The decision graph T decides a set $L^{(n)} \subseteq \Sigma^n$ iff for every $w \in \Sigma^n$ the sink at the end of the path p_w is labelled $L^{(n)}(w)$. [Throughout this work we make no difference between $L^{(n)}$ and its characteristic function denoted by $L^{(n)}$, too.] The size of a decision graph T , which we denote by $\text{SIZE}(T)$, is the number of branching vertices of T .

A decision graph is said to be *real-time*, if for every w the length of p_w is less than or equal to n .

A $\{0, 1\}$ -decision graph is called a *branching program*. Branching programs compute Boolean functions. They have been studied more extensively than decision graphs over a larger alphabet, although the latter ones are more adapted in many cases. However, the difference is not important.

The logarithm of the size of a smallest decision graph deciding a language is a lower bound on space requirement for any reasonable sequential model of computation.

Nonlinear lower bounds ($\Omega(n^2/\log^2 n)$) have already been given by Nechiporuk [7] (in the more general framework of contact schemes). In order to obtain larger lower bounds, bounded width branching programs have been studied in Borodin-Dolev-Fich-Paul [2], Chandra-Furst-Lipton [4], Pudlak [8] and Yao [11]. Another restricted model is the *one-time-only* branching program studied by Wegener [10], Zak [12], Dunne [5] and Ajtai *et al.* [1]. It imposes the constraint that any computation path may examine every input letter at most once. Wegener, Zak and Dunne have given $2^{\Omega(\sqrt{n})}$ lower bounds, whereas in [1] a 2^{cn} lower bound has been proved, for $c \approx 10^{-13}$. The Boolean functions studied in these works are determined by graph properties. The property given by Hajnal, Szemerédi and Turán in [1] is “ G has an even number of triangles”.

Clearly, the real-time model is more powerful than the one-time-only model. Again Zak [13] has proved a $2^{\Omega(\sqrt{n})}$ lower bound. In this paper we study the real-time decision graph complexity of the Dyck languages D_m^* . In view of the well-known Chomsky-Schützenberger Theorem, they are very interesting context-free languages. It is known that the membership problem for the Dyck language D_m^* is identical with the word problem of the free group on m distinct generators.

Comparing the Dyck language D_m^* with the graph property in [1] from the complexity point of view, we have:

(a) “ G has an even number of triangles” as well as D_m^* (see [6]) can be decided within logspace;

(b) D_m^* can be decided in realtime, whereas this is not clear for “ G has an even number of triangles”.

In section 2 we give several definitions from the theory of partially ordered sets. We make use of them in section 3, where a general lower bound for real-time decision graphs is derived. This result is applied in section 4 to prove $(2m)^{n/24}$ lower bounds for real-time decision graphs of the Dyck languages D_m^* . Finally, in section 5 we encode D_2^* and show a $2^{n/48}$ lower bound for real-time branching programs. As we are only interested in some basic grouptheoretic properties of D_m^* , we shall consider D_m^* to be the word problem of the free group of rank m .

2. PRELIMINARIES

A subset of a partially ordered set (poset) P is *descending* iff $x \in S$ and $y \leq x$ imply $y \in S$. An *ascending subset* of P is just a descending subset of the dual poset P^* obtained by reversing the order relation. If S is any subset of P , then

$$Cl_P(S) := \{x \mid \exists y \in S : x \leq y\} \quad (\text{resp. } \overline{Cl}_P(S) := \{x \mid \exists y \in S : x \geq y\})$$

is the smallest descending (resp. ascending) subset of P containing S . $Cl_P(S)$ is often called the *closure* of S in P .

A *chain* C is a totally ordered subset of P . A subset Q of P is said to be a *cutset* iff $C \subseteq P$, C maximal chain, implies $Q \cap C \neq \emptyset$.

The *product* $P \times Q$ of partially ordered sets P and Q is the set of all ordered pairs (p, q) , where $p \in P$ and $q \in Q$, endowed with the order $(p, q) \leq (r, s)$ whenever $p \leq r$ and $q \leq s$. The least upper bound $(p, q) \vee (r, s)$ exists iff both $p \vee r$ and $q \vee s$ exist. If they exist, then $(p, q) \vee (r, s) = (p \vee r, q \vee s)$. The dual assertion for the greatest lower bound also holds.

The product (f, g) of order-preserving maps $f: P \rightarrow P'$ and $g: Q \rightarrow Q'$ is the map $P \times Q \rightarrow P' \times Q'$ which assigns to each pair (p, q) the pair $(f(p), g(q))$. Clearly, (f, g) is order preserving.

In line with [3] let us introduce the partially ordered set $\text{Cond}(\Sigma^n)$ where Σ is a finite alphabet. The elements of $\text{Cond}(\Sigma^n)$, the so called *conditions*,

are all partial maps from $\{1, \dots, n\}$ into Σ including the empty condition $\hat{0}$ which is defined nowhere. Condition c_1 is a *subcondition* of condition c_2 (we write $c_1 \leq c_2$) iff the graph of c_1 is contained in the graph of c_2 . The *graph of a condition* c is defined to be $\text{graph}(c) := \{(i, \sigma) \mid i \in \text{dom } c \text{ and } c(i) = \sigma\}$, where $\text{dom } c = \{i \mid c(i) \text{ is defined}\}$.

It is very easy to check that $\text{Cond}(\Sigma^n)$ is the poset of faces of a simplicial complex. First this means that two conditions c_1 and c_2 have a greatest lower bound $c_1 \wedge c_2$. We remark that $\text{graph}(c_1 \wedge c_2) = \text{graph}(c_1) \cap \text{graph}(c_2)$. Secondly each segment $[\hat{0}, c] = \{c' \mid \hat{0} \leq c' \leq c\}$ is isomorphic to the Boolean algebra of all subsets of $\text{graph}(c)$. Consequently, if $c' \in [\hat{0}, c]$, then there is the complement $c - c'$ of c' in c .

The maximal elements of $\text{Cond}(\Sigma^n)$ are the ordinary words of length n over Σ , i.e. the elements of Σ^n . We extend the natural length function for words to a *rank function* r of the entire poset defining

$$r(c) := |\text{dom } c| = |\text{graph}(c)|$$

for any condition c . Then, of course, the empty condition $\hat{0}$ has rank 0, all atomic conditions (i.e., all conditions covering the empty condition) have rank 1, all maximal conditions (i.e., all words of length n) are of rank n .

If two conditions c_1 and c_2 have an upper bound, then they are called *compatible*. In that case they have a least upper bound $c_1 \vee c_2$. Obviously $\text{graph}(c_1 \vee c_2) = \text{graph}(c_1) \cup \text{graph}(c_2)$ holds.

We call a condition c a *piece* iff the domain of c is a segment $[i, j] \subseteq \{1, 2, \dots, n\}$. If $c \in \text{Cond}(\Sigma^n)$ is a piece, $\text{dom } c = [i, j]$, then we associate with c a word $w_c \in \Sigma^{j-i+1}$ by $w_c(k) = c(k+i-1)$. Condition $c' \leq c$ is said to be a *part* of condition c iff c' is a maximal element of $\{c'' \mid c'' \leq c, c'' \text{ is a piece}\}$. Clearly every condition is the least upper bound of its parts.

For $n, m \geq 1$, there is a natural order-preserving isomorphism $\text{Cond}(\Sigma^n) \times \text{Cond}(\Sigma^m) \rightarrow \text{Cond}(\Sigma^{n+m})$. If we denote by $c_1 c_2$ the image of (c_1, c_2) under this map, then $\text{dom } c_1 c_2 = \text{dom } c_1 \cup (\{n\} + \text{dom } c_2)$, and

$$c_1 c_2(i) := \begin{cases} c_1(i), & \text{if } i \leq n \\ c_2(i-n), & \text{if } i > n \end{cases}$$

provided that $i \in \text{dom } c_1 c_2$.

Let T be a Σ -decision graph. We assign to each edge e leading from vertex v_1 to vertex v_2 an element $(i, \sigma) \in \{1, 2, \dots, n\} \times \Sigma$ where i is the label of v_1 and σ is the label of e itself. Extending this assignment each path p in T is

mapped onto a subset $c(p)$ of $\{1, 2, \dots, n\} \times \Sigma$. A path p is called a *computation path* iff p starts at the source and $c(p)$ is the graph of a condition. It is plain that if p is a computation path, if $w \in \Sigma^n$, and if $c(p) \leq w$, then the path p_w along which the word is computed (see introduction) contains p as a prefix, i. e. as an initial subpath. In particular, we have that $c(p_w) = w$.

3. GENERAL LOWER BOUNDS

Let $L^{(n)}$ be a nonempty subset of Σ^n . The following three definitions are slight modifications of those occurring in [9], [10], respectively.

$L^{(n)}$ is called *k-sensitive* if for every condition c of rank $k-1$ there are words $w_1 \geq c$ and $w_2 \geq c$ such that $w_1 \in L^{(n)}$ and $w_2 \notin L^{(n)}$. A word $w \in L^{(n)}$ is said to be *critical* iff all other words $w' \in \Sigma^n$ with $r(w \wedge w') = n-1$ do not belong to $L^{(n)}$.

The language $L^{(n)}$ is called *critical* iff all of its elements are critical.

LEMMA 1: *If T is a real-time Σ -decision graph which decides $L^{(n)}$, and if a word $w \in L^{(n)}$ is critical, then the computation path in T corresponding to w examines each input exactly once.*

Proof: Let p be the path corresponding to w . Assume that there is an input which is not examined exactly once. We shall derive a contradiction. Since T is real-time, $r(c(p)) \leq n-1$. As w is critical, there is a word w' with $w' \geq c(p)$ and $w' \notin L^{(n)}$. Contradiction to the fact, that p is also computation path corresponding to w' . \square

LEMMA 2: (i) *If T is a Σ -decision graph deciding $L^{(n)}$, and if $L^{(n)}$ is k -sensitive, then no computation path of length less than k leads to a sink.*

(ii) *If moreover T is real-time and $L^{(n)}$ critical, then each computation path of length k examines each input at most once. Consequently, in this case all paths of length k are computation paths.*

Proof: (i) Exactly the definition of k -sensitivity.

(ii) Assume that there is a computation path p of length k which examines an input at least twice. Then $r(c(p)) \leq k-1$.

Since $L^{(n)}$ is k -sensitive, there is a word $w \geq c(p)$ belonging to $L^{(n)}$. Let q be the computation path in T corresponding to w . Then p is prefix of q . Consequently $r(c(q)) \leq n-1$, because T is real-time. Contradiction to lemma 1. \square

The conditions c_1 and c_2 are said to be $L^{(n)}$ -equivalent [we write $c_1 \sim c_2 \pmod{L^{(n)}}$] iff

(i) $\text{dom } c_1 = \text{dom } c_2$;

(ii) if w_1 and w_2 are words, if $w_1 \geq c_1$ and $w_2 \geq c_2$, and if

$$w_1 - c_1 = w_2 - c_2, \quad \text{then } w_1 \in L^{(n)} \iff w_2 \in L^{(n)}.$$

By definition two conditions are equivalent only if they have the same rank.

LEMMA 3: Let T be any real-time Σ -decision graph computing $L^{(n)}$. Furthermore, let $L^{(n)}$ be $2k+1$ -sensitive and critical, where $2k+1 \leq n$.

If p_1 and p_2 are computation paths of length k in T leading to one and the same node v , then $c(p_1) \sim c(p_2) \pmod{L^{(n)}}$.

Proof: (i) Let $c_1 := c(p_1)$, $c_2 := c(p_2)$. We claim that $\text{dom } c_1 = \text{dom } c_2$.

Define c'_1 (resp. c'_2) to be the maximal subcondition of c_1 (resp. c_2) which is compatible with c_2 (resp. c_1).

Then

$$\text{dom}(c_1 - c'_1) = \text{dom}(c_2 - c'_2),$$

$$\text{dom}(c'_1 \vee c_2) = \text{dom}(c_1 \vee c'_2)$$

and

$$r(c'_1 \vee c_2) = r(c_1 \vee c'_2) \leq 2k.$$

Furthermore $((c_1 \vee c'_2) - (c_1 - c'_1)) \vee (c_2 - c'_2) = c'_1 \vee c_2$.

Since $L^{(n)}$ is $2k+1$ -sensitive, there is a word $w_1 \geq c_1 \vee c_2$ belonging to $L^{(n)}$. The property $w_1 \geq c_1 \vee c'_2 \geq c_1$ implies that the path in T traced under w_1 to a sink labelled 1 equals $p_1 q$, where q leads from v to the sink. By lemma 1 $p_1 q$ examines each input exactly once. Hence $c(q) = w_1 - c_1$, and $\text{dom } c_1 = \{1, 2, \dots, n\} - \text{dom}(c(q))$.

Let $w_2 := (w_1 - (c_1 - c'_1)) \vee (c_2 - c'_2)$. Obviously w_2 is a word. An easy calculation reveals that $w_2 \geq c'_1 \vee c_2 \geq c_2$ as well as $w_2 \geq w_1 - c_1 = c(q)$. Hence $p_2 q$ is the path in T traced under w_2 . Therefore $w_2 \in L^{(n)}$ and again $p_2 q$ examines each input exactly once. Thus $\text{dom } c_2 = \{1, \dots, n\} - \text{dom}(c(q))$. So as claimed $\text{dom } c_1 = \text{dom } c_2$.

(ii) Let w_1 and w_2 be words such that $w_1 \geq c_1$, $w_2 \geq c_2$, $w_1 - c_1 = w_2 - c_2$, and $w_1 \in L^{(n)}$. We claim that $w_2 \in L^{(n)}$. Since $w_1 \geq c_1$, $p_1 q$ is the path traced under w_1 to a sink labelled 1. Analogously to (i) we get that $p_2 q$ is the computation path corresponding to w_2 . Hence $w_2 \in L^{(n)}$. \square

In order to formulate the general lower bound for real-time decision graphs we need some further notations. Let $Cl(S)$ denote the closure of S with reference to $Cond(\Sigma^n)$. The notation $\overline{Cl}(S)$ also refers to $Cond(\Sigma^n)$. Let $K^{(n)} \subseteq \Sigma^n$ be another non-empty subset and Q be a cutset of $Cl(K^{(n)})$. We define:

$$[c] := \{c' \in Cond(\Sigma^n) \mid c \sim c' \text{ mod } L^{(n)}\},$$

for any condition c ,

$$m_1(L^{(n)}, Q) := \max \{ |[c] \cap Q| \mid c \in Q \}$$

$$m_2(K^{(n)}, Q) := \max \{ |\overline{Cl}(\{c\}) \cap K^{(n)}| \mid c \in Q \}.$$

THEOREM 1: *Let $L^{(n)}$ be $2k+1$ -sensitive ($2k+1 < n$) and critical. Assume that $K^{(n)}$ is nonempty, and Q is a cutset of $Cl(K^{(n)})$ such that $c \in Q$ implies $r(c) \leq k$.*

If T is a real-time Σ -decision graph deciding $L^{(n)}$, then

$$SIZE(T) \geq \frac{|K^{(n)}|}{m_1(L^{(n)}, Q) m_2(K^{(n)}, Q)}.$$

Proof: Let π be the set of all computation paths p of T such that:

- length of p is less than or equal to k ;
- $c(p) \in Q$;
- $c(p') \notin Q$ for each proper prefix p' of p .

By lemma 2 $r(c(p))$ equals the length of p , for all $p \in \pi$. It is easy to see that for all $p, q \in \pi, p \neq q, c(p)$ and $c(q)$ are not compatible with each other and consequently $\overline{Cl}(c(p)) \cap \overline{Cl}(c(q)) = \emptyset$.

Let $w \in K^{(n)}, p_w$ the unique computation path corresponding to w . Again by lemma 2, the length of p_w is greater than k . Since Q is a cutset of $Cl(K^{(n)})$ and $r(c) \leq k$ for any $c \in Q$, there is a $p \in \pi$ such that p is a prefix of p_w . According to definitions each $p \in \pi$ is connected to at most $m_2(K^{(n)}, Q)$ words of $K^{(n)}$ in the above described way. Hence $|\pi| \geq |K^{(n)}| / m_2(K^{(n)}, Q)$.

Let V be the set of non-sink vertices of T . We consider the map $\Theta : \pi \rightarrow V$ which assigns to each p its terminal node. We claim that:

$$SIZE(T) = |V| \geq |\text{image } \Theta| \geq \frac{|K^{(n)}|}{m_1(L^{(n)}, Q) m_2(K^{(n)}, Q)}$$

Assume that $v \in V$ is a vertex such that $\Theta^{-1}(v)$ is maximal, and p is a fixed path mapped onto v via Θ . Then by lemma 3

$$\Theta^{-1}(v) \subseteq \{q \in \pi \mid c(q) \sim c(p) \bmod L^{(n)}\}.$$

Hence

$$|\Theta^{-1}(v)| \leq |[c(p)] \cap Q| \leq m_1(L^{(n)}, Q).$$

Since $\{\Theta^{-1}(v) \mid v \in \text{image } \Theta\}$ is a partition of π , our claim follows. \square

4. LOWER BOUNDS FOR DECISION GRAPHS

Let $X = \{x_1, \dots, x_m\}$, $m \geq 2$. Assume that $\langle X \rangle$ is the free group on X . Then each element of $\langle X \rangle$ can be represented as a word over $\underline{X} = X \cup \{x_1^{-1}, \dots, x_m^{-1}\}$. Given two words w_1 and w_2 over \underline{X} , it is well-known that w_1 is *freely equal* to w_2 , i. e. w_1 and w_2 define one and the same element in $\langle X \rangle$, iff w_1 can be transformed into w_2 by a finite sequence of the following rules: (1) replace $x_i x_i^{-1}$ by 1, (2) replace $x_i^{-1} x_i$ by 1, (3) the inverse of (1), (4) the inverse of (2), where 1 is the empty word ($i = 1, 2, \dots, m$).

The *word problem* of $\langle X \rangle$ is the following formal language:

$$W(\langle X \rangle) := \{w \in \underline{X}^* \mid w = 1 \text{ in } \langle X \rangle\}.$$

If

$$W^{(n)}(\langle X \rangle) := W(\langle X \rangle) \cap \underline{X}^n,$$

then $W^{(n)}(\langle X \rangle) = \emptyset$ if n is odd. Hence we assume n to be even.

A word w is called *reduced* iff neither rule (1) nor rule (2) can be applied to w . Obviously, each group element of $\langle X \rangle$ has a unique reduced representation over \underline{X} . It is plain that X is a set of reduced words, and $W^{(n)}(\langle X \rangle)$ as well as X^* are closed under cyclic permutation.

LEMMA 4: (i) *Two reduced words are equal iff they are freely equal.*

(ii) *Let $c \in \text{Cond}(\underline{X}^n)$, $r(c) \leq n/2$.*

Then there is a condition $c' \geq c$ such that:

- $r(c') \leq 2r(c)$;
- *if u is a part of c' , then the word w_u associated with u is freely equal to 1.*

Proof: Easy. \square

COROLLARY: $W^{(n)}(\langle X \rangle)$ is $(n/2) + 1$ -sensitive.

LEMMA 5: $W^{(n)}(\langle X \rangle)$ is critical.

Proof: The assertion follows from the trivial observation that

$$x_{i_1}^{e_1} x_{i_2}^{e_2} \dots x_{i_n}^{e_n} = 1$$

in $\langle X \rangle$, for $e_i = \pm 1$, implies

$$x_{i_j}^{e_j} = (x_{i_1}^{e_1} \dots x_{i_{j-1}}^{e_{j-1}})^{-1} (x_{i_{j+1}}^{e_{j+1}} \dots x_{i_n}^{e_n})^{-1} \quad \text{in } \langle X \rangle. \quad \square$$

Throughout the remainder of this section let

$$Cl(S) := Cl_{\text{Cond}(X^n)}(S)$$

$$\overline{Cl}(S) := \overline{Cl}_{\text{Cond}(X^n)}(S),$$

for $S \subseteq \text{Cond}(X^n)$.

Define $Q_{n,k} := \{c \in Cl(X^n); r(c) = k\}$.

LEMMA 6: (i) The sets $Q_{n,k}$ are cutsets of $Cl(X^n)$.

(ii) Assume that n is divisible by $6l$, and $l \geq 2$ is an integer. Then $m_1(W^{(n)}(\langle X \rangle), Q_{n,n/2l}) \leq m^{n/3l}$.

(iii) $m_2(X^n, Q_{n,k}) = m^{n-k}$.

Proof: Claim (i) and (iii) are obvious.

(ii) Let $c_1, c_2 \in Cl(X^n)$, $r(c_1) = r(c_2) = (n/2)l$, $c_1 \neq c_2$ such that $c_1 \sim c_2 \pmod{W^{(n)}(\langle X \rangle)}$. Let $I := \text{dom } c_1 = \text{dom } c_2$. Since both $W^{(n)}(\langle X \rangle)$ and X^n are invariant under cyclic permutations, we can restrict ourselves to the case that $|I \cap \{1, 2, \dots, n/3\}| \geq (1/6)l$.

It is sufficient to show that there is an $i \in I$, $i > n/3$, such that $c_1(i) \neq c_2(i)$. Suppose that this is not the case. We shall derive a contradiction. Let $c_1 = c'_1 d$ and $c_2 = c'_2 d$, where $c'_1, c'_2 \in \text{Cond}(X^{n/3})$ and $d \in \text{Cond}(X^{2n/3})$. Let $w'_1, w'_2 \in X^{n/3}$ be words such that

$$c'_1 \leq w'_1, \quad c'_2 \leq w'_2 \quad \text{and} \quad w'_2 - c'_2 = w'_1 - c'_1.$$

Let $d' \in \text{Cond}(X^{2n/3})$, $d' \geq d$ be a condition the existence of which is ensured by lemma 4(ii). Since $r(d') \leq (2/3)l \leq n/3$, all parts of d' are associated with words which are freely equal to 1, and $|w'_1| = n/3$, there is a $w_2 \in X^{2n/3}$ such that $w_2 \geq d'$ and $w'_1 w_2 = 1$ in X . The $W^{(n)}(\langle X \rangle)$ -equivalence of c_1 and c_2 implies $w'_2 w_2 = 1$ in $\langle X \rangle$. Then w'_1 and w'_2 are freely equal and consequently equal. It follows that $c'_1 = c'_2$. Contradiction to $c_1 \neq c_2$. \square

THEOREM 2: Assume that T is a real-time X -decision graph deciding $W^{(n)}(\langle X \rangle)$, for 12 dividing n and $|X| \geq 2$.

Then $\text{SIZE}(T) \geq |\underline{X}|^{tn/12} \geq |\underline{X}|^{n/24}$, where $t = \ln |X| / (\ln |X| + \ln 2)$.

Proof: Applying theorem 1 for

$$L^{(n)} = W^{(n)}(\langle X \rangle), \quad K^{(n)} = X^n \quad \text{and} \quad Q = Q_{n, n/4}$$

We obtain

$$\text{SIZE}(T) \geq \frac{m^n}{m^{n/6} m^{n-n/4}} = m^{n/12} = (2m)^{(\ln m / \ln 2m) \cdot n/12}. \quad \square$$

5. LOWER BOUNDS FOR BRANCHING PROGRAMS

Let

$$X = \{x_1, x_2\}, \quad X^{-1} = \{x_1^{-1}, x_2^{-1}\}, \quad \underline{X} = X \cup X^{-1}, \quad \varphi: \underline{X} \rightarrow \{0, 1\}^2$$

defined by

$$x_1 \mapsto 10, \quad x_2 \mapsto 01, \quad x_1^{-1} \mapsto 11, \quad x_2^{-1} \mapsto 00.$$

First we observe that φ can be extended to $\varphi_n: \underline{X}^n \rightarrow \{0, 1\}^{2n}$ in the straightforward way. Second, we can extend φ to an injective order-preserving map $\varphi_{1*}: \text{Cond}(\underline{X}^1) \rightarrow \text{Cond}(\{0, 1\}^2)$ simply by setting $\varphi_{1*}(\hat{0}) = \hat{0}$.

We know from section 2 that $\text{Cond}(\underline{X}^n)$ is isomorphic to $\text{Cond}(\underline{X}^1) \times \dots \times \text{Cond}(\underline{X}^1)$, and $\text{Cond}(\{0, 1\}^{2n})$ is isomorphic to $\text{Cond}(\{0, 1\}^2) \times \dots \times \text{Cond}(\{0, 1\}^2)$. We define the order-preserving injection $\varphi_{n*}: \text{Cond}(\underline{X}^n) \rightarrow \text{Cond}(\{0, 1\}^{2n})$ to be $(\varphi_{1*}, \dots, \varphi_{1*})$. It is obvious that $r(\varphi_{n*}(c)) = 2r(c)$. We remark that φ_{n*} restricted to \underline{X}^n is identical to φ_n .

LEMMA 7: *Assume that c and c' belong to the closure of X^n in $\text{Cond}(\underline{X}^n)$, i. e. to $\text{Cl}(X^n)$. Then $\varphi_{n*}(c \wedge c') = \varphi_{n*}(c) \wedge \varphi_{n*}(c')$.*

Proof: Notice that the Hamming-distance of $\varphi(x_1)$ and $\varphi(x_2)$ is equal to 2. This implies that

$$\varphi_{1*}(d \wedge d') = \varphi_{1*}(d) \wedge \varphi_{1*}(d') \quad \text{for } d, d' \in X \cup \{\hat{0}\} \subseteq \text{Cond}(\underline{X}^1).$$

It is known that an isomorphism of partially ordered sets respects least upper bounds as well as greatest lower bounds. Let

$$c = c_1 c_2 \dots c_n, \quad c' = c'_1 c'_2 \dots c'_n \quad \text{where } c_j, c'_j \in X \cup \{\hat{0}\},$$

for $i = 1, \dots, n$. Then

$$c \wedge c' = (c_1 \wedge c'_1)(c_2 \wedge c'_2) \dots (c_n \wedge c'_n).$$

Consequently

$$\begin{aligned} \varphi_{n^*}(c \wedge c') &= \varphi_{1^*}(c_1 \wedge c'_1) \varphi_{1^*}(c_2 \wedge c'_2) \dots \varphi_{1^*}(c_n \wedge c'_n) \\ &= \varphi_{n^*}(c) \wedge \varphi_{n^*}(c'). \quad \square \end{aligned}$$

We define $f_{2n} := W^{(n)}(\langle X \rangle) \cdot \varphi_n^{-1}$. (Remember, that we identify a formal language with its characteristic function.)

LEMMA 8: *The Boolean function f_{2n} , for even n , is $(n/2) + 1$ sensitive and critical.*

Proof: Immediate consequence of lemma 4 and 5, resp. \square

Throughout the remainder of this section let $Cl_1(A)$ denote the closure of A in $\text{Cond}(\underline{X}^n)$, whereas let $Cl_2(B)$ denote the closure of B in $\text{Cond}(\{0, 1\}^{2n})$. In section 4 we considered the subsets

$$Q_{n,k} = \{c \in Cl_1(X^n) \mid r(c) = k\}, \quad k \geq 1,$$

of $\text{Cond}(\underline{X}^n)$. Now we are interested in cutsets of the closure of $\varphi_{n^*}(\underline{X}^n)$ in $\text{Cond}(\{0, 1\}^{2n})$. Clearly, the sets $\varphi_{n^*}(Q_{n,k})$ do not have this property. Define

$$\begin{aligned} K^{(2n)} &:= \varphi_{n^*}(X^n) \\ R_{n,k} &:= Cl_2(\varphi_{n^*}(Q_{n,k})) - Cl_2(\varphi_{n^*}(Q_{n,k-1})). \end{aligned}$$

LEMMA 9: (i) *The sets $R_{n,k}$ are cutsets of $Cl_2(K^{(2n)})$.*

(ii) *If $6l$ divides n , for an integer $l \geq 2$, then*

$$m_1(f_{2n}, R_{n,n/2l}) \leq 2^{n/3l}$$

(iii) $m_2(K^{(2n)}, R_{n,k}) \leq 2^{n-k}$.

Proof: It follows from the definition of $R_{n,k}$ that if $d \in R_{n,k}$ then there is a $c \in Q_{n,k}$ such that $d \leq \varphi_{n^*}(c)$. We show first, that this condition c is uniquely determined. If $c, c' \in Q_{n,k}$ such that $d \leq \varphi_{n^*}(c)$, and $d \leq \varphi_{n^*}(c')$, then

$$d \leq \varphi_{n^*}(c) \wedge \varphi_{n^*}(c') = \varphi_{n^*}(c \wedge c')$$

by lemma 8. Since

$$d \notin Cl_2(\varphi_{n^*}(Q_{n,k-1})), \quad r(c \wedge c') > k - 1.$$

Hence $c = c'$. Notation: $\mu(d) = c$.

It is plain that, for $d, d' \in R_{n,k}$ with $\text{dom } d = \text{dom } d'$, $\mu(d) = \mu(d')$ implies $d = d'$.

(i) Obvious.

(ii) We shall show that if $d \sim d' \pmod{f_{2n}}$ for $d, d' \in R_{n, n/2}$ then $\mu(d) \sim \mu(d') \pmod{W^{(n)}(\langle X \rangle)}$.

Let $d = d_1 d_2 \dots d_n$ and $d' = d'_1 d'_2 \dots d'_n$ where, for $i = 1, \dots, n$, d_i and d'_i belong to $\text{Cond}(\{0, 1\}^2)$. Then there are $w, w' \in \{0, 1\}^{2n}$ such that:

- $w \geq \varphi_{n^*}(\mu(d)) \geq d, w' \geq d'$;
- $w \in K^{(2n)} (\Leftrightarrow \varphi_n^{-1}(w) \in X^n)$;
- $w - d = w' - d'$;
- $f_{2n}(w) = 1 (\Leftrightarrow \varphi_n^{-1}(w) = 1 \text{ in } \langle X \rangle)$;
- $f_{2n}(w') = 1 (\Leftrightarrow \varphi_n^{-1}(w') = 1 \text{ in } \langle X \rangle)$.

Let $w = w_1 w_2 \dots w_n$ and $w' = w'_1 w'_2 \dots w'_n$ where $w_i, w'_i \in \{0, 1\}^2$. Obviously, then $w_i \geq d_i$, and $w'_i \geq d'_i$, for $i = 1, \dots, n$. Set $I := \{i \mid r(d_i) = r(d'_i) = 1, d_i \neq d'_i\}$. Assume that $I \neq \emptyset$. Then, for any $i \in I$, $\varphi^{-1}(w_i) \in X$ (since $w > \varphi_{n^*}(\mu(d))$), and $\varphi^{-1}(w'_i) \in X^{-1}$. But this implies $\varphi_n^{-1}(w) \neq \varphi_n^{-1}(w') \pmod{N_-}$, where N_- is the normal closure of the element $x_1 x_2^{-1}$ in $\langle X \rangle$. Contradiction to $w = w' = 1$ in $\langle X \rangle$.

Hence we have proved: If $r(d_i) = r(d'_i) = 1$, then $d_i = d'_i$. The fact that $\mu(d) \sim \mu(d') \pmod{W^{(n)}(\langle X \rangle)}$ follows from the definition of $W^{(n)}(\langle X \rangle)$ -equivalence and f_{2n} -equivalence. Consequently,

$$m_1(f_{2n} R_{n, n/2}) \leq m_1(W^{(n)}(\langle X \rangle), Q_{n, n/2}).$$

But $m_1(W^{(n)}(\langle X \rangle), Q_{n, n/2}) \leq 2^{n/3}$ by lemma 6.

(iii) Let $d \in R_{n, k}$.

$$|\text{Cl}_2^*(\{d\}) \cap K^{(2n)}| = |\text{Cl}_1(\{\mu(d)\}) \cap X^n| = 2^{n-k}, \quad \square$$

THEOREM 3: Assume that T is a real-time branching program computing f_n for 24 dividing n .

Then $\text{SIZE}(T) \geq 2^{n/48}$.

Proof: Immediate consequence of theorem 1 and lemma 9. \square

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