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COMMUTATIVITY IN GROUPS PRESENTED BY FINITE CHURCH-ROSSER THUE SYSTEMS (*)

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Abstract. – *Let G be a group that can be presented by a finite Church-Rosser Thue system. Then, whenever two elements u and v of G commute, the subgroup $\langle u, v \rangle_G$ of G generated by u and v is finite, or it is infinite cyclic. In particular, each finitely generated abelian subgroup of G is either finite or isomorphic to* **Z**. *Further, if the center of G is non-trivial, then G itself is already finite or isomorphic to* **Z**.

Résumé. – *Soit G un groupe présenté par un système de Thue fini ayant la propriété Church-Rosser. Si deux éléments u et v de G commutent, alors le sous-groupe $\langle u, v \rangle_G$ de G qu'ils engendrent est fini, ou cyclique. En particulier, tout sous-groupe abélien finissent engendré de G est fini ou isomorphe à* **Z**. *De plus, si G possède un centre non trivial, alors G lui-même est fini ou isomorphe à* **Z**.

INTRODUCTION

Thue systems are string-rewriting systems that have been studied extensively in computability theory, combinatorial (semi-) group theory, and formal language theory. A Thue system T on Σ induces a congruence relation $\overset{*}{\underset{T}{\leftrightarrow}}$ on Σ^*, and hence, languages can be defined as unions of congruence classes. In addition, T presents a monoid $\mathscr{M}_T$ which is taken to be the factor monoid of the free monoid Σ^* modulo the congruence $\overset{*}{\underset{T}{\leftrightarrow}}$.

Thue systems that satisfy the Church-Rosser property [4, 5, 11] are of special interest, since a finite Church-Rosser Thue system defines a unique

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normal form for each of its congruence classes, and any word can be reduced in linear time to the normal form in its class [5]. Hence, it is only natural to ask which monoids can be presented by finite Church-Rosser Thue systems. Observe that we are only interested in finite systems, since every countable monoid can be presented by an infinite Church-Rosser Thue system.

So far only a few results could be obtained in this direction. Cochet [10] proved that a group G can be presented by a finite special Church-Rosser Thue system if and only if G is isomorphic to the free product of finitely many (finite or infinite) cyclic groups. Gilman [13] conjectured that a group G can be presented by a finite monadic Church-Rosser Thue system if and only if G is isomorphic to the free product of a finitely generated free group and finitely many finite groups, which is exactly the class of groups that have presentations with a simple reduced word problem [15]. For two-monadic Church-Rosser Thue systems this conjecture has been proved only recently by Avenhaus, Madlener, and Otto [3]; however, the general case is still open. Finally, Avenhaus, Book, and Squier [2] established that whenever M is an infinite commutative monoid that is cancellative, then M can be presented by a finite Church-Rosser Thue system if and only if M is either the free cyclic group or the free cyclic monoid. Diekert [12] has generalized this result. For groups his result states that whenever a group G presented by some finite Church-Rosser Thue system has an abelian subgroup S of finite index, then any abelian subgroup of G is either finite or isomorphic to $\mathbf{Z}$. However, $\mathbf{Z}$ and $\mathbf{Z}_2 * \mathbf{Z}_2$ are the only infinite groups meeting these requirements [12].

Here, we restrict our attention to groups presented by finite Church-Rosser Thue systems. We investigate under which conditions two elements u and v of such a group commute. Of course, if the subgroup $\langle u, v \rangle_G$ of G generated by u and v is cyclic, then u and v commute. However, if u and v commute, and if u has infinite order, then this already implies that $\langle u, v \rangle_G$ is cyclic (Theorem 2.3). In fact, it turns out that the centralizer $C_G(u)$ of u in G is isomorphic to $\mathbf{Z}$.

To prove this result we establish a lemma that may be of interest in its own right : Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathscr{M}_T$ presented by $(\Sigma; T)$ is a group. Then for each word $u \in \Sigma^*$, the language $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$ of irreducible descendants of powers of u is regular. Observe that the descendants of a regular set modulo a finite Church-Rosser Thue system may in general form a non-recursive set [22]. Further, the lemma is effective in that a regular expression presenting the set $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$ can be constructed effectively from u and T. Thus, given

a finite Church-Rosser Thue system T on Σ such that $\mathscr{M}_T$ is a group and a word $u \in \Sigma^*$, the order of u in $\mathscr{M}_T$ can be determined effectively.

From our characterization theorem we can easily derive that each finitely generated abelian subgroup that can be presented by a finite Church-Rosser Thue system is either finite or isomorphic to $\mathbf{Z}$. Further, if G can be presented in this way, and if the center of G is non-trivial, then G itself is already finite or isomorphic to $\mathbf{Z}$. Finally, if G contains a finitely generated abelian subgroup that is normal in G, then G is already either finite, isomorphic to $\mathbf{Z}$, or isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_2$. These results extend the ones obtained by Diekert considerably.

Finally, even though our result does not settle the problem of which groups can be presented by finite Church-Rosser Thue systems, it gives an easy criterion to verify that a given group does not have a presentation of this form.

Based on the techniques developed in this paper it can be shown that the groups in discussion are context-free groups. This and some related results will appear in a forthcoming paper. Hence, from Muller and Schupp's result [18] we can conclude that the groups presented by finite Church-Rosser Thue systems form a proper subclass of the class of groups that are finite extensions of free groups. Actually, we conjecture that this subclass is exactly the class of groups appearing in Gilman's conjecture.

1. MONOID PRESENTATIONS AND ELEMENTS OF FINITE ORDER

In the following the basic notions and definitions for this paper are given. For more details and for a thorough discussion of the various applications of Thue systems the reader may consult the excellent overview papers by Book [7, 8].

An *alphabet* Σ is a finite set the elements of which are called *letters*. The set Σ^* of words over Σ is the free monoid generated by Σ, where the *empty word* e serves as the identity. For a word $w \in \Sigma^*$, the *length* of w is denoted by $|w| : |e| = 0$, and $|wa| = |w| + 1$ for $w \in \Sigma^*$, $a \in \Sigma$. The identity of words is written as $=$, and the *concatenation* of words u and v is simply written as uv. Numerical superscripts are often used to abbreviate words : $w^0 = e$, and $w^{n+1} = w^n w$ for $w \in \Sigma^*$, $n \in \mathbf{N}$.

A *Thue system* T on Σ is a subset of $\Sigma^* \times \Sigma^*$. An element (l, r) of T is called a *rule*. For a Thue system T on Σ, $\mathrm{dom}(T) = \{l \in \Sigma^* \mid \exists r \in \Sigma^* : (l, r) \in T\}$ is the *domain* of T, and $\mathrm{range}(T) = \{r \in \Sigma^* \mid \exists l \in \Sigma^* : (l, r) \in T\}$ is its *range*. The

Thue congruence $\overset{*}{\underset{T}{\leftrightarrow}}$ induced by T is the reflexive transitive closure of the relation $\underset{T}{\leftrightarrow}$, which is defined as follows: $u \underset{T}{\leftrightarrow} v$ if and only if $\exists x, y \in \Sigma^*, (l, r) \in T: [u = x l y \text{ and } v = xry]$ or $[u = xry \text{ and } v = x l y]$. For a word $w \in \Sigma^*$, the *congruence class* $\{v \in \Sigma^* \mid w \overset{*}{\underset{T}{\leftrightarrow}} v\}$ of w is denoted by $[w]_T$.

The set of congruence classes $\{[w]_T \mid w \in \Sigma^*\}$ forms a monoid under the operation $[u]_T \circ [v]_T = [uv]_T$ with identity $[e]_T$. This monoid is denoted as $\mathcal{M}_T$. It is the factor monoid of the free monoid Σ^* modulo the Thue congruence $\overset{*}{\underset{T}{\leftrightarrow}}$. If M is a monoid such that $M \cong \mathcal{M}_T$, i. e., the monoids M and $\mathcal{M}_T$ are isomorphic, then the ordered pair $(\Sigma; T)$ is called a *presentation* of M with Σ being the set of *generators*, and T being the set of *defining relations*. The monoid M is called *finitely presented*, if there exists a finite presentation of M, i. e., a presentation $(\Sigma; T)$ with Σ and T both being finite. In this paper we will only be dealing with finite presentations.

Let T be a Thue system on Σ. We define a mapping $\mathrm{ord}_T: \Sigma^* \to \mathbf{N}$ as follows:

$$\mathrm{ord}_T(w) := \begin{cases} \min\{k \in \mathbf{N}_+ \mid \exists n \in \mathbf{N}: w^{n+k} \overset{*}{\underset{T}{\leftrightarrow}} w^n\} \\ \qquad \text{if there are integers } n \geqq 0 \text{ and } k \geqq 1 \\ \qquad \text{such that } w^{n+k} \overset{*}{\underset{T}{\leftrightarrow}} w^n \\ 0 \quad \text{otherwise.} \end{cases}$$

The value $\mathrm{ord}_T(w)$ is called the *order* of w modulo T. If $\mathrm{ord}_T(w) \neq 0$, then w is said to be an *element of finite order for* T, otherwise w is said to be an *element of infinite order* for T. Obviously, a word $w \in \Sigma^*$ is an element of finite (infinite) order for T if and only if w presents an element of finite (infinite) order of the monoid $\mathcal{M}_T$. In particular, if the monoid $\mathcal{M}_T$ is cancellative, then $w \in \Sigma^*$ is an element of finite order if and only if $w^k \overset{*}{\underset{T}{\leftrightarrow}} e$ for some integer $k \geqq 1$.

Two words $u, v \in \Sigma^*$ are called *cyclically equal* (modulo T) ($u \underset{T}{\approx} v$) if there are words $x, y \in \Sigma^*$ such that $u \overset{*}{\underset{T}{\leftrightarrow}} xy$ and $v \overset{*}{\underset{T}{\leftrightarrow}} yx$ [20]. We claim that the order of words is invariant under the relation of cyclic equality.

LEMMA 1.1: *Let T be a Thue system on Σ, and let $u, v \in \Sigma^*$. If $u \underset{T}{\approx} v$, then* $\mathrm{ord}_T(u) = \mathrm{ord}_T(v)$.

Proof: Let u and v be cyclically equal. Then there exist words $x, y \in \Sigma^*$ such that $u \overset{*}{\underset{T}{\leftrightarrow}} xy$ and $v \overset{*}{\underset{T}{\leftrightarrow}} yx$. Assume that u has finite order. Then we have integers $n \geqq 0$ and $k \geqq 1$ such that $u^{n+k} \overset{*}{\underset{T}{\leftrightarrow}} u^n$, and hence,

$$v^{n+k+1} \overset{*}{\underset{T}{\leftrightarrow}} (yx)^{n+k+1} = y(xy)^{n+k} x \overset{*}{\underset{T}{\leftrightarrow}} yu^{n+k} x \overset{*}{\underset{T}{\leftrightarrow}} yu^n x \overset{*}{\underset{T}{\leftrightarrow}} y(xy)^n x \overset{*}{\underset{T}{\leftrightarrow}} v^{n+1}.$$

Thus, v has finite order, too, and $\mathrm{ord}_T(v) \leqq \mathrm{ord}_T(u)$. By symmetry we can conclude that $\mathrm{ord}_T(u) = \mathrm{ord}_T(v)$. □

In the following we are interested in Thue systems that satisfy certain restrictions. A Thue system T on Σ is called *length-reducing*, if $|l| > |r|$ for each rule $(l, r) \in T$, and it is called *monadic*, if it is length-reducing and $\mathrm{range}(T) \subseteq \Sigma \cup \{e\}$. The *reduction relation* $\overset{*}{\underset{T}{\rightarrow}}$ defined by T is the reflexive transitive closure of the relation $\underset{T}{\rightarrow}$, which is defined through $u \underset{T}{\rightarrow} v$ if and only if $u \underset{T}{\leftrightarrow} v$ and $|u| > |v|$. So for a length-reducing Thue system T the relation $\underset{T}{\rightarrow}$ corresponds to the process of substituting an occurrence of the left-hand side of a rule by an occurrence of the corresponding right-hand side. A word $w \in \Sigma^*$ is called *irreducible*, if no reduction step $\underset{T}{\rightarrow}$ can be applied to w, otherwise it is called *reducible*. $\mathrm{IRR}(T)$ denotes the set of all irreducible words modulo T, $\Delta_T^*(w) = \{v \in \Sigma^* \mid w \overset{*}{\underset{T}{\rightarrow}} v\}$ denotes the set of all descendants of w modulo T, and for any language

$L \subseteq \Sigma^*$, $\Delta_T^*(L) = \bigcup_{w \in L} \Delta_T^*(w)$. A length-reducing Thue system T on Σ is called a *Church-Rosser Thue system*, if each congruence class contains a unique irreducible word, which can then be taken as the normal form for its class [5].

2. THE RESULT

Let T be a Thue system on Σ. The monoid $\mathbf{M}_T$ presented by $(\Sigma; T)$ is a group if and only if, for each word $w \in \Sigma^*$, there exists a word $w' \in \Sigma^*$ such that $w w' \overset{*}{\underset{T}{\leftrightarrow}} e$. Obviously, this is equivalent to saying that for each letter $a \in \Sigma$, there exists a word $a' \in \Sigma^*$ such that $a a' \overset{*}{\underset{T}{\leftrightarrow}} e$. In general, it is undecidable whether or not such words a' $(a \in \Sigma)$ exist, since it is undecidable in general whether or not the monoid $\mathscr{M}_T$ defined by a given presentation $(\Sigma; T)$ is a group. However, if we restrict our attention to finite presentations involving Church-Rosser Thue systems, then this problem becomes decidable [24]. In fact, given a finite Church-Rosser Thue system T on Σ such that $\mathscr{M}_T$ is a group, one can effectively determine an irreducible word a^{-1} for each letter $a \in \Sigma$ such that $a a^{-1} \overset{*}{\underset{T}{\rightarrow}} e$. We then extend the function $^{-1}: \Sigma \to \Sigma^*$ to all of Σ^* by defining $e^{-1} := e$ and $(aw)^{-1} := w^{-1} a^{-1}$, $w \in \Sigma^*$, $a \in \Sigma$. So in the following we will associate a fixed function $^{-1}: \Sigma^* \to \Sigma^*$ with each presentation $(\Sigma; T)$ provided T is Church-Rosser and $\mathscr{M}_T$ is a group. We then write w^{-k} to mean $(w^{-1})^k$, i. e., w^l is defined for all integers l.

Let the monoid $\mathscr{M}_T$ given by the presentation $(\Sigma; T)$ be a group, and let $u_1, u_2, \ldots, u_n \in \Sigma^*$. Then the *subgroup* $\langle u_1, \ldots, u_n \rangle_{\mathscr{M}_T}$ of $\mathscr{M}_T$ that is *generated by* $\{u_1, u_2, \ldots, u_n\}$ is the least subgroup of $\mathscr{M}_T$ containing all the elements presented by $u_1, u_2, \ldots, u_n$, i. e.,

$$\langle u_1, \ldots, u_n \rangle_{\mathscr{M}_T} = \{w \in \Sigma^* \mid \exists\, m \geq 0,\ v_1, \ldots, v_m \in \{u_1, \ldots, u_n\} \cup \{u_1^{-1}, \ldots, u_n^{-1}\} : w \overset{*}{\underset{T}{\leftrightarrow}} v_1 v_2 \ldots v_m\}.$$

A subgroup S of $\mathscr{M}_T$ is called *cyclic* if it is generated by a single element, *i. e.*, $S = \langle u \rangle_{\mathscr{M}_T}$ for some word $u \in \Sigma^*$.

Finally, recall that a word $w \in \Sigma^*$ is called *primitive* if there are no word $x \in \Sigma^*$ and integer $k > 1$ such that $w = x^k$; otherwise, w is called *imprimitive*. In either case, the shortest x such that $w = x^k$ is the *root* of w.

In this section we will only be dealing with finite Church-Rosser Thue systems presenting groups. We will derive a characterization for those pairs of words that commute modulo a Thue system of this type. We first state a special case of the characterization theorem we are aiming at. This special case will be very useful for proving the main result.

LEMMA 2.1: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathscr{M}_T$ is a group, and let $u \in \Sigma^*$ be a non-empty primitive word such that $\{u\}^* \subseteqq \mathrm{IRR}(T)$. Then for each word $v \in \Sigma^*$ and each integer $m \geqq 1$, if u^m and v commute, then $v \in \langle u \rangle_{\mathscr{M}_T}$.*

Proof: Let $v \in \Sigma^*$ and $m \geqq 1$ such that $u^m v \overset{*}{\underset{T}{\leftrightarrow}} v u^m$. If $v \overset{*}{\underset{T}{\leftrightarrow}} e$, then nothing has to be shown, and so we may assume that $v \overset{*}{\underset{T}{\not\leftrightarrow}} e$. Since $u^m v \overset{*}{\underset{T}{\leftrightarrow}} v u^m$, we have $u^{m.n} v \overset{*}{\underset{T}{\leftrightarrow}} v u^{m.n}$ for all $n \geqq 1$. Let w_n denote the irreducible descendant of $u^{m.n} v$. Then $u^{m.n} v \overset{*}{\underset{T}{\rightarrow}} w_n$ and $v u^{m.n} \overset{*}{\underset{T}{\rightarrow}} w_n$, since T is Church-Rosser.

CLAIM : *There exists an integer $n \geqq 1$ such that $u^m w_n = w_n u^m$.*

Proof: Let $k \geqq 1$. Then $u^{m \cdot k} v \overset{*}{\underset{T}{\rightarrow}} w_k$ implying $|w_k| \leqq m \cdot k \cdot |u| + |v|$. On the other hand, $u^{m \cdot k} \overset{*}{\underset{T}{\leftrightarrow}} w_k v^{-1}$, and hence, $w_k v^{-1} \overset{*}{\underset{T}{\rightarrow}} u^{m \cdot k}$ according to the choice of u, and so $m \cdot k \cdot |u| \leqq |w_k| + |v^{-1}|$. Let

$$\alpha_k := |w_k| - m \cdot k \cdot |u| + |v^{-1}|.$$

Then $\alpha_k \geqq 0$, and

$$|w_k| = m \cdot k \cdot |u| - |v^{-1}| + \alpha_k.$$

We have $u^m w_k \overset{*}{\underset{T}{\leftrightarrow}} u^m u^{m \cdot k} v = u^{m \cdot (k+1)} v \overset{*}{\underset{T}{\leftrightarrow}} w_{k+1}$, i. e., $u^m w_k \overset{*}{\underset{T}{\rightarrow}} w_{k+1}$ and analogously, $w_k u^m \overset{*}{\underset{T}{\rightarrow}} w_{k+1}$. This means that either $u^m w_k = w_{k+1} = w_k u^m$ or

$|w_{k+1}| < m \cdot |u| + |w_k|$. In the former case we are done, so assume the latter. Then

$$m \cdot (k+1) \cdot |u| - |v^{-1}| + \alpha_{k+1} < m \cdot |u| + |w_k| = m \cdot (k+1) \cdot |u| - |v^{-1}| + \alpha_k$$

implying that $\alpha_{k+1} < \alpha_k$. Since

$$\alpha_1 = |w_1| - m \cdot |u| + |v^{-1}| \leqq m \cdot |u| + |v| - m \cdot |u| + |v^{-1}| = |v| + |v^{-1}|,$$

we conclude that there exists an integer $n \geqq 1$ such that $u^m w_n = w_{n+1} = w_n u^m$. □

Since $u^m w_n = w_n u^m$, and since u is primitive, there exists an integer $l \geqq 0$ such that $w_n = u^l$. Hence, $u^{m \cdot n} v \overset{*}{\underset{T}{\leftrightarrow}} w_n = u^l$ implying that

$$v \overset{*}{\underset{T}{\leftrightarrow}} (u^{-1})^{m \cdot n} u^l \overset{*}{\underset{T}{\leftrightarrow}} u^{l-m \cdot n},$$

i. e., $v \in \langle u \rangle_{\mathcal{M}_T}$. □

Let T be a finite length-reducing Thue system on Σ, and let $R \subseteq \Sigma^*$ be a regular language. If T is monadic, then the language $\Delta_T^*(R)$ is also regular [9]. However, if T is non-monadic, then this language is not necessarily regular. In fact, even if T is Church-Rosser, this language can be non-recursive [22].

In what follows we are interested in languages of the form $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$, where u is an element of infinite order, and T is a finite Church-Rosser Thue system presenting a group.

LEMMA 2.2: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathcal{M}_T$ is a group. Then for each word $u \in \Sigma^*$, the language $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$ is regular.*

Proof: Observe that the language $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$ is finite if and only if the word u has finite order. So let $u \in \Sigma^*$ such that $\mathrm{ord}_T(u) = 0$. Without loss of generality we may assume that u is irreducible. For each $n \geqq 0$, let u_n denote the irreducible descendant of u^n. Then $u_m \neq u_n$ for all integers $n, m \geqq 0$, $n \neq m$.

Let $n \geqq 1$. Then $uu_n \overset{*}{\underset{T}{\rightarrow}} u_{n+1}$, and since $\mathcal{M}_T$ is a group, $u^{-1} u_{n+1} \overset{*}{\underset{T}{\rightarrow}} u_n$. Thus, $|u_n| - |u^{-1}| \leqq |u_{n+1}| \leqq |u_n| + |u|$ implying that

$$0 \leqq |uu_n| - |u_{n+1}| \leqq |u| + |u^{-1}|.$$

Hence, whenever $uu_n \overset{i}{\underset{T}{\rightarrow}} u_{n+1}$, then $i \leqq |u| + |u^{-1}|$. Analogously, $u_n u \overset{j}{\underset{T}{\rightarrow}} u_{n+1}$ also implies $j \leqq |u| + |u^{-1}|$.

Let $\lambda := \max \{|l| \mid l \in \mathrm{dom}(T)\}$, and let $\mu := 2 \cdot (|u| + |u^{-1}|) \cdot (\lambda - 1) + \lambda$. Since $u_m \neq u_n$ for all $m \neq n$, there exists an index $n(\mu)$ such that $|u_k| \geqq \mu$ for all $k \geqq n(\mu)$. Finally, let $p \geqq n(\mu)$ be chosen such that $|u_p| < |u_{p+1}|$. *Since* $uu_p \overset{*}{\underset{T}{\rightarrow}} u_{p+1} \overset{*}{\underset{T}{\leftarrow}} u_p u$, we have the following factorizations: $u_p = xt = sz$ and $u_{p+1} = vt = sw$, where $ux \overset{i}{\underset{T}{\rightarrow}} v$ and $zu \overset{j}{\underset{T}{\rightarrow}} w$. Since $u, u_p \in \mathrm{IRR}(T)$, and since $i, j \leqq |u| + |u^{-1}|$, we can conclude that $|x|, |z| \leqq (|u| + |u^{-1}|) \cdot (\lambda - 1)$. By the choice of p this yields that $t = yz$ and $s = xy$ for a word $y \in \Sigma^*$ satisfying $|y| \geqq \lambda$. Thus, we have the following situation:

$$u_p = xyz, \quad u_{p+1} = vyz = xyw, \quad \text{where } ux \overset{*}{\underset{T}{\rightarrow}} v \text{ and } zu \overset{*}{\underset{T}{\rightarrow}} w.$$

Since $\mathscr{M}_T$ is a group, and since $u \overset{*}{\underset{T}{\leftrightarrow}} e$, we obtain $v \neq x$ and $z \neq w$. Now four cases must be distinguished.

(i) $v = xx_1$ and $w = z_1 z$ for some non-empty words $x_1, z_1 \in \Sigma^*$.

Then $u_{p+1} = xx_1 yz = xyz_1 z$, which implies $x_1 y = yz_1$, $ux \overset{*}{\underset{T}{\rightarrow}} xx_1$, and $zu \overset{*}{\underset{T}{\rightarrow}} z_1 z$.

CLAIM. $u_{p+k} = xx_1^k yz$ for all $k \geqq 1$.

Proof: By induction on k:

$k = 1$: $u_{p+1} = xx_1 yz$ according to our assumptions.

$k \rightarrow k+1$: $uu_{p+k} = uxx_1^k yz$ by induction hypothesis.

Now $uxx_1^k yz \overset{*}{\underset{T}{\rightarrow}} xx_1^{k+1} yz = xx_1^k yz_1 z$. The word $xx_1^k y$ is a factor of u_{p+k}, and hence it is irreducible.

The word $yz_1 z$ is a factor of u_{p+1}, and hence it is irreducible, too. Since $|y| \geqq \lambda$, this means that the word $xx_1^{k+1} yz = xx_1^k yz_1 z$ is irreducible, i. e., $xx_1^{k+1} yz = xx_1^k yz_1 z$ is irreducible, i. e., $xx_1^{k+1} yz = u_{p+k+1}$. □

Thus, in this situation the language

$$\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T) = \{u_n \mid n \geqq 0\} = \{u_n \mid 0 \leqq n \leqq p\} \cup \{xx_1^k yz \mid k \geqq 0\}$$

is clearly regular.

(ii) $x = vv_1$ and $z = w_1 w$ for some non-empty words v_1, $w_1 \in \Sigma^*$. Then $u_p = vv_1 yw_1 w$ and $u_{p+1} = vyw_1 w = vv_1 yw$ implying $|u_p| > |u_{p+1}|$. This contradicts our choice of index p, i. e., case (ii) cannot occur.

(iii) $x = vv_1$ and $w = z_1 z$ for some non-empty words v_1, $z_1 \in \Sigma^*$. Then $u_{p+1} = vyz = vv_1 yz_1 z$ implying $y = v_1 yz_1$, which contradicts the fact that v_1 and z_1 are non-empty.

(iv) $v = xx_1$ and $z = w_1 w$ for some non-empty words x_1, $w_1 \in \Sigma^*$. Then $u_{p+1} = xx_1 yw_1 w = xyw$ giving the same contradiction as above.

Thus, only case (i) can occur, and the lemma is proved. □

Finally, we can state and prove our characterization theorem for commuting elements in finite Church-Rosser Thue systems presenting groups.

THEOREM 2.3: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathscr{M}_T$ is a group, and let $u \in \Sigma^*$ be a word of infinite order. Then for each word $v \in \Sigma^*$, the following two statements are equivalent:*

(i) u and v commute.

(iii) The subgroup $\langle u, v \rangle_{\mathscr{M}_T}$ of $\mathscr{M}_T$ generated by u and v is cyclic.

Proof: Let $u, v \in \Sigma^*$ such that $\mathrm{ord}_T(u) = 0$. If there exists a word $y \in \Sigma^*$ such that $u, v \in \langle y \rangle_{\mathscr{M}_T}$, then obviously u and v commute. To prove the converse implication assume that $uv \overset{*}{\underset{T}{\leftrightarrow}} vu$.

If $v \overset{*}{\underset{T}{\leftrightarrow}} e$, then $\langle u, v \rangle_{\mathscr{M}_T} = \langle u \rangle_{\mathscr{M}_T}$, and we are done. So assume that $v \overset{*}{\underset{T}{\not\leftrightarrow}} e$. For each $n \geqq 0$, let u_n denote the irreducible descendant of u^n. By Lemma 2.2 the set

$$R(u) : = \{u_n \mid n \geqq 0\} = \Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$$

is regular. Since it is also infinite, there exists a set $I(u) = \{xw^i z \mid i \geqq 0\} \subseteq R(u)$, where $x, w, z \in \Sigma^*$ are words such that $xz \neq e \neq w$.

Let y denote the root of w, i. e., y is a non-empty primitive word such that $w = y^n$ for some $n \geqq 1$. Obviously, we have $\{y\}^* \subseteq \mathrm{IRR}(T)$. Now for each integer $i \geqq 0$, there is an index $j_i \geqq 1$ such that $xw^i z = u_{j_i}$. We fix an integer $k \geqq 0$ such that $j_{k+1} > j_k$, and in order to simplify notation we define $m : = j_{k+1}$,

$j := j_k$, and $l := m-j$. This gives

$$xw^{k+1}z = u_m \overset{*}{\underset{T}{\leftrightarrow}} u^m = u^{j+l} \overset{*}{\underset{T}{\leftrightarrow}} u_j u^l = xw^k z u^l,$$

which in turn yields $u^l \overset{*}{\underset{T}{\leftrightarrow}} z^{-1}wz$, since $\mathcal{M}_T$ is a group.

Using this relation we obtain

$$y^n(zvz^{-1}) \overset{*}{\underset{T}{\leftrightarrow}} zz^{-1}wzvz^{-1} \overset{*}{\underset{T}{\leftrightarrow}} zu^l vz^{-1} \overset{*}{\underset{T}{\leftrightarrow}} zvu^l z^{-1} \overset{*}{\underset{T}{\leftrightarrow}} zvz^{-1}zu^l z^{-1} \overset{*}{\underset{T}{\leftrightarrow}} (zvz^{-1})y^n,$$

which yields $zvz^{-1} \in \langle y \rangle_{\mathcal{M}_T}$ by Lemma 2.1. Analogously,

$$y^n(zuz^{-1}) \overset{*}{\underset{T}{\leftrightarrow}} zz^{-1}wzuz^{-1} \overset{*}{\underset{T}{\leftrightarrow}} zu^{l+1}z^{-1} \overset{*}{\underset{T}{\leftrightarrow}} (zuz^{-1})y^n$$

implying $zuz^{-1} \in \langle y \rangle_{\mathcal{M}_T}$. Thus, we have $u, v \in \langle z^{-1}yz \rangle_{\mathcal{M}_T}$, i.e., $u \overset{*}{\underset{T}{\leftrightarrow}} y_0^k$ and $v \overset{*}{\underset{T}{\leftrightarrow}} y_0^l$ for some non-zero integers $k, l \in \mathbf{Z}$. Here y_0 stands for $z^{-1}yz$.

Let p denote the greatest common divisor of k and l. Then $p \neq 0$, and obviously, $u, v \in \langle y_0^p \rangle_{\mathcal{M}_T}$. On the other hand, there are integers $\lambda, \mu \in \mathbf{Z}$ such that

$$\lambda . k + \mu . l = p, \quad \text{i.e.,} \quad u^\lambda v^\mu \overset{*}{\underset{T}{\leftrightarrow}} y_0^{\lambda . k + \mu . l} = y_0^p.$$

Thus, $y_0^p \in \langle u, v \rangle_{\mathcal{M}_T}$. This means that $\langle u, v \rangle_{\mathcal{M}_T} = \langle y_0^p \rangle_{\mathcal{M}_T}$, and hence, the subgroup $\langle u, v \rangle_{\mathcal{M}_T}$ is in fact cyclic. □

Actually, the proof just given shows a bit more than we claimed, since the word $y_0 := z^{-1}yz$ only depends on u, but not on the word v. Thus, in addition to Theorem 2.3 we have shown the following.

COROLLARY 2.4: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathcal{M}_T$ is a group. Then for each word u of infinite order, the centralizer $C_T(u)$ of u in $\mathcal{M}_T$ is isomorphic to* $\mathbf{Z}$.

Proof: Let $u \in \Sigma^*$ such that $\mathrm{ord}_T(u) = 0$. Then there exists a word $y_0 \in \Sigma^*$ such that for each $v \in \Sigma^*$, *if* u and v commute, then $v \in \langle y_0 \rangle_{\mathcal{M}_T}$. Thus, $C_T(u) = \{ v \in \Sigma^* \mid uv \overset{*}{\underset{T}{\leftrightarrow}} vu \} \subseteq \langle y_0 \rangle_{\mathcal{M}_T}$, and so $C_T(u)$ is isomorphic to $\mathbf{Z}$. □

3. SOME CONCLUSIONS

From our characterization theorem we fairly easily obtain a number of conclusions regarding the order of commuting elements and the subgroups generated by them.

COROLLARY 3.1: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathscr{M}_T$ is a group, and let $u, v \in \Sigma^*$ be words such that $uv \overset{*}{\underset{T}{\leftrightarrow}} vu$, but neither $u \overset{*}{\underset{T}{\leftrightarrow}} e$ nor $v \overset{*}{\underset{T}{\leftrightarrow}} e$.*

(a) If u has finite order, then v and uv have finite order.

(b) If u has infinite order, then v has infinite order, and either $v \overset{*}{\underset{T}{\leftrightarrow}} u^{-1}$ or uv also has infinite order.

Proof: Let $u, v \in \Sigma^*$ such that $uv \overset{*}{\underset{T}{\leftrightarrow}} vu$, $u \overset{*}{\underset{T}{\not\leftrightarrow}} e$, and $v \overset{*}{\underset{T}{\not\leftrightarrow}} e$. If $\mathrm{ord}_T(u) = 0$, then by Theorem 2.3 there exists a word $y \in \Sigma^*$ such that $u \overset{*}{\underset{T}{\leftrightarrow}} y^i$ and $v \overset{*}{\underset{T}{\leftrightarrow}} y^j$ for some integers $i, j \neq 0$. Now $\mathrm{ord}_T(u) = 0$ implies that $\mathrm{ord}_T(y) = 0$, which then yields that $\mathrm{ord}_T(v) = 0$. Thus, either u and v both have infinite order, or they both have finite order.

(*a*) Let u and v have finite order, and let $m := \mathrm{ord}_T(v)$. Then $m \geqq 2$, $v^m \overset{*}{\underset{T}{\leftrightarrow}} e$, and $v^i \overset{*}{\underset{T}{\not\leftrightarrow}} e$ for all $i \{1, 2, \ldots, m-1\}$. If $\mathrm{ord}_T(uv) = 0$, then

$$(uv)\, v^{m-1} = uv^m \overset{*}{\underset{T}{\leftrightarrow}} u \overset{*}{\underset{T}{\leftrightarrow}} v^m u \overset{*}{\underset{T}{\leftrightarrow}} v^{m-1}(uv)$$

implies that $\mathrm{ord}_T(v) = 0$ according to the above remark. Thus, uv has finite order.

(*b*) Let u and v have infinite order. Then the subgroup $\langle u, v \rangle_{\mathscr{M}_T}$ is cyclic, and since it is infinite, we have $\langle u, v \rangle_{\mathscr{M}_T} \cong \mathbf{Z}$. Thus, $uv \in \langle u, v \rangle_{\mathscr{M}_T}$ implies that either $uv \overset{*}{\underset{T}{\leftrightarrow}} e$, i. e., $v \overset{*}{\underset{T}{\leftrightarrow}} u^{-1}$, or $\mathrm{ord}_T(uv) = 0$. □

A group is called *torsion-free*, if it does not contain any non-trivial elements of finite order. The following corollary gives a characterization for those groups presented by finite Church-Rosser Thue systems that contain elements of infinite order, and it states a simple observation about groups of this kind that are not torsion-free.

COROLLARY 3.2: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathcal{M}_T$ is a group.*

(a) The group $\mathcal{M}_T$ is infinite if and only if it contains an element of infinite order.

(b) If $\mathcal{M}_T$ is infinite but not torsion-free, then $\mathcal{M}_T$ contains infinitely many non-trivial elements of finite order.

Proof: (a) If $\mathcal{M}_T$ contains an element of infinite order, then obviously, $\mathcal{M}_T$ must be infinite. Now assume conversely that the group $\mathcal{M}_T$ is infinite. Then the set $\mathrm{IRR}(T)$ is an infinite regular set of representatives for $\mathcal{M}_T$. Hence, the pumping lemma for regular sets applies giving a subset $\{xy^kz \mid k \geq 0\}$ of $\mathrm{IRR}(T)$, where $y \neq e$. Since the set $\mathrm{IRR}(T)$ is subword-closed, this gives $\{y\}^* \subseteq \mathrm{IRR}(T)$. Thus, y describes an element of infinite order of $\mathcal{M}_T$.

(b) Since $\mathcal{M}_T$ is infinite, there exists a word $y \in \Sigma^*$ such that $\mathrm{ord}_T(y) = 0$, and since $\mathcal{M}_T$ is not torsion-free, there exists a word $x \in \mathrm{IRR}(T) - \{e\}$ such that $\mathrm{ord}_T(x) = m \geq 2$. For all $i \geq 1$, let $x_i := y^{-i}xy^i$. Then $\mathrm{ord}_T(x_i) = \mathrm{ord}_T(x) = m$ for all i.

Assume that $x_i \overset{*}{\underset{T}{\leftrightarrow}} x_{i+j}$ for some integers $i, j \geq 1$. Then $y^{-i}xy^i \overset{*}{\underset{T}{\leftrightarrow}} y^{-i-j}xy^{i+j}$ implying that $x \overset{*}{\underset{T}{\leftrightarrow}} y^{-j}xy^j$, i.e., $y^jx \overset{*}{\underset{T}{\leftrightarrow}} xy^j$. Since x has finite order, y^j must have finite order by Corollary 3.1. Hence, $j = 0$. Thus, for all $i \neq j$, $x_i \overset{*}{\underset{T}{\not\leftrightarrow}} x_j$, i.e., $\{x_i \mid i \geq 1\}$ presents an infinite subset of $\mathcal{M}_T$ of elements of finite order. □

Corollary 3.2 (a) solves the Burnside problem [1] for groups presented by finite Church-Rosser Thue systems. In fact, our proof is valid for each group that can be presented by a complete string-rewriting system the domain of which is a regular set. That we used length as an ordering is not important in this situation.

If T is a finite monadic Church-Rosser Thue system on Σ and $w \in \Sigma^*$, then the order $\mathrm{ord}_T(w)$ of w can be determined effectively. In fact, given a finite

monadic Church-Rosser Thue system T on Σ, it is decidable whether or not there exists a word $w \in \Sigma^*$, $w \overset{*}{\underset{T}{\leftrightarrow}} e$, such that $\mathrm{ord}_T(w) > 0$ [23]. However, for finite Church-Rosser Thue systems that are non-monadic, this problem is still open, while it is known to be undecidable for finite Thue systems in general. If we restrict our attention to letters only, then we do at least have the following result, which can be viewed as a generalization of [2, Lemma 4].

LEMMA 3.3: *Let T be a finite Church-Rosser Thue system on Σ, and let $a \in \Sigma$. Then the following two statements are equivalent:*

(i) $\mathrm{ord}_T(a) > 0$.

(ii) $\exists\, m \geqq 1 : a^m \in \mathrm{dom}(T)$.

Proof: If $\mathrm{ord}_T(a) > 0$, then there exist integers $n \geqq 0$ and $k \geqq 1$ such that $a^{n+k} \overset{*}{\underset{T}{\leftrightarrow}} a^n$. Since T is Church-Rosser, this implies that a^{n+k} and a^n have a common descendant. In particular, a^{n+k} is reducible modulo T, i.e., $a^m \in \mathrm{dom}(T)$ for some $m \in \{1, 2, \ldots, n+k\}$.

Now assume conversely that $a^m \in \mathrm{dom}(T)$ for some integer $m \geqq 1$. Then there exists a word $r \in \Sigma^*$, $|r| < m$, such that $(a^m, r) \in T$. For each $n \geqq m$, let u_n denote the irreducible descendant of a^n modulo T. If $u_n \in \{a\}^*$ for some $n \geqq m$, then $a^n \overset{*}{\underset{T}{\leftrightarrow}} u_n = a^k \in \mathrm{IRR}(T)$, which implies $n \geqq m > k$. Thus, $\mathrm{ord}_T(a) > 0$. So let us assume that $u_n \notin \{a\}^*$ for all $n \geqq m$.

CLAIM: $|u_n| < m$ *for all* $n \geqq m$.

Proof: By induction on n:

$n = m$: $a^m \to r \overset{*}{\underset{T}{\to}} u_m$ implying $|u_m| \leqq |r| < m$.

$n \to n+1$: We have $a^n \overset{*}{\underset{T}{\to}} u_n$, and by induction hypothesis $|u_n| < m$.

Now $a^{n+1} = a^n a \underset{T}{\overset{*}{\rightarrow}} u_n a$ and $a^{n+1} = aa^n \underset{T}{\overset{*}{\rightarrow}} au_n$, which yield $au_n \underset{T}{\overset{*}{\rightarrow}} u_{n+1} \underset{T}{\overset{*}{\leftarrow}} u_n a$. Since $u_n \notin \{a\}^*$, we conclude that $au_n \neq u_n a$, i.e., $au_n \underset{T}{\overset{+}{\rightarrow}} u_{n+1}$. Hence, $|u_{n+1}| \leqq |u_n| < m$. □

Since there exist only finitely many words $v \in \Sigma^*$ satisfying $|v| < m$, we see that there are integers $n_1 > n_2 \geqq m$ such that $u_{n_1} = u_{n_2}$. Hence, $a^{n_1} \underset{T}{\overset{*}{\leftrightarrow}} a^{n_2}$, i.e., $\mathrm{ord}_T(a) > 0$. □

If, however, $\mathscr{M}_T$ is a group, then by Lemma 2.2 $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$ is a regular set. Now either this set is finite, in which case there exists an integer $n \geqq 1$ such that $u^n \underset{T}{\overset{*}{\rightarrow}} e$, or this set is infinite, in which case there exists an integer $p \geqq 1$ such that

$$u^p \underset{T}{\overset{*}{\rightarrow}} u_p \in \mathrm{IRR}(T), \qquad u^{p+1} \underset{T}{\overset{*}{\rightarrow}} u_{p+1} \in \mathrm{IRR}(T),$$

and

$$|u_{p+1}| > |u_p| \geqq 2.(|u| + |u^{-1}|).(\lambda - 1) + \lambda,$$

where $\lambda := \max\{|l| \mid l \in \mathrm{dom}(T)\}$. From the proof of Lemma 2.2 we see that this condition is not only necessary but also sufficient for the set $\Delta_T^*(\{u\}^*) \cap \mathrm{IRR}(T)$ to be infinite. Thus, we have the following result.

COROLLARY 3.4: *Let T be a finite Church-Rosser Thue system on Σ such that the monoid $\mathscr{M}_T$ is a group. Then given a word $u \in \Sigma^*$, the order of u modulo T can be determined effectively.*

Finally, we have the following characterization for certain abelian subgroups of groups that are presented by finite Church-Rosser Thue systems.

COROLLARY 3.5: *Let G be a group that can be presented by a finite Church-Rosser Thue system. If S is an abelian subgroup of G such that S contains an element of infinite order, then S is isomorphic to* **Z**.

Proof: Let T be a finite Church-Rosser Thue system on Σ such that $(\Sigma; T)$ is a presentation of G, and let S be an abelian subgroup of G containing an

element u of infinite order. Then S is a subgroup of the centralizer $C_T(u)$ of u in G, which is isomorphic to $\mathbf{Z}$ by Corollary 2.4. Thus, $S \cong \mathbf{Z}$. □

If S is a finitely generated abelian subgroup of G, then S is infinite if and only if it contains an element of infinite order. Thus, Corollary 3.5 implies the following.

COROLLARY 3.6: *Let G be a group that can be presented by a finite Church-Rosser Thue system. Then every finitely generated abelian subgroup of G is either finite or isomorphic to $\mathbf{Z}$.*

Another application of Corollary 2.4 gives the following characterization of groups that can be presented by finite Church-Rosser Thue systems and that have a non-trivial center.

COROLLARY 3.7: *Let G be a group that can be presented by a finite Church-Rosser Thue system. If the center C of G is non-trivial, then G is either finite or isomorphic to $\mathbf{Z}$.*

Proof: Let T be a finite Church-Rosser Thue system on Σ such that $(\Sigma; T)$ presents the group G, and assume that the center of G is non-trivial, i. e.,

$$C = \{ u \in \Sigma^* \mid \forall v \in \Sigma^* : uv \overset{*}{\underset{T}{\leftrightarrow}} vu \} \supseteq [e]_T.$$

If C contains an element of finite order, then by Corollary 3.1 each element of G has finite order, and hence G is finite by Corollary 3.2(a).

Now assume that C contains an element u of infinite order. Then G equals the centralizer $C_T(u)$, which is isomorphic to $\mathbf{Z}$ by Corollary 2.4. Thus, $G \cong \mathbf{Z}$. □

For groups containing a finite normal subgroup we have the following observation.

COROLLARY 3.8: *Let G be a group that can be presented by a finite Church-Rosser Thue system. If G contains a non-trivial normal subgroup that is finite, then G itself is finite.*

Proof: Let T be a finite Church-Rosser Thue system on Σ such that $(\Sigma; T)$ is a presentation of G, and let N be a finite non-trivial normal subgroup of G. Then there are words $u_1, u_2, \ldots, u_n \in \mathrm{IRR}(T) - \{e\}$ such that the set $\{e, u_1, u_2, \ldots, u_n\}$ exactly describes the subgroup N.

Assume that G is infinite. Then there exists an irreducible word $u \in \Sigma^*$ such that $\mathrm{ord}_T(u) = 0$. In particular, $u^m \overset{*}{\not\leftrightarrow}_T e$ for all $m \geqq 1$. The mapping $u_i \to u^{-1} u_i u$ induces an automorphism on N. Thus, there exists an index $m \geqq 1$ such that $u^{-m} u_1 u^m \overset{*}{\leftrightarrow}_T u_1$, i.e., $u_1 u^m \overset{*}{\leftrightarrow}_T u^m u_1$. Since $\mathrm{ord}_T(u_1) \neq 0$ while $\mathrm{ord}_T(u) = \mathrm{ord}_T(u^m) = 0$, this contradicts Corollary 3.1. Thus, G is indeed finite. □

Using this observation we can now prove the following generalization of Diekert's characterization theorem [12].

COROLLARY 3.9: *Let G be a group that can be presented by a finite Church-Rosser Thue system. If G contains a finitely generated nontrivial abelian subgroup that is normal in G, then G is either finite, isomorphic to $\mathbf{Z}$ or isomorphic to the infinite dyhedral group $\mathbf{Z}_2 * \mathbf{Z}_2$.*

Proof: Let N be a finitely generated non-trivial abelian subgroup of G that is normal in G. By Corollary 3.6 N is either finite or infinite cyclic. If N is finite, then G is finite by Corollary 3.8. So let $N = \langle u \rangle_G$ for some element u of G of infinite order. Then the centralizer $C_G(N) = C_G(u)$ is an infinite cyclic subgroup (Corollary 2.4) that is normal in G. Let $C_G(u) = \langle v \rangle_G$ for some element v of G of infinite order. For every element $g \in G$, the mapping $\varphi_g : h \to g^{-1} h g$ induces an automorphism of $C_G(u) = \langle v \rangle_G$, i.e., $g^{-1} v g = v$ or $g^{-1} v g = v^{-1}$. If $g^{-1} v g = v$, i.e., g and v commute, then also g and u commute implying that $g \in C_G(u)$. Analogously, if $g^{-1} v g = v^{-1}$, then $g^2 \in C_G(u)$. Thus, for all $g_1, g_2 \in G$, if $g_1, g_2 \notin C_G(u)$, then $g_1 g_2^{-1} \in C_G(u)$, i.e., $C_G(u) g_1 = C_G(u) g_2$. Hence, the index $|G : C_G(u)|$ of $C_G(u)$ in G is at most two. Now Diekert's result applies giving $G \cong \mathbf{Z}$ or $G \cong \mathbf{Z}_2 * \mathbf{Z}_2$. □

While it is still an open problem to characterize those groups that can be presented by finite Church-Rosser Thue systems, our results can at least serve as criteria to verify that a given group does not have a presentation of this type.

Example 3.10: Let $G = F_2 \times \mathbf{Z}_2$, i.e., G is the direct product of the free group F_2 of rank 2 and the cyclic group $\mathbf{Z}_2$ of order 2. Then G is not finite, but $\mathbf{Z}_2$ is a finite normal subgroup of G. Thus, by Corollary 3.8 G cannot be presented by a finite Church-Rosser Thue system. □

In [16, 17] Jantzen investigates the class of groups $G_n (n \geqq 1)$, where G_n is given through the presentation $(\Sigma; S_n)$,

$$\Sigma = \{ a, b \}, \; S_n = \{ ((ab)^n ba, e) \}.$$

He shows that for no $n \geqq 1$, there is a finite canonical (semi-) Thue system T_n on Σ such that $|u| \geqq |v|$ for all rules $(u, v) \in T_n$, and $\overset{*}{\underset{T_n}{\leftrightarrow}} = \overset{*}{\underset{S_n}{\leftrightarrow}}$. Using our results we can prove that none of these groups can be presented by a finite Church-Rosser Thue system.

Example 3.11: (*a*) The group G_1 is neither finite nor isomorphic to $\mathbf{Z}$. However, its center C is non-trivial, since $e \overset{*}{\underset{S_1}{\leftrightarrow}} a^2 \in C$, as can be seen easily. Thus, Corollary 3.7 gives the intended result.

(*b*) Let $n \geqq 2$. Then G_n contains a subgroup that is isomorphic to $\mathbf{Z}(1/n) := \{ p . n^q \mid p, q \in \mathbf{Z} \}$ [16]. This subgroup is abelian, and it clearly contains an element of infinite order. However, it is not finitely generated, and so is not isomorphic to $\mathbf{Z}$. Thus, Corollary 3.5 applies. □

We conclude this paper with a look at Greendlinger's group G presented by $(\{ a, b, c, \bar{a}, \bar{b}, \bar{c} \}; \{ (a\bar{a}, e), (\bar{a}a, e), (b\bar{b}, e), (\bar{b}b, e), (c\bar{c}, e), (\bar{c}c, e), (abc, cba) \})$ [14].

Example 3.12: The elements presented by ab and by ca commute in G. However, the subgroup $\langle ab, ca \rangle_G$ generated by them is free abelian of rank 2 [14]. Thus, Corollary 3.6 implies that the Greendlinger group G cannot be presented by a finite Church-Rosser Thue system. □

This last example answers a question raised in [21].

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