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A semigroup characterization of dot-depth one languages


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A SEMIGROUP CHARACTERIZATION
OF DOT-DEPTH ONE LANGUAGES (*)

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Communicated by J.-F. Perrot

1. INTRODUCTION

Let $A$ be a non-empty finite set, called alphabet. $A^+$ (respectively $A^*$) is the free semigroup (respectively free monoid) generated by $A$. Elements of $A^*$ are called words. The empty word in $A^*$ is denoted by $\lambda$ (the identity of $A^*$). The concatenation of two words $x, y$ is denoted by $xy$. The length of a word $x$ is denoted by $|x|$.

Any subset of $A^*$ is called a language. If $L_1$ and $L_2$ are languages, then $L_1 \cup L_2$ is their union, $L_1 \cap L_2$ is their intersection, and $\overline{L}_1 = A^* - L_1$ is the complement of $L_1$ with respect to $A^*$. Also $L_1L_2 = \{ w \in A^* | w = xy, x \in L_1, y \in L_2 \}$ is the concatenation of $L_1$ and $L_2$.

Let $\sim$ be an equivalence relation on $A^*$. For $x \in A^*$ we denote by $[x]_{\sim}$ the equivalence class of $\sim$ containing $x$. An equivalence relation $\sim$ on $A^*$ is a congruence if for all $x, y \in A^*$, $x \sim y$ implies $uxv \sim uyv$ for any $u, v \in A^*$.

The syntactic congruence of a language $L$ is defined as follows: for $x, y \in A^*$, $x \equiv_L y$ if for all $u, v \in A^*(uxv \in L \text{ iff } uyv \in L)$. The syntactic semigroup of $L$ is the quotient semigroup $A^+/\equiv_L$.

Let $\eta$ be any family of languages. Then $\eta M (\eta B)$ will denote the smallest family of languages containing $\eta$ and closed under concatenation (finite union and complementation respectively).

(*) Received February 1981, revised May 1983.
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R.A.I.R.O. Informatique théorique/Theoretical Informatics, 0399-0540/1983/321/8 5.00
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Let \( \mathcal{L} = \{ \{ \lambda \}, \{ a \} ; a \in A \} \) be the family of elementary languages. Then define:
\[
\mathcal{B}_0 = \varepsilon B, \\
\mathcal{B}_k = \mathcal{B}_{k-1} MB \quad \text{for} \quad k \geq 1.
\]
This sequence \((\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_k, \ldots)\) is called the dot-depth hierarchy. A language \( L \) is of dot-depth at most \( k \) if \( L \in \mathcal{B}_k \).

The dot-depth hierarchy was introduced in [3]. It was proved in [2] that it is infinite if the alphabet has two or more letters. In [4] it was shown that \((\mathcal{B}_0, \mathcal{B}_1, \ldots)\) forms a hierarchy of \(+−\) varieties of languages. Therefore, in the rest of the paper we consider languages as subsets of \( A^* \). For an excellent and general presentation of problems related to this paper the reader is referred to Brzozowski's survey paper [1] or the above mentioned monograph of Eilenberg [4].

In [6] Simon conjectured that a language \( L \) is in \( \mathcal{B}_1 \) iff its syntactic semigroup \( S_L \) is finite and there exists an integer \( n > 0 \) such that for each idempotent \( e \) in \( S_L \), and any elements \( a, b \in S_L \):
\[
(eaeb)^n eae = (eaeb)^n e = ebe(aebe)^n.
\]
Simon also proved that \( L \in \mathcal{B}_1 \) implies this condition. By an example we show that this conjecture fails. We present a necessary and sufficient condition for a syntactic semigroup to be the syntactic semigroup of a language of dot-depth at most one. The main result is as follows: Let \( L \) be a language and let \( S_L \) be its syntactic semigroup. Then \( L \in \mathcal{B}_1 \) iff \( S_L \) is finite and there exists an integer \( n > 0 \) such that for all idempotents \( e_1, e_2 \) in \( S_L \) and any elements \( a, b, c, d \in S_L \):
\[
(e_1 ae_2 b)^n e_1 ae_2 de_1 (ce_2 de_1)^n = (e_1 ae_2 b)^n e_1 (ce_2 de_1)^n.
\]

We will refer to this as the "dot-depth one" condition. This semigroup characterization gives a decision procedure for testing whether or not a regular language is in \( \mathcal{B}_1 \).

In the proof of this characterization we use a theorem on graphs from [5].

We will say that a language \( L \subset A^* \) is a \( \sim \) language, if \( L \) is a union of congruence classes of \( \sim \). Let \( L \) be a language and let \( S_L \) be its syntactic semigroup. The class \([x] \equiv_L \), as an element of \( S_L \), will be also denoted by \( \underline{x} \), where \( x \in A^+ \). Then \( x \equiv_L y \) iff \( \underline{x} = \underline{y} \) in \( S_L \).

2. BASIC CONGRUENCE \(_m\sim \_k \) [6]

Let \( k, m \) be integers, \( k \geq 1, m \geq 0 \). Let \( \nu = (w_1, w_2, \ldots, w_m) \) be an \( m \)-tuple of words \( w_i \) of length \( k \), i.e. \( |w_i| = k, w_i \in A^* i = 1, 2, \ldots, m \). We say that \( \nu \) occurs in...
\( x, x \in A^* \) (we write \( v \in x \)), if \( x = u_iw_iv_i \), for some \( u_i, v_i \in A^*(i = 1, 2, \ldots, m) \) such that \( |u_j| < |u_{j+1}|, j = 1, 2, \ldots, m-1 \).

Let us set:

\[
\tau_{m, k}(x) = \{ v \mid v \in (A^k)^m \text{ and } v \subseteq x \}.
\]

By convention \( \tau_{0, k}x = \emptyset \).

For \( x \in A^* \) and \( n \geq 0 \) define \( f_n(x) \) as follows: if \( |x| \leq n \), then \( f_n(x) = x \); otherwise \( f_n(x) \) is the prefix of \( x \) of length \( n \). Similarly, \( t_n(x) = x \) if \( |x| \leq n \), and \( t_n(x) \) is the suffix of length \( n \) of \( x \) otherwise.

Now, for \( x, y \in A^* \) and \( k \geq 0, m \geq 0 \) we define:

\[
x_m \sim_k y \text{ iff } x = y \text{ if } |x| \leq m + k - 1 \text{ and } f_k(x) = f_k(y), t_k(x) = t_k(y) \text{ otherwise.}
\]

In the case \( k = 0 \) we write \( \tau_m \) instead \( \tau_{m, 0} \) and \( m \sim \) instead \( m \sim_0 \). If \( m = 1 \), we also write \( \tau \) instead \( \tau_1 \).

**Proposition 1:**

(a) \( m \sim_k \) is a congruence of finite index on \( A^* \); (b) \( x_m \sim_k y \) implies \( x_{m-1} \sim_k y \), for \( m \geq 1 \) and all \( x, y \in A^* \); (c) \( w(xw)_m \sim_k w(xw)_{m+1} \), for \( w, x \in A^* \) and \( |w| = k \); (d) \( (w_1xw_2y)_m \sim_k (w_1xw_2y)_m \), for \( w_1, w_2, x, y, u, v \in A^* \) and \( |w_1| = |w_2| = k \).

**Proof:** The verification of (a), (b) and (c) is straightforward.

(d) By (b):

\[
\tau_{m, k+1}(x) = \tau_{m, k+1}(y)
\]

implies:

\[
\tau_{j, k+1}(x) = \tau_{j, k+1}(y),
\]

for all \( x, y \in A^* \) and \( j \in \{0, 1, \ldots, m\} \). If

\[
v_1 = (w_1, \ldots, w_i) \in (A^{k+1})^i
\]

and

\[
v_2 = (v_1, \ldots, v_j) \in (A^{k+1})^j,
\]

we denote by \( (v_1, v_2) \) the \( i+j\)-tuple \( (w_1, \ldots, w_i, v_1, \ldots, v_j) \in (A^{k+1})^{i+j} \).

Evidently:

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Using (c), we have:

\[ \tau_{m, k+1}((w_1 \times w_2 y)^m w_1 x w_2) \subseteq \tau_{m, k+1}((w_1 \times w_2 y)^w_1 x w_2) \]

\[ \equiv \tau_{m, k+1}((w_1 \times w_2 y)^{m+1} w_1). \]

Similarly:

\[ \tau_{m, k+1}((w_1 \times w_2 y)^m w_1 x w_2) = \tau_{m, k+1}((w_1 \times w_2 y)^m w_1). \]

Since \(|w_1| = |w_2| = k\), by the above conclusions from (b) and (c):

\[ \tau_{m, k+1}((w_1 \times w_2 y)^m w_1 x w_2 v w_1 (u w_2 v w_1)^m) = \bigcup_{i+j=m} \{ (v_1, v_2) \mid v_1 \in \tau_{i, k+1}((w_1 \times w_2 y)^m w_1 x w_2), v_2 \in \tau_{j, k+1}(w_1 v w_1 (u w_2 v w_1)^m) \} \]

\[ = \bigcup_{i+j=m} \{ (v_1, v_2) \mid v_1 \in \tau_{i, k+1}((w_1 \times w_2 y)^m w_1), v_2 \in \tau_{j, k+1}(w_1 v w_1 (u w_2 v w_1)^m) \} \]

\[ = \tau_{m, k+1}((w_1 \times w_2 y)^m w_1 (u w_2 v w_1)^m). \]

**Theorem 2** (Simon [6]): A language \( L \) is of dot-depth at most one, \( L \in \mathcal{B}_1 \), if and only if \( L \) is an \( m \sim_k \) language for some \( m, k \geq 0 \).

### 3. Graphs and the Induced Syntactic Graph Congruence

First we briefly recall Eilenberg's terminology for graphs [4]. A directed graph \( G \) consists of two sets, an alphabet \( A \) and a set of vertices \( V \), along with two functions: \( \alpha, \omega : A \to V \). Elements of \( A \) are also called edges in this case.

Two letters (or edges) \( a, b \in A \) are called consecutive if \( \alpha(a) = \omega(b) \). Let \( D \subset A^2 \) be the set of all words \( ab \) such that \( a \) and \( b \) are non-consecutive. Then the set of all paths of \( G \) is:

\[ P = A^+ - A^* DA^*. \]

Functions \( \alpha, \omega \) can be extended to \( \alpha, \omega : P \to V \) in the following way: if \( p = a_1 a_2 \ldots a_n \in P, a_1, a_2, \ldots, a_n \in A \), then \( \alpha(a_1, a_2, \ldots, a_n) = a_n \omega \). For each vertex \( v \) we adjoint to \( P \) a trivial path \( 1_v \), where \( 1_v \alpha = 1_v, \omega = v \). If \( p = a_1 a_2 \ldots a_n \in P \), then the length of \( p \), \(|p| = n \).

A path \( p \) is called a loop if \( p \alpha = p \omega \). We say that two paths \( p_1 \) and \( p_2 \) are consecutive if \( p_1 \omega = p_2 \alpha \). In this case the concatenation \( p_1 p_2 \) is again a path. Two paths \( p_1 \) and \( p_2 \) are coterminal if \( p_1 \alpha = p_2 \alpha \) and \( p_1 \omega = p_2 \omega \).

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An equivalence relation $\sim$ on $P$ is called a graph congruence if it satisfies the following conditions:

(i) if $p_1 \sim p_2$, then $p_1$ and $p_2$ are coterminel; 
(ii) if $p_1 \sim p_2$ and $p_3 \sim p_4$ and $p_1, p_3$ are consecutive, then $p_1 p_3 \sim p_2 p_4$.

For trivial paths, by convention we set $\tau_m(1_v) = \emptyset$. Thus the relation $\sim (\sim_1)$ is also defined on $P$. In [5] the following theorem is proved:

**Theorem 3:** Let $\sim$ be a graph congruence of finite index on $P$ satisfying the condition:

$$(p_1 p_2)^n p_1 p_4 p_3 p_4)^n \sim (p_1 p_2)^n (p_3 p_4)^n,$$

for some $n \geq 1$ and $p_1, p_2, p_3, p_4 \in P$. (Note that $p_1, p_2$ and $p_3, p_4$ must be loops about the same vertex).

Then there exists an integer $m \geq 1$ such that for any two coterminel paths $x$ and $y$, $x \sim y$ implies $x \sim y$.

We will use this theorem in proving the semigrouè characterization of languages of dot-depth at most one ($\mathcal{B}_1$).

Let $A$ be a finite alphabet. Define a graph $G_k = (V, E, \alpha, \omega)$ for $k \geq 0$ as follows:

- $V = \{ w \mid w \in A^* \text{ and } |w| = k \}$ is the set of vertices,
- $E = \{ (w_1, \sigma, w_2) \mid \sigma \in A, w_1, w_2 \in V \text{ and } t_k(w_1, \sigma) = w_2 \}$, is the set of edges (letters)
- $\alpha, \omega : E \to V, (w_1, \sigma, w_2) \alpha = w_1, (w_1, \sigma, w_2) \omega = w_2$.

Let $P$ be the set of all paths in $G_k$, including the empty path over each vertex from $V$. Now, let us define the mapping:

$$: A^k A^* \to P,$$

recursively as follows:

- $\overline{x} = 1_x$ if $x \in A^k$,
- $\overline{x} \sigma = \overline{x(t_k(x), \sigma, t_k(x \sigma))}$.

For $k = 0$, by convention $A^0 = \{ \lambda \}$. One can verify that the mapping $\overline{\cdot}$ is bijective. It follows from the definition that $|x| = k + h$, $h \geq 0$ iff $|\overline{x}| = h$.

If $\rho$ is a congruence relation on $A^*$, then by $\overline{\rho}$ we will denote the induced congruence on $P$ defined in the following way: for $\overline{x}, \overline{y} \in P, x, y \in A^k A^*$, $x \rho y$ if $x, y$ are coterminel paths and $x \rho y$. One can verify that $\overline{\rho}$ is a graph congruence on $P$. 

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PROPOSITION 4: Let $G_k$ be a graph for $k \geq 1$ and $P$ be the set of all paths of $G_k$. Let $x \in A^k A^*$. If $x = x_1 x_2$, then $\bar{x} = \bar{x}_1 t_k(x_1) \bar{x}_2$, for $|x_1| \geq k$.

Proof: If $|x| = k$, then the only decomposition possible is $x = x\lambda$. But $\bar{x} = 1_x = 1_x 1_x = \bar{x} x \lambda = \bar{x} t_k(x) \lambda$. Induction assumption: the proposition is true for $x$ such that $|x| = k + h$, $h \geq 0$. Suppose $x = x_1 x_2 \sigma$, where $|x_1 x_2| = k + h$ and $|x_1| \geq k$. By definition:

$$\bar{x} = \bar{x}_1 x_2 (t_k(x_1 x_2), \sigma, t_k(x_1 x_2 \sigma))$$

By the induction assumption:

$$x_1 x_2 = x_1 t_k(x_1) x_2.$$

Hence:

$$\bar{x} = \bar{x}_1 t_k(x_1) x_2 (t_k(x_1 x_2), \sigma, t_k(x_1 x_2 \sigma)).$$

Again by definition:

$$t_k(x_1) x_2 \sigma = t_k(x_1) x_2 (t_k(x_1) x_2, \sigma, t_k(x_1 x_2 \sigma)).$$

Thus $\bar{x} = \bar{x}_1 t_k(x_1) x_2 \sigma$ because $t_k(x_1 x_2) = t_k(t_k(x_1) x_2)$. Thus the induction step holds. 

LEMMA 5: Let $x \in A^k A^*$ and $\bar{x} = a_1 a_2 \ldots a_n$, $a_j \in E$, $j = 1, 2, \ldots, n$. Then for $i \in \{1, 2, \ldots, n\}$ $a_i = (w, \sigma, t_k(w \sigma))$ iff $x = x_1 w \sigma x_2$ for some $x_1$, $x_2 \in A^*$ and $|x_1 w \sigma| = k + i$.

Proof: Suppose $f_{k+i}(x) = x_1 w \sigma$. By Proposition 3 $\bar{x} = \bar{x}_1 w \sigma \bar{x}_2$. By the definition of it follows from Proposition 3 that $\bar{x}_1 w \sigma \bar{x}_2 = (w, \sigma, t_k(w \sigma)) t_k(w \sigma) x_2$. Also by the definition of $|x_1 w| = k + i - 1$, because $|x_1 w| = k + i - 1$. Hence $a_i = (w, \sigma, t_k(w \sigma))$.

The converse follows in the similar way. 

PROPOSITION 6: For any $x$, $y \in A^k A^*$:

$$x_m \sim_k y \implies \bar{x}_m \sim \bar{y},$$

where $\bar{x}$, $\bar{y} \in P$ of $G_k$.

Proof: If $|x| \leq m + k$, then $x = y$ and consequently, $\bar{x}_m \sim \bar{y}$. Otherwise, let $\tau_{m, k+1}(x) = \tau_{m, k+1}(y) \neq \emptyset$. It follows from Lemma 5 that $(w_1, \sigma_1, v_1), \ldots, (w_m, \sigma_m, v_m) \in \tau_m(x)$ implies $(w_1 \sigma_1, \ldots, w_m \sigma_m) \in \tau_{m, k+1}(x) = \tau_{m, k+1}(y)$. Hen-
ce, again by Lemma 4 \(((w_1, \sigma_1, v_1), \ldots, (w_m, \sigma_m, v_m)) \in \tau_m(y)\). Thus, \(\tau_m(x) \subseteq \tau_m(y)\). By symmetry, \(\tau_m(y) \subseteq \tau_m(x)\).

Since \(f_k(x) = f_k(y)\) and \(t_k(x) = t_k(y)\), then \(x\) and \(y\) are coterminally.

Consequently, \(x \sim y\). \(\square\)

**PROPOSITION 7:** Let \(L \subseteq A^+\) and let \(S_L\) be the finite syntactic semigroup of \(L\), satisfying the condition: there exists \(m, m > 0\), such that for all idempotents \(e_1, e_2\) in \(S_L\) and any elements \(a, b, c, d \in S_L\):

\[(e_1 a e_2 b)^m e_1 a e_2 d (c e_2 d e_1)^m = (e_1 a e_2 b)^m e_1 (c e_2 d e_1)^m.\]

Then the congruence \(\equiv_L\) on \(P\) of \(G_k\) for \(k = \text{card } S_L + 1\), induced by the syntactic congruence \(\equiv_L\) satisfies condition \((\text{A})\) of Theorem 2 and is of finite index on \(P\).

**Proof:** Since \(G_k\) is finite and \(\equiv_L\) is of finite index on \(A^+\), then \(\equiv_L\) is of finite index on \(P\).

We have to show that there is an integer \(n, n > 0\) such that:

\[\left(\begin{array}{c}
(p_1 p_2)^n p_1 p_4 (p_3 p_4)^n \\
\equiv_L (p_1 p_2)^n (p_3 p_4)^n
\end{array}\right),\]

for \(p_1, p_2, p_3, p_4 \in P\).

Since \(p_1 p_2\) and \(p_3 p_4\) are loops about the same vertex and since paths \(p_1\) and \(p_4\) are consecutive by \((\text{A})\), then \(p_1 \alpha = p_2 \omega = p_3 \alpha = p_4 \omega = w,\) and \(p_1 \omega = p_2 \alpha = p_3 \omega = p_4 \alpha = v\) for some \(w, v \in A^k\). Therefore we may assume that \(p_1 = wu_1, p_2 = vu_2, p_3 = wu_3, p_4 = vu_4\) for some \(u_1, u_2, u_3, u_4 \in A^*\) such that \(t_k(wu_1) = t_k(wu_3) = v, t_k(vu_2) = t_k(vu_4) = w\). Consequently:

\[(p_1 p_2)^n p_1 p_4 (p_3 p_4)^n = w(u_1 u_2)^n u_1 u_4 (u_3 u_4)^n.\]

Similarly:

\[(p_1 p_2)^n (p_3 p_4)^n = w(u_1 u_2)^n u_3 u_4.\]

By the definition of \(\equiv_L\) it is sufficient to show that there exists \(n, n > 0\), such that:

\[w(u_1 u_2)^n u_1 u_4 (u_3 u_4)^n = w(u_1 u_2)^n u_3 u_4,\]

i.e.:

\[(1)\]

\[w(u_1 u_2)^n u_1 u_4 (u_3 u_4)^n = w(u_1 u_2)^n u_3 u_4.\]

Let \(s \in S_L\). Since \(S_L\) is finite, then \(s^r\) is an idempotent for some \(r \geq 1\) ([4], vol. 17, n° 4, 1983)
Proposition 4.2, p. 68). Now, since $S_L$ satisfies the dot-depth one condition, there is $m \geq 1$ such that:

$$s^r (ss^r)^m = s^r (ss^r)^{m+1}$$

i.e. $s^r s^m = s^r s^m s$. It follows that there exists an integer $q$ such that for any $s \in S_L$

$$s^q = s^{q+1}$$

i.e. $S_L$ is aperiodic.

We claim that (1) holds for $n > m, q$. First we will show that if $|u_1 u_2| > 0$ ($|u_3 u_4| > 0$) then we may consider $u_1, u_2 (u_3, u_4$ respectively) such that $|u_1|, |u_2| \geq k$ ($|u_3|, |u_4| > k$ respectively). Since $n > q$, then by the aperiodicity of $S_L$:

$$w(u_1 u_2)^n = w(u_1 u_2)^{n(2k+1)}$$

Let us define:

$$\tilde{u}_1 = (u_1 u_2)^k u_1, \tilde{u}_2 = (u_1 u_2)^k$$

Evidently:

$$|\tilde{u}_1|, |\tilde{u}_2| \geq k, \quad t_k(w \tilde{u}_1) = v, \quad t_k(v \tilde{u}_2) = w$$

and:

$$w(u_1 u_2) = w(\tilde{u}_1 \tilde{u}_2)^n.$$

Similarly, we may proceed for $u_3$ and $u_4$.

Now, we consider the full case if $|u_1 u_2|, |u_3 u_4| > 0$. The other cases if $|u_1 u_2| = 0$ or $|u_3 u_4| = 0$ follow in the same way. By the above, instead of proving (1) it is sufficient to show that:

(2) $$w(u_1 v u_2 w)^n u_1 v u_3 w = w(u_1 v u_2 w)^n (u_3 v u_4 w)^n,$$

holds.

Now, since $|w| = |v| = k > \text{card } S_L + 1$, then $w = w_1 w_2 w_3$ and $v = v_1 v_2 v_3$ for $w_1, w_3, v_1, v_3 \in A, w_2, v_2 \in A^+$ such that $w_i = w_i^j w_i^j, v_i = v_i^j v_i^j$ for any $i \geq 0$. So as before, we can choose $i$ such that $w_i^j$ and $v_i^j$ are idempotents in $S_L$. Thus (2) can be rewritten in a form:

$$w_1 e_1 (ae_1 b e_1)^n a e_2 d e_1 (c e_2 d e_1)^n w_3 = w_1 e_1 (ae_2 b e_1)^n (c e_2 d e_1)^n w_3,$$

where:

$$e_1 = w_2^i, \quad e_2 = v_2^j, \quad a = w_3^i u_1,\quad v_1,$$

$$b = v_3^i u_2 w_1, \quad c = w_3 v_3^i v_1$$

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and \( d = v_3 u_4 w_1 \). Thus by the dot-depth one condition, (2) holds. \( \square \)

4. SEMIGROUP CHARACTERIZATION OF \( \mathfrak{A} \):

Now we are in a position to prove our main result.

**Theorem 8**: Let \( L \) be a language, \( L \subseteq A^+ \) and let \( S_L \) be its syntactic semigroup. Then the following are equivalent:

(i) \( L \in \mathfrak{A}_1 \);

(ii) \( L \) is a \( m \sim_k \) language for some \( m, k \geq 1 \);

(iii) \( S_L \) is finite and there is an integer \( n > 0 \) such that for all idempotents \( e_1, e_2 \) in \( S_L \) and any elements \( a, b, c, d \) in \( S_L \):

\[
(\text{e}_1 \text{a}_2 \text{b})^n \text{e}_1 \text{a}_2 \text{d}_1 (\text{c}_2 \text{d}_1)^n = (\text{e}_1 \text{a}_2 \text{b})^n \text{e}_1 (\text{c}_2 \text{d}_1)^n.
\]

**Proof**: (i) \( \iff \) (ii) by Theorem 2;

(ii) \( \Rightarrow \) (iii) : by (a) of Proposition 1 \( S_L \) is finite.

Now, let \( e_1 = z_1, e_2 = z_2, a = x, b = y, c = u, d = v \) for some \( z_1, z_2, x, y, u, v \in A^+ \).

Define \( w_1 = z_1^h, w_2 = z_2^h \) for \( h \) such that \( |w_1|, |w_2| \geq k \). Consequently, \( e_1 = w_1 \), \( e_2 = w_2 \). By (d) of Proposition 1 for \( m \sim_k \):

\[
(w_1 x w_2 y)^m w_1 x w_2 w_1 (u w_2 v w_1)^m = (w_1 x w_2 y)^m w_1 (u w_2 v w_1)^m.
\]

Thus \( S_L \) satisfies the dot-depth one condition with \( n = m \).

(iii) \( \Rightarrow \) (ii): suppose \( S_L \) satisfies the dot-depth one condition with \( n \). Let \( k = \text{card} S + 1 \). By Proposition 7 the induced syntactic congruence \( \equiv_L \) on \( P \) of \( G_k \), satisfies the condition (A) of the theorem on graphs with some \( n_1 > n, q \), and is of finite index on \( P \). Hence by Theorem 3 there exists \( m \) such that for any two coterminall paths \( x, y \):

\[
\overline{x} \sim^L \overline{y} \implies \overline{x} \equiv_L \overline{y}.
\]

Now, consider \( x, y \in A^k A^* \), and the congruence \( m \sim_k \). We have that \( x \sim_k y \) implies \( x \sim^L \overline{y} \) and that \( x, \overline{y} \) are coterminall. Hence, \( x \sim_k y \) implies \( \overline{x} \equiv_L \overline{y} \) and consequently, \( x \equiv_L y \). If \( |x| \leq k \), then \( x \sim_k y \) implies \( x = y \) and consequently, \( x \equiv_L y \). Thus \( L \) is a \( m \sim_k \) language. \( \square \)

It is easy to see that if a syntactic semigroup satisfies the dot-depth one condition, then it also satisfies the condition: there exists an integer \( n > 0 \) such that for any idempotent \( e \) in \( S_L \) and any elements \( a, b, c, d \),

\[
(eaeb)^n eae = (eaeb)^n e = ebe(aebe)^n.
\]
The following example shows that the converse is not true.

Let $A = \{0, 1, 2, 3\}$ and let $L = (01^+ \cup 02^+)01^+ 3(2^+ 3 \cup 1^+ 3)^*$. The syntactic semigroups $S_L$ of $L$ satisfies the above condition, but it fails the dot-depth one condition. By Theorem 8 $L \notin \mathcal{B}_1$. On the other hand one can verify that $L \notin \mathcal{B}_1$, apart from Theorem 8, using (d) of Proposition 1 and proving that for any $m, k L$ cannot be a $m \sim_k$ language.

REFERENCES


