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RELATIONAL MORPHISMS AND OPERATIONS ON RECOGNIZABLE SETS (*)

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Communiqué par J.-F. Perrot

Abstract. — Relational morphisms between finite monoids (a notion due to Tilson) are used to study the effect certain operations on recognizable sets have on the syntactic monoids of those sets. This leads to concise proofs of a number of known results concerning the product operation, and a new result concerning the star operation.

Résumé. — On utilise les morphismes relationnels (dus à Tilson) pour étudier l’effet que certaines opérations sur les langages reconnaissables produisent sur les monoïdes syntactiques de ces langages. On obtient ainsi des démonstrations simples pour plusieurs résultats déjà connus sur l’opération de produit et un résultat nouveau sur l’opération étoile.

1. INTRODUCTION

Some recent research in the theory of automata has been devoted to describing the effect various operations on recognizable sets have on the syntactic monoids of the sets involved. A particularly simple example of such a description (this one treating the operation of intersection) is the following: If \( \Sigma \) is a finite alphabet and \( A \) and \( B \) are recognizable subsets of \( \Sigma^* \) (the free monoid on \( \Sigma \)), then \( M(A \cap B) \trianglelefteq M(A) \times M(B) \) [Here \( M(X) \) denotes the syntactic monoid of \( X \), and \( M_1 \trianglelefteq M_2 \) means \( M_1 \) divides \( M_2 \) — that is, \( M_1 \) is a quotient of a submonoid of \( M_2 \).] (See Eilenberg [1] for a detailed explanation of the terminology of this paper.) More complex examples treat the product operation (Schützenberger [8]), unambiguous product (Schützenberger [9]), the shuffle product (Perrot [2]), \( n \)-fold products (Straubing [11]), and images under length-preserving morphisms from one free monoid to another (Reutenauer [7], Straubing [12], Pin [5]).

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This paper presents a general method for studying such questions. The method uses relational morphisms between finite monoids, a concept introduced by Tilson [13].

Relational morphisms are discussed in section 2. The method is applied in section 3 to give new, brief proofs of a number of known results concerning the product operation. In section 4 I use it to prove a new result, which concerns the operation $A \to A^*$ in the case that $A^*$ is a pure submonoid of $\Sigma^*$.

2. RELATIONAL MORPHISMS

Let $M_1$ and $M_2$ be finite monoids. A relation $\rho : M_1 \to M_2$ is a map from $M_1$ into $\mathcal{P}(M_2)$ (the power set of $M_2$). If $m \in M_1$ then $m \rho$ denotes the image of $m$ under this map. The graph of $\rho$, denoted $\# \rho$, is the set $\{(m_1, m_2) \in M_1 \times M_2 | m_2 \in m_1 \rho\}$. The inverse of $\rho$, denoted $\rho^{-1}$, is the unique relation $\eta : M_1 \to M_2$ such that $\# \eta = \{(m_2, m_1) \in M_2 \times M_1 | (m_1, m_2) \in \# \rho\}$. The domain of $\rho$, denoted $\text{dom} \rho$, is the set $\{m \in M_1 | m \rho \neq \emptyset\}$.

A relation $\rho : M_1 \to M_2$ is said to be a relational morphism if the following two conditions hold:

(i) $\# \rho$ is a submonoid of $M_1 \times M_2$;
(ii) $\text{dom} \rho = M_1$.

Condition (i) is equivalent to:

(i)' $1 \in 1 \rho$ and for all $m, m' \in M_1, (m \rho)(m' \rho) \subseteq (mm') \rho$. [Here $(m \rho)(m' \rho)$ is the usual product of subsets of $M_2 : (m \rho)(m' \rho) = \{st \in M_2 | s \in m \rho, t \in m' \rho\}\}.

An ordinary morphism $\varphi : M_1 \to M_2$ is just a relational morphism that is also a function from $M_1$ into $M_2$. Such a morphism will sometimes be called a functional morphism for clarity.

A relation $\rho : M_1 \to M_2$ is said to be surjective if $\bigcup_{m \in M_1} m \rho = M_2$ and injective if $m \rho \cap m' \rho = \emptyset$ for any pair of distinct elements $m$ and $m'$ of $M_1$. If $\rho : M_1 \to M_2$ is a surjective relational morphism, then $\rho^{-1} : M_2 \to M_1$ is a relational morphism.

If $\rho : M_1 \to M_2$ and $\eta : M_2 \to M_3$ are relations, then $\rho \eta : M_1 \to M_3$ is the relation defined by $m(\rho \eta) = \bigcup_{m' \in mp} m'$ for all $m \in M_1$. It is easy to check that $(\rho \eta)^{-1} = \eta^{-1} \rho^{-1}$, and that if $\rho$ and $\eta$ are relational morphisms, then $\rho \eta$ is a relational morphism.

To each relational morphism $\rho : M_1 \to M_2$ there is associated a functional morphism $\overline{\rho} : \# \rho \to M_2$ defined by $(m_1, m_2) \overline{\rho} = m_2$ for all $(m_1, m_2) \in \# \rho$. $M_1$ itself is the image of $\# \rho$ under the functional morphism $\pi : \# \rho \to M_1$ defined by $(m_1, m_2) \pi = m_1$ for all $(m_1, m_2) \in \# \rho$. Observe that $\rho = \pi^{-1} \overline{\rho}$. 

R.A.I.R.O. Informatique théorique/Theoretical Informatics
Let $\mathcal{V}$ be a collection of finite semigroups. A functional morphism $\psi : M_1 \to M_2$ will be called a functional $\mathcal{V}$-morphism if for each idempotent $e \in M_2$, the semigroup $e \psi^{-1}$ is a member of $\mathcal{V}$. Similarly, a relational morphism $\rho : M_1 \to M_2$ will be called a relational $\mathcal{V}$-morphism if for each idempotent $e \in M_2$, the semigroup $e \rho^{-1}$ is a member of $\mathcal{V}$. The collection $\mathcal{V}$ is said to be an $S$-variety if $\mathcal{V}$ is closed under division and finite direct products. (Similarly, a collection $\mathcal{V}$ of finite monoids closed under division and finite direct products is called an $M$-variety.)

**Lemma 1:** Let $\mathcal{V}$ be an $S$-variety:

(a) if $\rho : M_1 \to M_2$ is a relational $\mathcal{V}$-morphism, then $\bar{\rho} : \#\rho \to M_2$ is a functional $\mathcal{V}$-morphism;

(b) if $\psi : M \to M_2$ is a functional $\mathcal{V}$-morphism, and $M \leq M$, then there is a relational $\mathcal{V}$-morphism $\rho : M_1 \to M_2$.

**Proof:** (a) Let $e \in M_2$ be idempotent. Then $e \bar{\rho}^{-1} = \#\rho \cap (M_1 \times \{e\})$. The projection $\pi : \#\rho \to M_1$ is injective when restricted to $e \bar{\rho}^{-1}$, so $e \rho^{-1} = (e \bar{\rho}^{-1})\pi$ is isomorphic to $e \bar{\rho}^{-1}$. Since $e \rho^{-1}$ is, by assumption, a member of $\mathcal{V}$, it follows that $e \bar{\rho}^{-1} \in \mathcal{V}$. Thus $\bar{\rho}$ is a functional $\mathcal{V}$-morphism. (b) Let $e \in M_2$ be idempotent. By assumption, $e \psi^{-1} \in \mathcal{V}$. Since $M_1 \leq M$, there is a submonoid $M'$ of $M$ and a surjective functional morphism $\varphi : M' \to M_1$. Let $\rho = \varphi^{-1} \psi$. Then $e \rho^{-1} = (e \psi^{-1}) \varphi$. Now $(e \psi^{-1}) \varphi < e \psi^{-1}$, since $(e \psi^{-1}) \varphi$ is the image of $e \psi^{-1} \cap M'$ under the functional morphism $\varphi$. Since $e \psi^{-1} \in \mathcal{V}$, it follows that $e \rho^{-1} \in \mathcal{V}$. Thus $\rho$ is a relational $\mathcal{V}$-morphism. □

In this paper I will be concerned with $\mathcal{V}$-morphisms for two particular choices of the $S$-variety $\mathcal{V}$.

The variety $Ap$ of aperiodic semigroups consists of all finite semigroups which contain no nontrivial groups. Equivalently, $S \in Ap$ if and only if for each $s \in S$, $s^n = s^{n+1}$ for all sufficiently large positive integers $n$. Relational $Ap$-morphisms and functional $Ap$-morphisms will be called aperiodic relational morphisms and aperiodic functional morphisms, respectively.

The variety $\mathcal{D}$ of generalized-definite semigroups consists of all finite aperiodic semigroups all of whose idempotents lie in the unique minimal ideal. Equivalently, $S \in \mathcal{D}$ if and only if for all sufficiently large positive integers $n$,

$$s, r_1, \ldots, r_m, t_1, \ldots, t_n \in S$$

implies:

$$r_1 \ldots r_m s t_1 \ldots t_n = r_1 \ldots r_m t_1 \ldots t_n.$$
Relational $D$-morphisms and functional $D$-morphisms will be called generalized-definite relational morphisms and generalized-definite functional morphisms, respectively.

3. THE PRODUCT OPERATION

Let $A$ and $B$ be subsets of $\Sigma^*$, the free monoid generated by a finite alphabet $\Sigma$. The product $AB$ is defined by:

$$AB = \{ uv \in \Sigma^* | u \in A, v \in B \}.$$

Let $M(A)$, $M(B)$ and $M(AB)$ denote the syntactic monoids of $A$, $B$ and $AB$ respectively.

Theorem 2: There is an aperiodic relational morphism:

$$\rho : M(AB) \to M(A) \times M(B).$$

The proof will be given shortly. Theorem 2 is due, in a somewhat different form, to Schützenberger [8]. He showed, given two finite monoids $M_1$ and $M_2$, how to construct a finite monoid $M_1 \diamond M_2$ (the Schützenberger product of $M_1$ and $M_2$) with the following property: If $A$ and $B$ are recognizable subsets of $\Sigma^*$ then:

$$M(AB) \leq M(A) \diamond M(B).$$

As it turns out, there is an aperiodic functional morphism from $M_1 \diamond M_2$ onto $M_1 \times M_2$. Theorem 2 now follows from these facts and lemma 1 (b).

The proof of theorem 2 which I give below avoids the construction of $M_1 \diamond M_2$ altogether. Recall that the syntactic monoid $M(A)$ of a subset $A$ of $\Sigma^*$ is the quotient of $\Sigma^*$ by the congruence $\sim$, where $u_1 \sim u_2$ if and only if:

$$u_1 A = u_2 A \quad \text{for all} \quad u, v \in \Sigma^*.$$ 

Define a relational morphism:

$$\varphi : M(AB) \to M(A) \times M(B)$$

by $w \varphi = (w \eta_A, w \eta_B)$ for all $w \in \Sigma^*$. (Put otherwise, $\varphi = \eta_A \times \eta_B$). Define a relational morphism:

$$\varphi : M(AB) \to M(A) \times M(B)$$

Proof of theorem 2: Define a (functional) morphism:

$$\varphi : \Sigma^* \to M(A) \times M(B)$$

by $w \varphi = (w \eta_A, w \eta_B)$ for all $w \in \Sigma^*$. (Put otherwise, $\varphi = \eta_A \times \eta_B$). Define a relational morphism:
by ρ = η_{A^2B}^{-1} φ. [Observe that the fact that η_{AB} is surjective is needed to insure that the domain of ρ is M(AB).]

\[\Sigma^* \xrightarrow{\eta_{AB}} \xrightarrow{\varphi} M(AB) \xrightarrow{\rho} M(A) \times M(B)\]

I will show that ρ is an aperiodic relational morphism. Let e ∈ M(A) × M(B) be idempotent. Then e = (e', e''), where e' and e'' are idempotents in M(A) and M(B), respectively. Let e ∈ e^{-1}. Then there exists w ∈ Σ^* such that:

\[w \eta_{AB} = s, \quad w \eta_a = e' \quad \text{and} \quad w \eta_b = e''.\]

I will show that w^3 \sim w^4. Suppose uw^3 v ∈ AB for some u, v ∈ Σ^*. Then uw^3 v = xy, where x ∈ A and y ∈ B. Then either x = uwx', where x' y = w^2 v, or y = y' uv where xy' = uw^2. In the first case, since e' = w \eta_a is idempotent, w \sim w^2; thus uwx' ∈ A implies uw^2 x' ∈ A. Thus uw^4 v = uw^2 x' y ∈ AB. In the second case, since e'' = w \eta_b is idempotent, w \sim w^2, and it follows again that uw^4 v ∈ AB. Thus, uw^3 v ∈ AB implies uw^4 v ∈ AB. Conversely, suppose uw^4 v ∈ AB. Then uw^4 v = xy, where x ∈ A and y ∈ B. Either x = uw^2 x', where x' y = w^2 v or y = y' w^2 v and xy' = uw^2. In the first case, since w \sim w^2, uw^2 x' ∈ A implies uwx' ∈ A. In the second case, since w \sim w^2, y' w^2 v ∈ B implies y' wv ∈ B. In either case, uw^3 v ∈ AB. Thus uw^3 v ∈ AB ⇔ uw^4 v ∈ AB, so w^3 \sim w^4, and s^3 = w^3 \eta_{AB} = w^4 \eta_{AB} = s^4. Since this holds for all s in e \rho^{-1}, e \rho^{-1} is aperiodic. ■

The product AB is said to be of bounded ambiguity if there exists a positive integer k such that any w ∈ AB admits at most k distinct factorizations of the form w = xy, where x ∈ A, y ∈ B. AB is unambiguous is there is only one such factorization for each w ∈ AB. For example, if either A or B is finite, AB is of bounded ambiguity. On the other hand, if A = B is the set of all words in Σ* of even length, then AB is of unbounded ambiguity.

Unambiguous products and products of bounded ambiguity were studied by Schützenberger [9]. He showed, using an adaptation of the \(M_1 \diamond M_2\) vol. 15, n°2, 1981
construction, that if $AB$ is of bounded ambiguity, then there exists a monoid $M'$ such that $M(AB) < M'$, and a generalized-definite functional morphism $\varphi : M' \rightarrow M(A) \times M(B)$. By Lemma 1 (b), this is equivalent to:

**Theorem 3:** If $AB$ is of bounded ambiguity, then there is a generalized-definite relational morphism $\rho : M(AB) \rightarrow M(A) \times M(B)$.

**Proof:** As in the proof of theorem 2, let $\rho = \eta_{AB}^{-1} \varphi$, where $\varphi = \eta_A \times \eta_B$. Let $e = (e', e'') \in M(A) \times M(B) (B)$ be idempotent, and let $r, s, t \in e \rho^{-1}$. I will show $rst = rt$; in particular, $e \rho^{-1}$ is generalized-definite. Since $r, s, t \in e \rho^{-1}$ there exist $w, x, y \in \Sigma^*$ such that:

$$
\begin{align*}
w \eta_{AB} &= r, \\
x \eta_{AB} &= s, \\
y \eta_{AB} &= t,
\end{align*}
$$

Thus $w \sim w^2 \sim x \sim x^2 \sim y \sim y^2$ and $w \sim w^2 \sim x \sim x^2 \sim y \sim y^2$. Now suppose $uwxyv \in AB$ for some $u, v \in \Sigma^*$. Then either $uwz' \in A$, $z'' \in B$ and $z'z'' = yv$, or $z''yv \in B$, $z' \in A$, and $z'z'' = uw$. In the first case, $uwz' \in A \Rightarrow uwz' \in A \Rightarrow uwz' \in A$. In the second case, $z''yv \in B \Rightarrow z''yv \in B \Rightarrow z''xvy \in B$. In either case, $uwxyv \in AB$, thus $uwxyv \in AB \Rightarrow uwxyv \in AB$.

Conversely, suppose $uwxyv \in AB$. There are three possibilities: (i) $uwz' \in A$, $z'' \in B$ and $z'z'' = yv$; (ii) $z' \in A$, $z''xyv \in B$, and $z'z'' = uw$; (iii) $uwz' \in A$, $z''yv \in B$, and $z'z'' = x$. In case (i), $uwz' \in A \Rightarrow uw^2z' \in A \Rightarrow uwz' \in A$, and thus $uwxyv \in AB$. Case (ii) is identical. Thus in either of these cases, $uwxyv \in AB \Rightarrow uwxyv \in AB$. I will now show that case (iii) cannot arise: $uwz' \in A \Rightarrow uwz' \in A \Rightarrow uwz' \in A \Rightarrow \cdots \Rightarrow uwx^n z' \in A$ for any nonnegative integer $n$. Similarly, $z''yv \in B \Rightarrow z''x^n yv \in B$ for any nonnegative integer $n$. Now the word $uwx^n yv$ can be factored in $n$ distinct ways:

$$uwx^n yv = (uwz')(z''x^{n-1} yv) = (uwz')(z''x^{n-2} yv) = \cdots = (uwz')^{n-1} z')(z'' yv),$$

where in each factorization, the left-hand factor is in $A$ and the right-hand factor is in $B$. This contradicts the assumption of bounded ambiguity—thus case (iii) cannot arise. (It is conceivable that $x = 1$, the empty word of $\Sigma^*$, in which case the above argument does not yield $n$ distinct factorizations. However, if $x = 1$, then $s = 1$, so $rst = rt$ trivially.)

It has been shown that $uwxyv \in AB \Leftrightarrow uwxyv \in AB$. Thus $wy \sim wy$, so $rst = (wxy) \eta_{AB} = (wy) \eta_{AB} = rt$. This completes the proof. ■

R.A.I.R.O. Informatique théorique/Theoretical Informatics
Before proceeding to the star operation, I will mention, without giving the proof, another application of this technique to the product operation. In [11], I used a generalized version of the Shützenberger product to study the $n$-fold product $A_1 \ldots A_n$ of $n$ recognizable sets $A_1, \ldots, A_n$. A principal result of that paper can be stated as follows: There exists a relational $LJ$-morphism $\rho : M(A_1 \ldots A_n) \to M(A_1) \times \ldots \times M(A_n)$, where $LJ$ is the $S$-variety consisting of those finite semi-groups $S$ such that for each idempotent $e \in S$, the monoid $e S e$ is $J$-trivial. A different proof of this theorem can be given using the methods of theorems 2 and 3: One forms the relational morphism:

$$\eta_{A_1}^{-1} \times \ldots \times \eta_{A_n}^{-1} M(A_1 \ldots A_n) \to M(A_1) \times \ldots \times M(A_n)$$

and shows that it is an $LJ$-morphism.

4. THE STAR OPERATION

Let $A \subseteq \Sigma^*$. $A^*$ denotes the submonoid of $\Sigma^*$ generated by $A$. If $A$ is recognizable, then $A^*$ is as well, however there is no simple description of the effect of the star operation on syntactic monoids. This is because $M(A^*)$ may be arbitrarily complicated even when $M(A)$ has a very simple structure. Indeed, Pin [4] has shown that if $M$ is any finite monoid, then there exists a finite subset $A$ of $\Sigma^*$, for some alphabet $\Sigma$, such that $M < M(A^*)$. However, some meaningful results are possible if one places some restrictions on when the star operation is to be applied. A submonoid $T$ of $\Sigma^*$ is said to be pure if for every $w \in \Sigma^*$ and positive integer $n$, $w^n \in T$ implies $w \in T$.

**THEOREM 4:** Let $A \subseteq \Sigma^*$ be recognizable. If $A^*$ is a pure submonoid of $\Sigma^*$, then there is an aperiodic relational morphism $\rho : M(A^*) \to M(A)$.

This generalizes some previous results: Restivo [6] showed that if $A^*$ is pure and $M(A)$ is aperiodic, then $M(A^*)$ is aperiodic. Perrot [2] extended this to show that if $H$ is any $M$-variety consisting exclusively of groups, and if every group in $M(A)$ belongs to $H$, then every group in $M(A^*)$ belongs to $H$, provided $A^*$ is pure.

The proof of theorem 4 is an adaptation of an argument in [1] (theorem X.5.2). I require a preliminary lemma.

**LEMMA 5:** Let $B \subseteq \Sigma^*$ be recognizable, and let $w \in \Sigma^*$. Suppose there exists a positive integer $k$ such that for all $u, v \in \Sigma^*$, $uw^k v \in B \Rightarrow uw^{k+1} v \in B$. Then $w^s \sim w^{s+1}$ for all sufficiently large $s$. 

vol. 15, n°2, 1981
Proof: Since $B$ is recognizable, $M(B)$ is finite. Thus there exist positive integers $s'$ and $r$ such that if $m \in M(B)$ and $s \geq s'$, then $m^{s+r} = m^s$. Let $s = \max \{ s', k \}$. If $u, v \in \Sigma^*$, then:

$$uw^sv \in B \Rightarrow uw^k w^{s-k}v \in B \Rightarrow uw^{k+1} w^{s-k}v \in B \Rightarrow uw^{s+1}v \in B.$$ 

By the same argument:

$$uw^{s+1}v \in B \Rightarrow uw^{s+2}v \in B \Rightarrow \ldots \Rightarrow uw^{s+r}v \in B.$$ 

Now since $w^s \eta_B = (w \eta_B)^s = w^{s+r} \eta_B$, $w^s \sim w^{s+r}$, and thus

$$uw^{s+r}v \in B \Rightarrow uw^sv \in B.$$ 

Thus $uw^sv \in B \iff uw^{s+1}v \in B$, so $w^s \sim w^{s+1}$.

Proof of theorem 4: Let $\rho = \eta_A^{-1} \eta_A : M(A^*) \to M(A)$. Let $e \in M(A)$ be idempotent. I will show that $e \rho^{-1}$ is an aperiodic semigroup. That is, $s^{k+1} = s^k$ for all $s \in e \rho^{-1}$ and all sufficiently large $k$. Let $s \in e \rho^{-1}$. Then there exists $w \in \Sigma^*$ such that $w \eta_A = s$ and $w \eta_A = e$. By lemma 5, it is sufficient to show that there is a positive integer $k$ such that:

$$(*) \quad uw^k v \in A^* \Rightarrow uw^{k+1} v \in A^* \quad \text{for all } u, v \in \Sigma^*.$$ 

Let $k > |w|$ (the length of the word $w$) and suppose $uw^k v \in A^*$. Then $uw^k v = a_1 \ldots a_m$, where $a_i \in A - \{ 1 \}$ for each $i$. Let $1 \leq r \leq k$; I will say that the $r$th occurrence of $w$ is cut if for some $j$, $1 \leq j \leq m$, $uw^{r-1}$ is an initial segment of $a_1 \ldots a_j$, and $a_1 \ldots a_j$ is a proper initial segment of $uw^r$. (That is, $a_1 \ldots a_j \in uw^{r-1} \Sigma^*$, and $uw^r \in a_1 \ldots a_j \Sigma^*$.)

$$\begin{array}{c}
\underline{u} \quad \underline{w^{r-1}} \quad \underline{w} \quad \underline{w^{k-r}v} \\
\hline
a_1 \ldots a_j \quad a_{j+1} \ldots a_m
\end{array}$$

There are now two cases to consider:

Case 1: Every occurrence of $w$ is cut. Then for each $r$, $1 \leq r \leq k$, there exists $j_r$, $1 \leq j_r \leq m$, such that $a_1 \ldots a_{j_r} \in uw^{r-1} \Sigma^*$ and $uw^r \in a_1 \ldots a_{j_r} \Sigma^*$. Thus:

$$\begin{cases}
\{ a_1 \ldots a_j = uw^{r-1} b_r, \quad b_r \in \Sigma^* \\
\{ a_{j_r+1} \ldots a_m = c_r w^{k-r} v, \quad c_r \in \Sigma^* \\
b_r c_r = w
\end{cases}$$

for $r = 1, \ldots, k$. 

R.A.I.R.O. Informatique théorique/Theoretical Informatics
Now there are $k$ occurrences of $w$ altogether, and $|w| < k$ factorizations of $w$ of the form $w = bc$, where $b \in \Sigma^*$, $c \in \Sigma^+$. Thus some pair $b, c$ must appear twice in \((\star \star)\) — that is, $b_r = b_r', c_r = c_r'$, with $r \neq r'$. This yields:
\[
\begin{align*}
    a_1 \ldots a_i &= uw^{r-1} b, \\
    a_{i+1} \ldots a_j &= cw^s b, \\
    a_{j+1} \ldots a_m &= cw^t v
\end{align*}
\]
where $r \geq 1$, $0 \leq s$, $0 \leq t$, $r + s + t + 1 = k$, and $bc = w$.

Thus:
\[
\begin{align*}
    cw^s b &= c(bc)^sb = (cb)^{s+1}. & \text{(since $cw^s b \in A^*$, and since $A^*$ is pure, $cb \in A^*$.)}
\end{align*}
\]

Thus $\star$ holds in this case.

Case 2: Some occurrence of $w$ is not cut. If the $r$th occurrence of $w$ is not cut, then:
\[
\begin{align*}
    a_1 \ldots a_{j-1} c &= uw^{r-1}, \\
    a_j &= cw^t b, \\
    ba_{j+1} \ldots a_m &= w^q v,
\end{align*}
\]
where $b, c \in \Sigma^*$, $t \geq 1$, and $r + t + q - 1 = k$. Since $w \sim u^2$, and since $t \geq 1$,
\[
cw^t b = cw^{t-1} b \sim cw^t w^{t-1} b = cw^{t+1} b.
\]

Since $cw^t b \in A$, it follows that $cw^{t+1} b \in A$. Thus:
\[
\begin{align*}
    uw^{k+1} v &= a_1 \ldots a_{j-1} cw^{t+1} b a_{j+1} \ldots a_m \in A^*,
\end{align*}
\]
so $\star$ holds in this case as well. $\blacksquare$

Theorem 4 provides a connection between the operation $A \rightarrow A^*$ when $A^*$ is pure, and the product operation. Let $V$ be an $M$-variety. $V$ is said to be closed under product if for any finite alphabet $\Sigma$ and recognizable subsets $A$ and $B$ of $\Sigma^*$, $M(A) \in V$ and $M(B) \in V$ implies $M(AB) \in V$. $V$ is said to be closed under inverse images of aperiodic morphisms if for any finite monoids $M$ and $M'$, if $M' \in V$ and $\varphi : M \rightarrow M'$ is an aperiodic functional morphism, then $M \in V$. In [10] I showed that a nontrivial $M$-variety (that is, an $M$-variety which contains a monoid with
more than one element) is closed under product if and only if it is closed under inverse images of aperiodic morphisms. Now suppose $\mathcal{V}$ is a nontrivial $M$-variety closed under product. Let $\Sigma$ be a finite alphabet, $A$ a recognizable subset of $\Sigma^*$, and $M(A) \in \mathcal{V}$. If $A^*$ is pure, then by theorem 4, there is an aperiodic relational morphism $\rho : M(A^*) \to M(A)$. By lemma 1, $M(A^*) \not< \# \rho$, and $\bar{\rho} : \# \rho \to M(A)$ is an aperiodic functional morphism. By the theorem just cited, $\mathcal{V}$ is closed under inverse images of aperiodic morphisms, so $\# \rho \in \mathcal{V}$, and thus $M(A^*) \in \mathcal{V}$. This proves:

**Theorem 6:** $\mathcal{V}$ be a nontrivial $M$-variety closed under product. If $A \subseteq \Sigma^*$ is a recognizable set, $M(A) \in \mathcal{V}$, and $A^*$ is pure, then $M(A^*) \in \mathcal{V}$.

Put otherwise, nontrivial $M$-varieties closed under product are also closed under the operation $A \to A^*$ when $A^*$ is pure. It would be interesting to know if the converse is true: that is, if $\mathcal{V}$ is closed under the operation $A \to A^*$ when $A^*$ is pure, must $\mathcal{V}$ be closed under product?

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**References**


13. B. TILSON, Chapter XII in Reference [1].