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RELATIONAL MORPHISMS AND OPERATIONS ON RECOGNIZABLE SETS (*)

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Communiqué par J.-F. Perrot

Abstract. — *Relational morphisms between finite monoids (a notion due to Tilson) are used to study the effect certain operations on recognizable sets have on the syntactic monoids of those sets. This leads to concise proofs of a number of known results concerning the product operation, and a new result concerning the star operation.*

Résumé. — *On utilise les morphismes relationnels (dus à Tilson) pour étudier l'effet que certaines opérations sur les langages reconnaissables produisent sur les monoïdes syntactiques de ces langages. On obtient ainsi des démonstrations simples pour plusieurs résultats déjà connus sur l'opération de produit et un résultat nouveau sur l'opération étoile.*

1. INTRODUCTION

Some recent research in the theory of automata has been devoted to describing the effect various operations on recognizable sets have on the syntactic monoids of the sets involved. A particularly simple example of such a description (this one treating the operation of intersection) is the following: If Σ is a finite alphabet and A and B are recognizable subsets of Σ^* (the free monoid on Σ), then $M(A \cap B) \prec M(A) \times M(B)$ [Here $M(X)$ denotes the syntactic monoid of X, and $M_1 \prec M_2$ means M_1 divides M_2 — that is, M_1 is a quotient of a submonoid of M_2.] (See Eilenberg [1] for a detailed explanation of the terminology of this paper.) More complex examples treat the product operation (Schützenberger [8]), unambiguous product (Schützenberger [9]), the shuffle product (Perrot [2]), n-fold products (Straubing [11]), and images under length-preserving morphisms from one free monoid to another (Reutenauer [7], Straubing [12], Pin [5]).

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This paper presents a general method for studying such questions. The method uses *relational morphisms* between finite monoids, a concept introduced by Tilson [13].

Relational morphisms are discussed in section 2. The method is applied in section 3 to give new, brief proofs of a number of known results concerning the product operation. In section 4 I use it to prove a new result, which concerns the operation $A \to A^*$ in the case that A^* is a pure submonoid of Σ^*.

2. RELATIONAL MORPHISMS

Let M_1 and M_2 be finite monoids. A *relation* $\rho : M_1 \to M_2$ is a map from M_1 into $\mathcal{P}(M_2)$ (the power set of M_2). If $m \in M_1$ then $m\rho$ denotes the image of m under this map. The *graph* of ρ, denoted $\#\rho$, is the set $\{(m_1, m_2) \in M_1 \times M_2 \mid m_2 \in m_1 \rho\}$. The *inverse* of ρ, denoted ρ^{-1}, is the unique relation $\eta : M_1 \to M_2$ such that $\#\eta = \{(m_2, m_1) \in M_2 \times M_1 \mid (m_1, m_2) \in \#\rho\}$. The domain of ρ, denoted $\operatorname{dom}\rho$, is the set $\{m \in M_1 \mid m\rho \neq \emptyset\}$.

A relation $\rho : M_1 \to M_2$ is said to be a *relational morphism* if the following two conditions hold:

(i) $\#\rho$ is a submonoid of $M_1 \times M_2$;

(ii) $\operatorname{dom}\rho = M_1$.

Condition (i) is equivalent to:

(i)′ $1 \in 1\rho$ and for all $m, m' \in M_1$, $(m\rho)(m'\rho) \subseteq (mm')\rho$. [Here $(m\rho)(m'\rho)$ is the usual product of subsets of M_2 : $(m\rho)(m'\rho) = \{st \in M_2 \mid s \in m\rho,\ t \in m'\rho\}$.]

An ordinary morphism $\varphi : M_1 \to M_2$ is just a relational morphism that is also a function from M_1 into M_2. Such a morphism will sometimes be called a *functional morphism* for clarity.

A relation $\rho : M_1 \to M_2$ is said to be *surjective* if $\bigcup_{m \in M_1} m\rho = M_2$ and *injective* if $m\rho \cap m'\rho = \emptyset$ for any pair of distinct elements m and m' of M_1. If $\rho : M_1 \to M_2$ is a surjective relational morphism, then $\rho^{-1} : M_2 \to M_1$ is a relational morphism.

If $\rho : M_1 \to M_2$ and $\eta : M_2 \to M_3$ are relations, then $\rho\eta : M_1 \to M_3$ is the relation defined by $m(\rho\eta) = \bigcup_{m' \in m\rho} m'\eta$ for all $m \in M_1$. It is easy to check that $(\rho\eta)^{-1} = \eta^{-1}\rho^{-1}$, and that if ρ and η are relational morphisms, then $\rho\eta$ is a relational morphism.

To each relational morphism $\rho : M_1 \to M_2$ there is associated a functional morphism $\bar{\rho} : \#\rho \to M_2$ defined by $(m_1, m_2)\bar{\rho} = m_2$ for all $(m_1, m_2) \in \#\rho$. M_1 itself is the image of $\#\rho$ under the functional morphism $\pi : \#\rho \to M_1$ defined by $(m_1, m_2)\pi = m_1$ for all $(m_1, m_2) \in \#\rho$. Observe that $\rho = \pi^{-1}\bar{\rho}$.

Let $\underline{V}$ be a collection of finite semigroups. A functional morphism $\psi : M_1 \to M_2$ will be called a *functional $\underline{V}$-morphism* if for each idempotent $e \in M_2$, the semigroup $e\psi^{-1}$ is a member of $\underline{V}$. Similarly, a relational morphism $\rho : M_1 \to M_2$ will be called a *relational $\underline{V}$-morphism* if for each idempotent $e \in M_2$, the semigroup $e\rho^{-1}$ is a member of $\underline{V}$. The collection $\underline{V}$ is said to be an *S-variety* if $\underline{V}$ is closed under division and finite direct products. (Similarly, a collection $\underline{V}$ of finite *monoids* closed under division and finite direct products is called an *M-variety*.)

LEMMA 1: *Let $\underline{V}$ be an $\underline{S}$-variety:*

(*a*) *if $\rho : M_1 \to M_2$ is a relational $\underline{V}$-morphism, then $\bar{\rho} : \# \rho \to M_2$ is a functional $\underline{V}$-morphism;*

(*b*) *if $\psi : M \to M_2$ is a functional $\underline{V}$-morphism, and $M_1 \prec M$, then there is a relational $\underline{V}$-morphism $\rho : M_1 \to M_2$.*

Proof: (*a*) let $e \in M_2$ be idempotent. Then $e\bar{\rho}^{-1} = \# \rho \cap (M_1 \times \{e\})$. The projection $\pi : \# \rho \to M_1$ is injective when restricted to $e\bar{\rho}^{-1}$, so $e\rho^{-1} = (e\bar{\rho}^{-1})\pi$ is isomorphic to $e\bar{\rho}^{-1}$. Since $e\rho^{-1}$ is, by assumption, a member of $\underline{V}$, it follows that $e\bar{\rho}^{-1} \in \underline{V}$. Thus $\bar{\rho}$ is a functional $\underline{V}$-morphism. (*b*) Let $e \in M_2$ be idempotent. By assumption, $e\psi^{-1} \in \underline{V}$. Since $M_1 \prec M$, there is a submonoid M' of M and a surjective functional morphism $\varphi : M' \to M_1$. Let $\rho = \varphi^{-1}\psi$. Then $e\rho^{-1} = (e\psi^{-1})\varphi$. Now $(e\psi^{-1})\varphi \prec e\psi^{-1}$, since $(e\psi^{-1})\varphi$ is the image of $e\psi^{-1} \cap M'$ under the functional morphism φ. Since $e\psi^{-1} \in \underline{V}$, it follows that $e\rho^{-1} \in \underline{V}$. Thus ρ is a relational $\underline{V}$-morphism. ■

In this paper I will be concerned with $\underline{V}$-morphisms for two particular choices of the S-variety $\underline{V}$.

The variety Ap of *aperiodic* semigroups consists of all finite semigroups which contain no nontrivial groups. Equivalently, $S \in Ap$ if and only if for each $s \in S$, $s^n = s^{n+1}$ for all sufficiently large positive integers n. Relational Ap-morphisms and functional Ap-morphisms will be called *aperiodic relational morphisms* and *aperiodic functional morphisms*, respectively.

The variety $\underline{\tilde{D}}$ of *generalized-definite* semigroups consists of all finite aperiodic semigroups all of whose idempotents lie in the unique minimal ideal. Equivalently, $S \in \underline{\tilde{D}}$ if and only if for all sufficiently large positive integers n,

$$s, r_1, \ldots, r_n, t_1, \ldots, t_n \in S$$

implies:

$$r_1 \ldots r_n s t_1 \ldots t_n = r_1 \ldots r_n t_1 \ldots t_n.$$

(See Eilenberg [1], Capters V and VIII.) Relational $\tilde{D}$-morphisms and functional $\tilde{D}$-morphisms will be called *generalized-definite relational morphisms* and *generalized-definite functional morphisms*, respectively.

3. THE PRODUCT OPERATION

Let A and B be subsets of Σ^*, the free monoid generated by a finite alphabet Σ. The product AB is defined by:

$$AB=\{uv\in\Sigma^* \mid u\in A,\ v\in B\}.$$

Let $M(A)$, $M(B)$ and $M(AB)$ denote the syntactic monoids of A, B and AB respectively.

THEOREM 2: *There is an aperiodic relational morphism:*

$$\rho:\quad M(AB)\to M(A)\times M(B).$$

The proof will be given shortly. Theorem 2 is due, in a somewhat different form, to Schützenberger [8]. He showed, given two finite monoids M_1 and M_2, how to construct a finite monoid $M_1 \diamond M_2$ (the *Schützenberger product* of M_1 and M_2) with the following property: If A and B are recognizable subsets of Σ^* then:

$$M(AB)\prec M(A)\diamond M(B).$$

As it turns out, there is an aperiodic functional morphism from $M_1 \diamond M_2$ onto $M_1 \times M_2$. Theorem 2 now follows from these facts and lemma 1 (b).

The proof of theorem 2 which I give below avoids the construction of $M_1 \diamond M_2$ altogether. Recall that the syntactic monoid $M(A)$ of a subset A of Σ^* is the quotient of Σ^* by the congruence $\underset{A}{\sim}$, where $w_1 \underset{A}{\sim} w_2$ if and only if:

$$uw_1 v\in A \quad\Leftrightarrow\quad uw_2 v\in A$$

for all u, $v\in\Sigma^*$. Let $\eta_A:\Sigma^*\to M_A$ denote the morphism induced by this congruence.

Proof of theorem 2: Define a (functional) morphism:

$$\varphi:\quad \Sigma^*\to M(A)\times M(B)$$

by $w\varphi=(w\eta_A,\ w\eta_B)$ for all $w\in\Sigma^*$. (Put otherwise, $\varphi=\eta_A\times\eta_B$). Define a relational morphism:

$$\varphi:\quad M(AB)\to M(A)\times M(B)$$

by $\rho = \eta_{AB}^{-1}\varphi$. [Observe that the fact that η_{AB} is surjective is needed to insure that the domain of ρ is $M(AB)$.]

$$\begin{tikzcd}
 & \Sigma^* \arrow[dl, "\eta_{AB}"'] \arrow[dr, "\varphi"] & \\
M(AB) \arrow[rr, "\rho"] & & M(A)\times M(B)
\end{tikzcd}$$

I will show that ρ is an aperiodic relational morphism. Let $e \in M(A)\times M(B)$ be idempotent. Then $e=(e', e'')$, where e' and e'' are idempotents in $M(A)$ and $M(B)$, respectively. Let $s \in e\rho^{-1}$. Then there exists $w \in \Sigma^*$ such that:

$$w\eta_{AB}=s, \qquad w\eta_A=e' \qquad \text{and} \qquad w\eta_B=e''.$$

I will show that $w^3 \underset{AB}{\sim} w^4$. Suppose $uw^3v \in AB$ for some $u, v \in \Sigma^*$. Then $uw^3v = xy$, where $x \in A$ and $y \in B$. Then either $x = uwx'$, where $x'y = w^2v$, or $y = y'wv$ where $xy' = uw^2$. In the first case, since $e' = w\eta_A$ is idempotent, $w \underset{A}{\sim} w^2$; thus $uwx' \in A$ implies $uw^2x' \in A$. Thus $uw^4v = uw^2x'y \in AB$. In the second case, since $e'' = w\eta_B$ is idempotent, $w \underset{B}{\sim} w^2$, and it follows again that $uw^4v \in AB$. Thus, $uw^3v \in AB$ implies $uw^4v \in AB$. Conversely, suppose $uw^4v \in AB$. Then $uw^4v = xy$, where $x \in A$ and $y \in B$. Either $x = uw^2x'$, where $x'y = w^2v$ or $y = y'w^2v$ and $xy' = uw^2$. In the first case, since $w \underset{A}{\sim} w^2$, $uw^2x' \in A$ implies $uwx' \in A$. In the second case, since $w \underset{B}{\sim} w^2$, $y'w^2v \in B$ implies $y'wv \in B$. In either case, $uw^3v \in AB$. Thus $uw^3v \in AB \Leftrightarrow uw^4v \in AB$, so $w^3 \underset{AB}{\sim} w^4$, and $s^3 = w^3\eta_{AB} = w^4\eta_{AB} = s^4$. Since this holds for all s in $e\rho^{-1}$, $e\rho^{-1}$ is aperiodic. ■

The product AB is said to be of *bounded ambiguity* if there exists a positive integer k such that any $w \in AB$ admits at most k distinct factorizations of the form $w = xy$, where $x \in A$, $y \in B$. AB is *unambiguous* is there is only one such factorization for each $w \in AB$. For example, if either A or B is finite, AB is of bounded ambiguity. On the other hand, if $A = B$ is the set of all words in Σ^* of even length, then AB is of unbounded ambiguity.

Unambiguous products and products of bounded ambiguity were studied by Schützenberger [9]. He showed, using an adaptation of the $M_1 \diamond M_2$

construction, that if AB is of bounded ambiguity, then there exists a monoid M' such that $M(AB) \prec M'$, and a generalized-definite functional morphism $\varphi : M' \to M(A) \times M(B)$. By Lemma 1 (*b*), this is equivalent to:

THEOREM 3: *If AB is of bounded ambiguity, then there is a generalized-definite relational morphism $\rho : M(AB) \to M(A) \times M(B)$.*

Proof: As in the proof of theorem 2, let $\rho = \eta_{AB}^{-1} \varphi$, where $\varphi = \eta_A \times \eta_B$. Let $e = (e', e'') \in M(A) \times M(B)$ be idempotent, and let $r, s, t \in e\rho^{-1}$. I will show $rst = rt$; in particular, $e\rho^{-1}$ is generalized-definite.

Since $r, s, t \in e\rho^{-1}$ there exist $w, x, y \in \Sigma^*$ such that:

$$w\eta_{AB} = r, \qquad x\eta_{AB} = s, \qquad y\eta_{AB} = t,$$
$$w\eta_A = x\eta_A = y\eta_A = e',$$
$$w\eta_B = x\eta_B = y\eta_B = e''.$$

Thus $w \underset{A}{\sim} w^2 \underset{A}{\sim} x \underset{A}{\sim} x^2 \underset{A}{\sim} y \underset{A}{\sim} y^2$ and $w \underset{B}{\sim} w^2 \underset{B}{\sim} x \underset{B}{\sim} x^2 \underset{B}{\sim} y \underset{B}{\sim} y^2$. Now suppose $uwyv \in AB$ for some $u, v \in \Sigma^*$. Then either $uwz' \in A$, $z'' \in B$ and $z'z'' = yv$, or $z''yv \in B$, $z' \in A$, and $z'z'' = uw$. In the first case, $uwz' \in A \Rightarrow uwwz' \in A \Rightarrow uwxz' \in A$. In the second case, $z''yv \in B \Rightarrow z''yyv \in B \Rightarrow z''xyv \in B$. In either case, $uwxyv \in AB$, thus $uwyv \in AB \Rightarrow uwxyv \in AB$.

Conversely, suppose $uwxyv \in AB$. There are three possibilities: (i) $uwxz' \in A$, $z'' \in B$ and $z'z'' = yv$; (ii) $z' \in A$, $z''xyv \in B$, and $z'z'' = uw$; (iii) $uwz' \in A$, $z''yv \in B$, and $z'z'' = x$, In case (i), $uwxz' \in A \Rightarrow uw^2z' \in A \Rightarrow uwz' \in A$, and thus $uwyv \in AB$. Case (ii) is identical. Thus in either of these cases, $uwxyv \in AB \Rightarrow uwyv \in AB$. I will now show that case (iii) cannot arise: $uwz' \in A \Rightarrow uw^2z' \in A \Rightarrow uwxz' \in A \Rightarrow \ldots \Rightarrow uwx^nz' \in A$ for any nonnegative integer n. Similarly, $z''yv \in B \Rightarrow z''x^nyv \in B$ for any nonnegative integer n. Now the word uwx^nyv *can be factored in n* distinct ways:

$$uwx^nyv = (uwz')(z''x^{n-1}yv) = (uwxz')(z''x^{n-2}yv) = \ldots = (uwx^{n-1}z')(z''yv),$$

where in each factorization, the left-hand factor is in A and the right-hand factor is in B. This contradicts the assumption of bounded ambiguity — thus case (iii) cannot arise. (It is conceivable that $x = 1$, the empty word of Σ^*, in which case the above argument does not yield n distinct factorizations. However, if $x = 1$, then $s = 1$, so $rst = rt$ trivially.)

It has been shown that $uwxyv \in AB \Leftrightarrow uwyv \in AB$. Thus $wy \underset{AB}{\sim} wxy$, so $rst = (wxy)\eta_{AB} = (wy)\eta_{AB} = rt$. This completes the proof. ■

Before proceeding to the star operation, I will mention, without giving the proof, another application of this technique to the product operation. In [11], I used a generalized version of the Shützenberger product to study the n-fold product $A_1 \ldots A_n$ of n recognizable sets $A_1, \ldots, A_n$. A principal result of that paper can be stated as follows: There exists a relational $\underline{LJ}$-morphism $\rho : M(A_1 \ldots A_n) \to M(A_1) \times \ldots \times M(A_n)$, where $\underline{LJ}$ is the $\underline{S}$-variety consisting of those finite semi-groups S such that for each idempotent $e \in S$, the monoid $e S e$ is J-trivial. A different proof of this theorem can be given using the methods of theorems 2 and 3: One forms the relational morphism:

$$\eta_{A_1 \ldots A_n}^{-1} (\eta_{A_1} \times \ldots \times \eta_{A_n})\ M(\mathrm{A}_1 \ldots \mathrm{A}_n) \to M(A_1) \times \ldots \times M(A_n)$$

and shows that it is an $\underline{LJ}$-morphism.

4. THE STAR OPERATION

Let $A \subseteq \Sigma^*$. A^* denotes the submonoid of Σ^* generated by A. If A is recognizable, then A^* is as well, however there is no simple description of the effect of the star operation on syntactic monoids. This is because $M(A^*)$ may be arbitrarily complicated even when $M(A)$ has a very simple structure. Indeed, Pin [4] has shown that if M is any finite monoid, then there exists a *finite* subset A of Σ^*, for some alphabet Σ, such that $M \prec M(A^*)$. However, some meaningful results are possible if one places some restrictions on when the star operation is to be applied. A submonoid T of Σ^* is said to be *pure* if for every $w \in \Sigma^*$ and positive integer n, $w^n \in T$ implies $w \in T$.

THEOREM 4: *Let $A \subseteq \Sigma^*$ be recognizable. If A^* is a pure submonoid of Σ^*, then there is an aperiodic relational morphism $\rho : M(A^*) \to M(A)$.*

This generalizes some previous results: Restivo [6] showed that if A^* is pure and $M(A)$ is aperiodic, then $M(A^*)$ is aperiodic. Perrot [2] extended this to show that if $\underline{H}$ is any $\underline{M}$-variety consisting exclusively of groups, and if every group in $M(A)$ belongs to $\underline{H}$, then every group in $M(A^*)$ belongs to $\underline{H}$, provided A^* is pure.

The proof of theorem 4 is an adaptation of an argument in [1] (theorem X.5.2). I require a preliminary lemma.

LEMMA 5 : *Let $B \subseteq \Sigma^*$ be recognizable, and let $w \in \Sigma^*$. Suppose there exists a positive integer k such that for all $u, v \in \Sigma^*$, $uw^k v \in B \Rightarrow uw^{k+1} v \in B$. Then $w^s \underset{B}{\sim} w^{s+1}$ for all sufficiently large s.*

Proof: Since B is recognizable, $M(B)$ is finite. Thus there exist positive integers s' and r such that if $m \in M(B)$ and $s \geqq s'$, then $m^{s+r} = m^s$. Let $s \geqq \max\{s', k\}$. If u, $v \in \Sigma^*$, then:

$$uw^s v \in B \;\Rightarrow\; uw^k w^{s-k} v \in B \;\Rightarrow\; uw^{k+1} w^{s-k} v \in B \;\Rightarrow\; uw^{s+1} v \in B.$$

By the same argument;

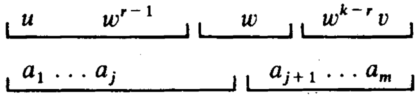
$$uw^{s+1} v \in B \;\Rightarrow\; uw^{s+2} v \in B \;\Rightarrow \dots \Rightarrow\; uw^{s+r} v \in B.$$

Now since $w^s \eta_B = (w\eta_B)^s = (w\eta_B)^{s+r} = w^{s+r}\eta_B$, $w^s \underset{B}{\sim} w^{s+r}$, and thus $uw^{s+r} v \in B \Rightarrow uw^s v \in B$. Thus $uw^s v \in B \Leftrightarrow uw^{s+1} v \in B$, so $w^s \underset{B}{\sim} w^{s+1}$. ■

Proof of theorem 4: Let $\rho = \eta_{A^*}^{-1}\eta_A : M(A^*) \to M(A)$. Let $e \in M(A)$ be idempotent. I will show that $e\rho^{-1}$ is an aperiodic semigroup. That is, $s^{k+1} = s^k$ for all $s \in e\rho^{-1}$ and all sufficiently large k. Let $s \in e\rho^{-1}$. Then there exists $w \in \Sigma^*$ such that $w\eta_{A^*} = s$ and $w\eta_A = e$. By lemma 5, it is sufficient to show that there is a positive integer k such that:

$$(\star) \qquad uw^k v \in A^* \;\Rightarrow\; uw^{k+1} v \in A^* \quad \text{for all } u, v \in \Sigma^*.$$

Let $k > |w|$ (the length of the word w) and suppose $uw^k v \in A^*$. Then $uw^k v = a_1 \dots a_m$, where $a_i \in A - \{1\}$ for each i. Let $1 \leqq r \leqq k$; I will say that the *rth occurrence of w is cut* if for some j, $1 \leqq j \leqq m$, uw^{r-1} is an initial segment of $a_1 \dots a_j$, and $a_1 \dots a_j$ is a *proper* initial segment of uw^r. (That is, $a_1 \dots a_j \in uw^{r-1}\Sigma^*$, and $uw^r \in a_1 \dots a_j \Sigma^+$.)



There are now two cases to consider:

Case 1: *Every occurrence of w is cut*. Then for each r, $1 \leqq r \leqq k$, there exists j_r, $1 \leqq j_r \leqq m$, such that $a_1 \dots a_{j_r} \in uw^{r-1}\Sigma^*$ and $uw^r \in a_1 \dots a_{j_r}\Sigma^+$. Thus:

$$(\star\star) \qquad \begin{cases} a_1 \dots a_j = uw^{r-1} b_r, & b_r \in \Sigma^*, \\ a_{j_{r+1}} \dots a_m = c_r w^{k-r} v, & c_r \in \Sigma^+, \\ b_r c_r = w & \end{cases}$$

for $r = 1, \dots, k$.

Now there are k occurrences of w altogether, and $|w|<k$ factorizations of w of the form $w=bc$, where $b\in\Sigma^*$, $c\in\Sigma^+$. Thus some pair b, c must appear twice in $(\star\star)$ – that is, $b_r=b_{r'}$, $c_r=c_{r'}$, with $r\neq r'$. This yields:

$$a_1\ldots a_i=uw^{r-1}b,$$
$$a_{i+1}\ldots a_j=cw^s b,$$
$$a_{j+1}\ldots a_m=cw^t v$$

where $r\geqq 1$, $0\leqq s$, $0\leqq t$, $r+s+t+1=k$, and $bc=w$.

Now $cw^s b=c(bc)^s b=(cb)^{s+1}$. Since $cw^s b\in A^*$, and since A^* is pure, $cb\in A^*$. Thus:

$$uw^{k+1}v=uw^{r-1}ww^s www^t v=(uw^{r-1}b)(cw^s b)(cb)(cw^t v)$$
$$=(a_1\ldots a_i\ldots a_j)(cb)(a_{j+1}\ldots a_m)\in A^*.$$

Thus $(\star)$ holds in this case.

Case 2: Some occurrence of w is not cut. If the *rth* occurrence of w is not cut, then:

$$a_1\ldots a_{j-1}c=uw^{r-1},$$
$$a_j=cw^t b,$$
$$ba_{j+1}\ldots a_m=w^q v,$$

where b, $c\in\Sigma^*$, $t\geqq 1$, and $r+t+q-1=k$. Since $w\underset{A}{\sim}w^2$, and since $t\geqq 1$,

$$cw^t b=cww^{t-1}b\underset{A}{\sim}cw^2w^{t-1}b=cw^{t+1}b.$$

Since $cw^t b\in A$, it follows that $cw^{t+1}b\in A$. Thus:

$$uw^{k+1}v=a_1\ldots a_{j-1}cw^{t+1}ba_{j+1}\ldots a_m\in A^*,$$

so $(\star)$ holds in this case as well. ■

Theorem 4 provides a connection between the operation $A\to A^*$ when A^* is pure, and the product operation. Let $\underline{V}$ be an $\underline{M}$-variety. *$\underline{V}$ is said to be closed under product* if for any finite alphabet Σ and recognizable subsets A and B of Σ^*, $M(A)\in\underline{V}$ and $M(B)\in\underline{V}$ implies $M(AB)\in\underline{V}$. $\underline{V}$ is said to be *closed under inverse images of aperiodic morphisms* if for any finite monoids M and M', if $M'\in\underline{V}$ and $\varphi: M\to M'$ is an aperiodic functional morphism, then $M\in\underline{V}$. In [10] I showed that a nontrivial $\underline{M}$-variety (that is, an $\underline{M}$-variety which contains a monoid with more than one element) is closed under product if and only if it is closed under inverse images of aperiodic morphisms. Now suppose $\underline{V}$ is a nontrivial $\underline{M}$-variety closed under product. Let Σ be a finite alphabet, A a recognizable subset of Σ^*, and $M(A) \in \underline{V}$. If A^* is pure, then by theorem 4, there is an aperiodic relational morphism $\rho : M(A^*) \to M(A)$. By lemma 1, $M(A^*) \prec \# \rho$, and $\bar{\rho} : \# \rho \to M(A)$ is an aperiodic functional morphism. By the theorem just cited, $\underline{V}$ is closed under inverse images of aperiodic morphisms, so $\# \rho \in \underline{V}$, and thus $M(A^*) \in \underline{V}$. This proves:

THEOREM 6: *$\underline{V}$ be a nontrivial $\underline{M}$-variety closed under product. If $A \subseteq \Sigma^*$ is a recognizable set, $M(A) \in \underline{V}$, and A^* is pure, then $M(A^*) \in \underline{V}$.*

Put otherwise, nontrivial $\underline{M}$-varieties closed under product are also closed under the operation $A \to A^*$ when A^* is pure. It would be interesting to know if the converse is true: that is, if $\underline{V}$ is closed under the operation $A \to A^*$ when A^* is pure, must $\underline{V}$ be closed under product?

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