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A THEORY OF COMPLEXITY
OF MONADIC RECURSION SCHEMES (*)

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Abstract. — Complexity of a monadic recursion scheme is defined through numerical characteristics of trees representing its computations. A class of such complexity characteristics of trees essentially unlike the computation time: so called mimeoinvariant complexity measures, is introduced which induce several dense hierarchies of complexity classes of monadic recursion schemes of unbounded complexity and infinite hierarchies of bounded complexity classes. Simple conditions are found under which a function is a nonreducible upper bound of complexity of a monadic recursion scheme.

Résumé. — On définit la complexité d’un schéma récursif monadique à l’aide de propriétés numériques des arbres qui représentent ses calculs. On introduit une classe de telles propriétés de complexité des arbres, appelées les mesures de complexité miméoinvariantes, qui sont essentiellement différentes du temps de calcul, et qui induisent plusieurs hiérarchies denses de classes de complexité pour les schémas récursifs monadiques de complexité non bornée et des hiérarchies infinies de classes de complexité bornée. On donne des conditions simples qui assurent qu’une fonction est une borne supérieure irréductible pour la complexité d’un schéma récursif monadique.

1. INTRODUCTION

There were several recent attempts to find a reasonable computer-free concept of computational complexity for program schemes. In particular, three different definitions may be mentioned: by R. Constable [1], by K. Weihrauch [2], and by Y. Igarashi [3]. All the three definitions have an essential common feature: they model computation time. We propose a concept of complexity of absolutely different nature. Our complexity measures characterize combinatorial complexity of objects representing computations of schemes. Moreover, the computation time is an illegal measure in our model. The mentioned papers differ also by the classes of schemes under study. Constable and Weihrauch treat standard (iterative) program schemes, while in Igarashi’s and our papers

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monadic recursion schemes are considered. However, we are dealing with classical monadic recursion schemes, whereas Igarashi exhibits his time hierarchy in a class of generalized monadic recursion schemes substantially broader than that of the classical schemes; furthermore, this hierarchy degenerates for classical monadic recursion schemes.

As it is well known [4, 5] there is a tight relation between monadic recursion schemes and cf-grammars: with each scheme \( E \) a cf-grammar \( G(E) \) is associated in a natural way, every computation of \( E \) under an interpretation \( I \) is represented by a rightmost derivation of \( G(E) \) "controlled" by \( I \). We make next step and consider the trees of these derivations as representations of the corresponding computations. After this it is rather natural to introduce complexity measures as integer-valued functions on trees.

This relation is the base for replantation of already existing complexity theory of cf-grammars as outlined in our earlier papers [6-9] to monadic recursion schemes. In particular, we apply (with minory changes) to monadic recursion schemes a central concept of this theory: the notion of a mimeoinvariant complexity measure (section 5). The characteristic feature of these measures is their invariance under a class of transformations of trees preserving on the whole their "topology". A complexity measure \( m \) has been chosen, we associate with each monadic recursion scheme \( E \) its \( m \)-complexity function \( m_E \) and thus to any nondecreasing total function \( f \) relate the class \( \mathcal{E}_f^m \) of all schemes whose complexity functions do not exceed \( f \). Mimeoinvariance of \( m \) implies that all quasirational monadic recursion schemes fall into bounded complexity class \( \mathcal{E}_\text{ Const}^m \). As it turns out all mimeoinvariant complexity measures have high classificational capacities. In section 6 we find simple conditions under which a function \( f \) or a constant \( c \) become nonreducible upper bounds of \( m \)-complexity of a monadic recursion scheme (thms. 6.1, 6.2). The main result of section 7 (thm. 7.1) gives a condition under which for a mimeoinvariant complexity measure \( m \) there is an infinitely decreasing sequence of functions \( f_1 > f_2 > f_3 > \ldots \) where each \( f_i \) is a nonreducible upper bound of \( m \)-complexity of a monadic recursion scheme. The theorem 8.1, a simplified version of this hierarchy theorem, shows that for any mimeoinvariant complexity measure \( m \) there is an infinitely decreasing sequence of functions \( f_1 > f_2 > f_3 > \ldots \) such that \( \mathcal{E}_{f_i}^m - \mathcal{E}_{f_{i+1}}^m \neq \emptyset \) for all \( i \). Hence all mimeoinvariant measures provide nondegenerate classifications of monadic recursion schemes. Finally, in the ninth section we formulate a definition of a monadic recursion scheme of maximal complexity and show that under reasonable conditions all unambiguous monadic recursion schemes are either of maximal complexity or of bounded density.

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2. PRELIMINARIES

We choose for the sequel two countable disjoint alphabets $E$ and $W$ (of terminal and nonterminal symbols respectively).

**Definitions and Notation 2.1:** Let $z$ be a string. A prefix (suffix) of $z$ is any string $v$ such that $z = vu$ (respectively $z = uv$) for a string $u$. Let $N$ and $Z^+$ denote the set of all numbers and all nonnegative integers respectively. A *(finite labelled rooted)* tree is a pair $T = (\Delta, l)$, where: (1) $\Delta$ is a finite nonempty subset of $(Z^+)^*$; (2) $\Delta$ is prefix closed, i.e. with every string $z$ it contains all its prefixes; (3) $l$ is a function (called a *labelling*) from $\Delta$ to $\Sigma \cup W \cup \{ \Lambda \}$ ($\Lambda$ is the empty string).

Strings in $\Delta$ are called *nodes* of $T$. For a node $v$ of $T$, $l(v)$ is called a *label* of $v$. Let $v, v'$ be two nodes of $T$ such that $v = va$ for some $a$ in $Z^+$. Then $v$ is called an *immediate successor* of $v'$ (in $T$). The set of all immediate successors of a node $v$ is denoted by $i(v)$. The cardinality of $i(v)$ is called a *width* of $v$. The maximal width of nodes of $T$ is called a *width* of $T$. A sequence $p = (v_1, v_2, \ldots, v_n)$, $n \geq 1$, of nodes of $T$ is called a path from $v_1$ to $v_n$ ($n$ being its length) if $v_{i+1}$ is in $i(v_i)$ for all $1 \leq i < n$. A path $(v_1, v_2)$ is called an arrow from $v_1$ to $v_2$. A node $v'$ is called a successor of a node $v$ if $v' \in i(v)$ and there is a path from $v$ to $v'$.

The node $\Lambda$ is called a root of $T$. A node $v$ is *bottom* if there is no arrow from $v$ to a node of $T$, and nonbottom otherwise. A node $v$ is preterminal if $l(v)$ is in $W \cup \{ \Lambda \}$ and either $v$ is bottom or all immediate successors of $v$ are bottom and are labelled by symbols in $E$.

We consider the following natural *(partial)* order (denoted by $<$) on the set of nodes of $T$. $v_1 < v_2$ holds iff there are nodes $v, v_1, v_2$ in $\Delta$ such that $v_1, v_2$ are in $i(v)$, $v_1 = va, v_2 = vb$ for some $a, b$ in $Z^+$ such that $a < b$, and $v_1, v'_1(v_2, v'_2)$ either coincide, or $v_1(v_2)$ is a successor of $v'_1(v'_2)$. This natural order is complete on the set of bottom nodes of $T$. If $v_1 < v_2 < \ldots < v_n$ is the sequence of all bottom nodes of $T$ in their natural order then the string $l(v_1)l(v_2)\ldots l(v_n)$ [denoted by $t(T)$] is the yield of $T$. The length $s$ of this string is denoted by $|T|$.

**Definition 2.2:** Two trees $T_1 = (\Delta_1, l_1)$ and $T_2 = (\Delta_2, l_2)$ are isomorphic (notation $T_1 \equiv T_2$) if there is a one-to-one correspondence $h$ between $\Delta_1$ and $\Delta_2$ such that for any two nodes $v_1, v_2$ of $T_1$: (1) $v_2$ is a successor of $v_1$ iff $h(v_2)$ is a successor of $h(v_1)$ in $T_2$, (2) $v_1 < v_2$ iff $h(v_1) < h(v_2)$ in $T_2$, (3) $l_1(v_1) = l_2(h(v_1))$.

---

(1) In our definitions and theorems we admit empty labels, empty right sides of productions, empty equations, and so on. However in the proofs of theorems we don't consider such cases for the reasons of space and because of triviality or routine character of the corresponding arguments.

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**Definition 2.3**: Let $T = (\Delta, l)$ be a tree and $v$ be a node of $T$. A tree $T' = (\Delta', l')$, where $\Delta' \subseteq \Delta \setminus \Delta$ and $l'(z) = l(vz)$ for all $z$ in $\Delta'$ is called a \textit{(v-)subtree} of $T$. $B(T') = v \Delta'$ is a base set of $T'$. The $v$-subtree (denoted by $T(v)$) whose base set $B(T(v))$ is the least subset of $\Delta$ containing $v$ and all its successors is called a \textit{full} \textit{(v-)subtree} of $T$. The $A$-subtree of $T$ (denoted by $CT(v)$) with the base set $(\Delta - B(T(v))) \cup \{v\}$ is called a \textit{complementary v-subtree} of $T$.

**Definition 2.4**: Let $T$, $T_1$, and $T_2$ be some trees and $v$ be a node of $T$. We say that $T$ is a \textit{composition} of $T_1$ and $T_2$ at $v$ (notation $T = \text{com}(T_1, v, T_2)$) if

**Definition 2.5**: Let $T = (\Delta, l)$ be a tree. A \textit{covering} of $T$ is a system of its subtrees $C = \{T_1, \ldots, T_r\}$ such that (1) $B(T_i) \cap B(T_j) = \emptyset$ for all $i \neq j$, $1 \leq i$, $j \leq r$, and (2) $\Delta = \bigcup_{i=1}^{r} B(T_i)$.

**Definition 2.6**: Let $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$ be two alphabets. $T = (\Delta, l)$ is a \textit{(syntactic) structure tree} (abbreviated \textit{s-tree}) over $\Sigma'$, $W'$ if: (1) all nonbottom nodes of $T$ are labelled by symbols in $W'$, (2) all bottom nodes of $T$ are labelled by symbols in $\Sigma' \cup W' \cup \{\Lambda\}$, (3) each nonbottom node possessing an immediate successor labelled by $\Lambda$ is of width 1. A $s$-tree is \textit{complete} (\textit{cs-tree} for short) if every its bottom node is labelled by an element of $\Sigma \cup \{\Lambda\}$. A $s$-tree $T$ is \textit{linear} if every its nonbottom node has no more than one immediate successor labelled by a nonterminal, $T$ is \textit{trivial} if the width of $T$ is $\leq 1$.

We introduce several binary relations on the set of $s$-trees which in a sense preserve their "topology".

**Definition 2.7**: Let $T_1 = (\Delta_1, l_1)$ and $T_2 = (\Delta_2, l_2)$ be two $s$-trees. $T_2$ is \textit{mimeomorphic} (strictly mimeomorphic) to $T_1$ (notation $T_1 \preceq T_2$ and respectively $T_1 \preceq^* T_2$) if there is a covering $C = \{T_{21}, \ldots, T_{2r}\}$ of $T_2$ and a one-to-one correspondence (mimeomorphism) $\varphi$ between $\Delta_1$ and $C$ with the following properties. Let $v_1, v_2$ be two nodes of $T_1$, $\varphi(v_1)$ be a $u_1$-subtree and $\varphi(v_2)$ be a $u_2$-subtree of $T_2$ for some $u_1, u_2$. Then: (1) if $v_2$ is in $l_1(v_1)$ in $T_1$, there is a node (respectively a preterminal node) $u$ of the tree $\varphi(v_1)$ such that $u_2$ is in $l(u)$ in $T_2$, (2) if $v_1 \prec v_2$ in $T_1$ then for any nodes $w_1$ of $\varphi(v_1)$ and $w_2$ of $\varphi(v_2)$, $u_1 w_1 \prec u_2 w_2$ holds in $T_2$, (3) $l_1(v_1)$ is in $\Sigma \cup \{\Lambda\}$ iff $\varphi(v_1)$ is a one-node $u_1$-subtree and $l_2(u_1)$ is in $\Sigma \cup \{\Lambda\}$. If every tree in $C$ is linear (trivial) we say that the mimeomorphism is \textit{linear} (\textit{trivial}) (notation $T_1 \preceq^l T_2$ and respectively $T_1 \preceq^t T_2$); if the mimeomorphism is both strict and linear (trivial) we say that it is \textit{strictly linear} (\textit{trivial}) (notation $T_1 \preceq^s T_2$ and respectively $T_1 \preceq^s T_2$).

\(^{(')}\) For a language $L$ and a string $z$, $z \backslash L$ denotes the quotient language $\{w | zw \text{ is in } L\}$.

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REMARK: If $T_1 \leq^s T_2$ and all subtrees in $C$ are either one-node or their bottom nodes are labelled by nonterminals then $T_2$ is homeomorphic to $T_1$ in the graph-theoretic sense (cf. [10]) (notation $T_1 \leq^h T_2$).

NOTATION: Let $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$. The set of all $s$-trees (cs-trees) of width $\leq k$ over $\Sigma'$, $W'$ is denoted by $\mathcal{S}(\Sigma', W', k)$ [respectively by $\mathcal{S}^c(\Sigma', W', k)$]. Let

$$\mathcal{S}(\Sigma', W') = \bigcup_{k=0}^{\infty} \mathcal{S}(\Sigma', W', k),$$

$$\mathcal{S}^c(\Sigma', W') = \bigcup_{k=0}^{\infty} \mathcal{S}^c(\Sigma', W', k),$$

$$\mathcal{S}(\Sigma, W) = \mathcal{S}, \quad \mathcal{S}^c(\Sigma, W) = \mathcal{S}^c.$$

DEFINITION 2.8: A set $S$ such that $S \subseteq \mathcal{S}^c(\Sigma_1, W_1, k)$ for some $k$ and finite $\Sigma_1 \subseteq \Sigma$, $W_1 \subseteq W$ is called a structure set (abbreviated s-set) if no two trees in $S$ are isomorphic. $L(S) = \{ t(T) \mid T \in S \}$ is the language characterized by $S$. $S$ is unambiguous if for each $z$ in $L(S)$ there is at most one cs-tree $T$ in $S$ such that $z = t(T)$; otherwise $S$ is ambiguous. A s-set $S$ is free if for any two s-trees $T_1 = (\Delta_1, l_1)$ and $T_2 = (\Delta_2, l_2)$ in $S$ and any nodes $v_1$ of $T_1$ and $v_2$ of $T_2$ such that $l_1(v_1) = l_2(v_2)$ $S$ contains a s-tree com$(CT_1(v_1), v_1, T_2(v_2))$.

NOTATION: Let $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$ be two alphabets. The class of all $cf$-grammars $G = (\Sigma_1, W_1, A, P)$ such that $\Sigma_1 \subseteq \Sigma'$ and $W_1 \subseteq W'$, is denoted by $\mathcal{G}(\Sigma', W')$. For a $cf$-grammar $G$ in $\mathcal{G}(\Sigma', W')$ we denote its structure set, i.e. the set of all its complete phrase-structure trees, by $S(G)$.

PROPOSITION 2.1:

$$\{ S(G) \mid G \text{ is in } \mathcal{G}(\Sigma', W') \} = \{ S \subseteq \mathcal{S}^c(\Sigma', W') \mid S \text{ is a free s-set} \}$$

for all $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$.

This well known proposition will provide a grammar-free form to our notion of complexity and to related concepts, convenient for applications to monadic recursion schemes as well as to $cf$-grammars.

3. COMPLEXITY MEASURES AND STRUCTURE SETS

The concepts presented in this section are introduced in [6, 7]. They form the framework within which we study there complexity of syntactic structures and derivation trees in $cf$-grammars. In section 5 below these concepts will be applied to monadic recursion schemes.
Définition 3.1: A complexity measure is a computable total function \( m \) from \( \mathcal{F} \) onto an infinite subset of \( \mathbb{Z}_+ \) such that \( m(T_1) = m(T_2) \) whenever \( T_1 \equiv T_2 \).

We cite a few examples of complexity measures (4).

Examples 1: Density of a s-tree [11, 12]. We define this measure by induction on full subtrees of a tree. Let \( T \) be a s-tree and \( v \) be a node of \( T \).

1. If \( v \) is a bottom node of \( T \) then \( \mu(T(v)) = 0 \).
2. Let \( v \) be a nonbottom node and:
   \[
   \mu_v = \max \{ \mu(T(v')) \mid v' \text{ is in } i(v) \}.
   \]
   Then:
   \[
   \mu(T(v)) = \begin{cases} 
   \mu_v, & \text{if } (\exists v_1, v_2 \text{ in } i(v)) \ [v_1 \neq v_2 \& \mu(T(v_1)) = \mu(T(v_2)) = \mu_0], \\
   \mu_v + 1, & \text{else.}
   \end{cases}
   \]
   \( \mu(T) = \mu(T(A)) \) is the density of \( T \).

2. Branching of a s-tree [6, 7] is the number \( b(T) \) of preterminal nodes of \( T \).

3. Capacity of a s-tree \( T \) is the number \( c(T) \) of all nodes of \( T \).

Définition 3.2: Let \( m \) be a complexity measure and \( S \) be a s-set. By \((m-) \) complexity of \( S \) we mean the function \( \lambda n.m_S(n) \) where:

\[
\lambda n.m_S(n) = \max \{ 0, m_S(T) \mid T \text{ in } S, |T| \leq n \},
\]
and:

\[
m_S(T) = \min \{ m(T') \mid T' \text{ in } S, t(T') = t(T) \}
\]
for all \( T \) in \( S \).

Besides this for \( z = t(T) \), \( T \) in \( S \), we set \( m_S(z) = m_S(T) \).

Définition 3.3: Let \( m \) be a complexity measure, \( f \) be a total nondecreasing function, and \( S \) be a s-set. We say that \( f \) is \( m \)-limiting \( S \) if:

(a) (\( \exists c \)) (\( \forall T \) in \( S \)) \( [m(T) \leq cf(|T|)] \) and

(b) there is a sequence of cs-trees \( T_1, T_2, \ldots \) in \( S \) (a fundamental sequence)
   such that the set \( \{ |T_i| \mid i > 0 \} \) is infinite and (3d) (\( \forall i \)) \( [dm(T_i) \geq f(|T_i|)] \).

This concept is very close to the notion of constructable function in automata

\[
\text{Remark As we observed in [6] the functions } \lambda n.n \text{ and } \log n \text{ (5) are respectively } b\text{-limiting and } \mu\text{-limiting the least free s-set containing:}
\]

\[
\begin{array}{c}
A \\
b \rightarrow A & A \rightarrow b \\
\end{array}
\]

(4) Some other examples of measures of importance for cf-grammar theory such as index, Yngve

\[
\text{measures, dispersion, selfembedding index, and so on, may be found in [6, 7].}
\]

(5) Functions that we use for measuring complexity are total nondecreasing functions from \( \mathbb{Z}_+ \)

\[
\text{into } \mathbb{Z}_+. \text{ For example, } \log n \text{ denotes the function } \lambda n.[\log_2(n+1)].
\]

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DEFINITION 3.4: Let \( f \) be a total nondecreasing unbounded function and \( m \) be a complexity measure. We say that \( f \) is \textit{limiting} \( m \) if:

(a) \((\forall k) (\exists c) (\forall T \in \mathcal{S}(\Sigma, W, k)) [m(T) \leq cf(|T|)]\) and

(b) there are \( k_0 \) and a sequence of s-tress \((T_1, \ldots, T_n, \ldots)\) in \( \mathcal{S}(\Sigma, W, k_0) \) (a fundamental sequence of \( f \)) and \( d > 0 \) such that \( dm(T_i) \geq f(|T_i|) \) for all \( i \).

For example, \( \log n \) is limiting \( \mu [6, 9] \) and \( \lambda n. n \) is limiting \( b \) and \( c \).

4. MONADIC RECURSION SCHEMES

There is an unsubstantial différence between the notion of a monadic recursion scheme under study and that of [4] and [5]. Nevertheless we outline here both their syntax and semantics.

I. SYNTAX: Treating monadic recursion schemes we give to symbols of \( \Sigma \) and \( W \) new names: \textit{basic} and \textit{defined function symbols} respectively.

Let \( \langle \mathcal{P}_i \rangle_{i \in N} \) be a system of countable pairwise disjoint alphabets (of switch function symbols) \( \mathcal{P}_i = \{ p_j | j \in N \} \) such that \( \mathcal{P}_i \cap (\Sigma \cup W) = \emptyset \) for all \( i \), \( \mathcal{P} = \bigcup_{i \in N} \mathcal{P}_i \), and \( x \) be a symbol not in \( \mathcal{P} \cup \Sigma \cup W \) (a variable symbol). Let \( \Sigma' \subseteq \Sigma \), \( W' \subseteq W \). A string \( z \) in \( (\Sigma' \cup W')^* x \) is a (monadic) term (over \( \Sigma', W' \)). A term \( z \) over \( \Sigma', W' \) is \textit{basic} if it doesn’t contain occurrences of defined function symbols.

DEFINITION 4.1: A monadic recursion scheme (MR-scheme) (over \( \Sigma, W \)) is a system \( E = (\Sigma_1, W_1, F_1, \{ e_1, \ldots, e_k \}) \) meeting the conditions:

1. \( \Sigma_1 \subseteq \Sigma \) and \( W_1 = \{ F_1, \ldots, F_k \} \subseteq W \) are finite alphabets;
2. \( e_i (1 \leq i \leq k) \) is a formal equation of the form:

\[
e_i: F_i x = (p^{n(i)}_{m(i)} x | u_{i1} x, \ldots, u_{in(i)} x),
\]

where \( p^{n(i)}_{m(i)} \) is a switch function symbol, \( u_{i1} x, \ldots, u_{in(i)} x \) are monadic terms over \( \Sigma_1, W_1, 1 \leq i \leq k \). We say that \( E \) \textit{defines} \( F_1 \). The set \( \{ p^{n(1)}_{m(1)}, \ldots, p^{n(k)}_{m(k)} \} \) is denoted by \( \mathcal{P}(E) \) and the class of all MR-schemes by \( \mathcal{E} \).

With the MR-scheme \( E \) in the definition 4.1 we associate a cf-grammar \( G(E) \) in the following regular way:

\[
G(E) = (\Sigma_1, W_1, F_1, R), \quad \text{where} \quad R = \bigcup_{i=1}^{k} R(e_i),
\]

and

\[
R(e_i) = \{ F_i \to u_{i1}, \ldots, F_i \to u_{in(i)} \}, \quad 1 \leq i \leq k.
\]
The associated cf-grammar will serve as a base for a semantic notion of a computation of a MR-scheme. Besides this it is a convenient means of syntactic classification of MR-schemes. For example, we call a MR-scheme $E$ linear if its associated grammar $G(E)$ is linear.

**II. SEMANTICS:** Let $E=(\Sigma_1, W_1, F_1, \{e_1, \ldots, e_k\})$ be a MR-scheme. An interpretation of $E$ is a system $I=(J, D)$, where $D$ is a set called a domain of $I$ and $J$ is a functional on $\Sigma_1 \cup \mathcal{P}(E) \cup \{x\}$ such that:

1. $J(f)$ is a total function from $D$ into $D$ for each $f$ in $\Sigma_1$;
2. $J(x)$ is an element of $D$, and
3. for each $n$ and each $p^n$ in $\mathcal{P}_n \cap \mathcal{P}(E)$, $J(p^n)$ is a total function from $D$ into $\{1, \ldots, n\}$. $I$ is naturally extendable to the set of basic terms:

\[
\begin{align*}
I(x) & = J(x), \\
I(f \cdot x) & = J(f)(I(x))
\end{align*}
\]

for all $f$ in $\Sigma_1$, $v$ in $\Sigma_1^*$.

An interpretation $I=(J, D)$ of $E$ is free (or Herbrand) if $D=\Sigma_1^*$, $J(x)=\Lambda$, and $J(f)(t) = ft$ for all $f$ in $\Sigma_1$ and $t$ in $D$.

A computation of a MR-scheme $E$ may be considered as a rightmost derivation of the grammar $G(E)$ controlled by an interpretation $I$ in the following sense.

**DEFINITION 4.2:** Let $I=(J, D)$ be an interpretation of the MR-scheme above. Let $X=y_1F_1y_2$ and $Y$ be two strings in $(\Sigma_1 \cup W_1)^*$ and $y_2$ be in $\Sigma_1^*$. Then:

\[X \Rightarrow_{EI} Y \text{ if } J(p^n(i))(I(y_2x)) = j\]

for some $j \leq n(i)$, $Y = y_1u_1y_2$ and the equation $e_i$ in $E$ is of the form:

\[e_i: \quad F_1x = (p^n(i)x | u_{i1}x, \ldots, u_{ij}x, \ldots, u_{in(i)}x).\]

A sequence $c(E, I)=(X_0, X_1, X_2, \ldots)$ is called an $I$-computation sequence of $E$ if

(a) $X_0 = F_1$, and (b) $X_i \Rightarrow_{EI} X_{i+1}$ for all $i \geq 0$. If the $I$-computation sequence $c(E, I)$ is finite, i.e. $c(E, I) = (X_0, X_1, \ldots, X_r)$ for some $r$, and its last string $X_r$ is in $\Sigma_1^*$ then it is called an $(I-)$ computation of $E$.

It is evident that for all interpretations $I$ such that $c(E, I)$ is a computation it is at the same time a rightmost complete derivation of the cf-grammar $G(E)$. The tree of this derivation [denoted $T(E, I)$] is called a tree of the $I$-computation $c(E, I)$. The set of all trees of computations of $E$, i.e. the set \{ $T(E, I) | I$ is an
interpretation of $E$ is denoted by $S(E)$. $E$ is unambiguous (ambiguous) if $S(E)$ is unambiguous (ambiguous). We will consider two partial value functions:

$$\text{TERMVAL}(E, I) = \begin{cases} t(T(E, I)) & \text{if } c(E, I) \text{ is a computation of } E, \\ \text{undefined otherwise} \end{cases}$$

$$\text{VAL}(E, I) = I(\text{TERMVAL}(E, I)x).$$

The set:

$$\text{TL}(E) = \{ \text{TERMVAL}(E, I) | I \text{ is an interpretation of } E \}$$

is called a term language of $E$.

**Définition 4.3:** We say that MR-schemes $E_1$, $E_2$ are termally (strongly) equivalent (notation $E_1 \equiv_r E_2$ and $E_1 \equiv_s E_2$ respectively) if $\text{TL}(E_1) = \text{TL}(E_2)$ [respectively $\lambda I. \text{VAL}(E_1, I) = \lambda I. \text{VAL}(E_2, I)$].

**Définition 4.4:** An equivalence $\equiv_r$ on $\mathcal{E}$ is called reasonable if $E_1 \equiv \equiv_r E_2$ implies $E_1 \equiv \equiv_s E_2$ for all $E_1, E_2$ in $\mathcal{E}$.

**Remark:** Strong equivalence of MR-schemes is of course a reasonable one. This follows directly from the fact that if $E_1 \equiv_r E_2$ then $\text{VAL}(E_1, I) = \text{VAL}(E_2, I)$ for each free interpretation $I$.

**Définition 4.5:** Let $\mathcal{E}_1$, $\mathcal{E}_2$ be two classes of MR-schemes and $\equiv_r$ be a reasonable equivalence on $\mathcal{E}$. We say that $\mathcal{E}_1$ is termally (strongly, $r$-) translatable into $\mathcal{E}_2$ (notation $\mathcal{E}_1 \Rightarrow_r \mathcal{E}_2$, $\mathcal{E}_1 \Rightarrow_s \mathcal{E}_2$, and $\mathcal{E}_1 \Rightarrow_r \mathcal{E}_2$ respectively) if for each MR-scheme $E_1$ in $\mathcal{E}_1$ there is a termally (strongly, $r$-) equivalent to $E_1$ MR-scheme $E_2$ in $\mathcal{E}_2$.

5. COMPLEXITY CLASSES AND MIMEOINVARIANT MEASURES

Application of complexity measures to structure sets leads in a straightforward manner to a natural notion of computation complexity of MR-schemes. In fact, we measure the complexity of trees in $S(E)$ bearing in mind that these are tree representations of the corresponding computations of $E$. So, we arrive at the following definition.

**Définition 5.1:** Let $m$ be a complexity measure and $E$ be a MR-scheme. By $m$-complexity of $E$ we mean the function $m_E = \lambda n.m_{S(E)}(n)$.

So that to stratify the class $\mathcal{E}$ of all MR-schemes into complexity classes we consider the following relations on the set of total functions on $Z_+$. vol. 15, n°1, 1981
NOTATION: Let \( g \) and \( f \) be total functions. \( g < f \) means \( \lim_{n \to \infty} g(n)/f(n) = 0 \), \( g \leq f \) means \( \lim_{n \to \infty} g(n)/f(n) \leq c \) for some \( c \) in \( \mathbb{Z}_+ \), i.e. \((\exists c) (\forall n) [g(n) \leq cf(n)]\), \( g \geq f \) means \( g \leq f \) & \( f \leq g \), and \( g \succ f \) means that \( g \leq f \) but not \( g < f \).

**Definition 5.2:** Let \( f \) be a total nondecreasing function. The set \( \mathcal{E}^m_f = \{ E \mid E \text{ is a } MR\text{-scheme, } m_E \leq f \} \) is a \((m-)\) complexity class. Let \( c \geq 0 \) be an integer. The set \( \mathcal{E}^m_c = \{ E \mid E \text{ is a } MR\text{-scheme, } (\forall n) [m_E(n) \leq c] \} \) is a \( c \)-bounded \((m-)\) complexity class. \( \mathcal{E}^m_{\text{Const.}} = \bigcup_{i=0}^{\infty} \mathcal{E}^m_i \) is called a bounded \((m-)\) complexity class.

Of course, \( \mathcal{E}^m_f = \mathcal{E} \) for any function \( f \) limiting a complexity measure \( m \).

**Remark:** All these notions can be (and they were) applied to \( cf\)-grammars. For example, the complexity function of a \( cf\)-grammar \( G \) is defined as \( m_G = \lambda n. m_{S(G)}(n) \) for all complexity measures \( m \).

**Definition 5.3:** A complexity measure \( m \) is nondegenerate if there is an unambiguous \( MR\)-scheme \( E \) whose \( m \)-complexity function \( m_E \) is unbounded. Meanwhile, the definition 5.1 is too general to be workable. We are looking for a reasonable class of measures which (1) make the complexity stratifications \( \{ \mathcal{E}^m_f \} \) and \( \{ \mathcal{E}^m_c \} \) nontrivial, and (2) have close values on “topologically” similar trees. We attain both objectives imposing simple conditions on complexity measures. These conditions formalize a vague formulation of our second objective in terms of mimeomorphisms. In fact we assume that a \( s \)-tree \( T \) (strictly) linear mimeomorphic to another \( s \)-tree \( T' \) is only negligibly different from it from the complexity point of view. The complexity measures meeting this condition are called mimeoinvariant.

**Definition 5.4 (main definition):** A nondegenerate complexity measure \( m \) is mimeoinvariant if it satisfies the axioms:

\[
A \quad \left\{ \begin{array}{l}
(\exists c_A) \ (\forall T_1, T_2 \text{ in } \mathcal{S}) \smallskip
T_1 \leq^l T_2 \Rightarrow m(T_2) \leq c_A m(T_1),
\end{array} \right.
\]

\[
B \quad \left\{ \begin{array}{l}
(\exists d_B) \ (\forall T_1, T_2 \text{ in } \mathcal{S}) \smallskip
[T_1 \leq T_2 \Rightarrow m(T_1) \leq d_B m(T_2)],
\end{array} \right.
\]

and it is asymptotically mimeoinvariant if it meets the conditions:

\[
A^a \quad \left\{ \begin{array}{l}
(\exists c \geq 0) \ (\forall T_1, T_2 \text{ in } \mathcal{S}) \smallskip
[T_1 \leq^l T_2 \Rightarrow m(T_2) \leq m(T_1) + c],
\end{array} \right.
\]

\[
B^a \quad \left\{ \begin{array}{l}
(\forall T_1, T_2 \text{ in } \mathcal{S}) \smallskip
[T_1 \leq T_2 \Rightarrow m(T_1) \leq m(T_2)].
\end{array} \right.
\]
The measures $\mu$ and $b$ are obviously mimeoinvariant and asymptotically mimeoinvariant, while $c$ is not (it must be noted that $c$ does not depend on form of $s$-trees). Note also that if $m$ is (asymptotically) mimeoinvariant then all linear MR-schemes fall into $E^m_{\text{Const.}}$.

6. RIGHT-NORMAL FORM cf-GRAMMARS AND COMPLEXITY OF INDIVIDUAL MR-SCHEMES

Let $m$ be a complexity measure and $f$ be an unbounded nondecreasing function. Let us say that a cf-grammar $G$ is of nonreducible $m$-complexity $f$ if: (1) $m_G \leq f$ and (2) for no cf-grammar $G'$ such that $m_G < f L(G) = L(G')$.

A similar notion may be introduced for MR-schemes. This notion however relates upon a choice of equivalence relation among JVTR-schemes. We will consider only reasonable equivalences. So let $\equiv_r$ be some reasonable equivalence relation on $\mathcal{S}$ and $\Rightarrow_r$ be the corresponding translatability relation.

**Definition 6.1:** A MR-scheme $E$ is of $r$-nonreducible $m$-complexity $f$ if: (1) $E$ is in $E^m_f$ and (2) for no MR-scheme $E'$ such that $m_E < f E' \equiv_r E$.

Remark: If there is a MR-scheme of $r$-nonreducible $m$-complexity $f$ then the class $E^m_f$ is unempty and is $r$-translatable neither into any class $E^m_g$ such that $g < f$, nor into $E^m_{\text{Const.}}$.

In this section we give the conditions sufficient for a function and a constant to be nonreducible MR-scheme complexity bounds.

We start with a few simple observations.

**Definition 6.2:** A cf-grammar is in a right-normal form if all its productions are of the form $A \rightarrow \varphi u$, $u$ in $\Sigma^+$. 

**Proposition 6.1:** For each cf-grammar $G$ in right-normal form there is a MR-scheme $E$ such that $S(E) = S(G)$ and thus $m_E = m_G$.

[Of course, this is a scheme such that $G(E) = G$.]

The next proposition follows directly from the proof of the theorem 2.5 in [4].

**Proposition 6.2:** For every MR-scheme $E$ there are a MR-scheme $\hat{E}$ (unambiguous if $E$ is unambiguous) with $G(\hat{E})$ in right-normal form and a bijection $q: \mathcal{I} \rightarrow \mathcal{I}$ on the set $\mathcal{I}$ of all interpretations, such that $T_1 = T(E, I)$ exists iff $T_2 = T(\hat{E}, q(I))$ exists, and in the case they exist $t(T_1) = t(T_2)$ and $T_2 \leq t T_1$.

From these propositions we have:

**Corollary 6.1:** For every mimeoinvariant complexity measure $m$ and for every MR-scheme $E$ there is a cf-grammar $G_E$ in right-normal form such that $m_{G_E} \preceq m_E$ and $L(G_E) = TL(E)$.
COROLLARY 6.2: For any mimeoinvariant complexity measure \( m \), any reasonable equivalence relation \( \equiv_r \), and any cf-grammar \( G \) in right-normal form of nonreducible \( m \)-complexity \( f \) there is a MR-scheme \( E \) of \( r \)-nonreducible \( m \)-complexity \( f \).

In our papers [6, 8] we have developed a technics of constructing cf-grammars of nonreducible \( m \)-complexities. The above-stated propositions permit reconstruction of these cf-grammars into MR-schemes of nonreducible \( m \)-complexity in the case they are in right-normal form.

DEFINITION 6.3: A function \( f \) is semihomogeneous if

\[
(\forall c_1) (\exists c_2) (\forall n_1, n_2) \\
[n_1 \leq c_1 \land n_2 > f(n_1) \leq c_2 f(n_2)].
\]

REMARK: A semihomogeneous function cannot of course be superexponential.

THEOREM 6.1: Let \( \equiv_r \) be a reasonable equivalence relation on \( E \), \( m \) be a mimeoinvariant complexity measure, and \( f \) be a nondecreasing unbounded semihomogeneous function \( m \)-limiting the s-set \( S(E) \) for a MR-scheme \( E \). Then there is a MR-scheme \( E_f \) of \( r \)-nonreducible \( m \)-complexity \( f \).

Proof: Let \( m, r, f \) and \( E \) be as above. First of all we associate with \( E \) the MR-scheme \( \hat{E} \) as in the proposition 6.2 and consider the cf-grammar \( G(\hat{E}) \). Then we carry out the following construction originating from [6, 8]. Let \( G(\hat{E}) = (\Sigma_1, W_1, A, P) \). We choose four new symbols \( a, b, c, d \) in \( \Sigma - \Sigma_1 \) and choose a symbol \( F_n \) in \( W_1 \) for each production \( n \) in \( P \). After this we set:

\[
W_1' = \{ F_n | \pi \text{ in } P \} \cup W_1 \quad \text{and} \quad P' = \bigcup_{\pi \text{ in } P} P',
\]

where for each \( \pi = F \rightarrow \phi \) in \( P \):

\[
P_{\pi}' = \{ F \rightarrow cF_d, \ F \rightarrow aF_b, \ F \rightarrow a\phi b \}.
\]

As a result, we obtain the cf-grammar:

\[
\Gamma [G(\hat{E})] = (\Sigma_1 \cup \{ a, b, c, d \}, W_1', A, P').
\]

Since \( m \) is mimeoinvariant we infer from the proposition 6.2 that the function \( f \) is \( m \)-limiting the s-set \( S(\hat{E}) = S(G(\hat{E})) \). This being clear, we use the following fact proven in [6] (thm. 9.4) and in [8] (thm. 1).

PROPOSITION 6.3: Let \( m \) be a mimeoinvariant complexity measure, \( f \) be a nondecreasing semihomogeneous function \( m \)-limiting the set \( S(G) \) of a cf-grammar \( G \). Then the cf-grammar \( \Gamma [G] \) associated with \( G \) as above is of nonreducible \( m \)-complexity \( f \).
Thus we see that $\Gamma[G(\hat{E})]$ is a cf-grammar in right-normal form and of nonreducible $m$-complexity $f$. Hence by the corollary 6.2 we associate with $\Gamma[G(\hat{E})]$ the needed MR-scheme $E_f$ of $r$-nonreducible $m$-complexity $f$.

Q.E.D.

**Remark:** Consider the MR-scheme:

$$E_0: \quad Fx = (px | FFfx, gx).$$

It is evident that the function $\log n$ is $\mu$-limiting the $s$-set $S(E_0)$ and the function $\lambda n \cdot n$ is $b$-limiting this $s$-set. Since both these measures are mimeoinvariant, both functions are semihomogeneous, and $G(E_0)$ is in right-normal form the construction of the theorem 6.1 delivers a MR-scheme $E_1$ of $r$-nonreducible $\mu$-complexity $\log n$ and of $r$-nonreducible $b$-complexity $\lambda n \cdot n$ for each reasonable equivalence relation $\equiv_r$. Though the proof of the theorem 6.1 defines this MR-scheme entirely we cite it out here:

$$E_1: \begin{cases} 
Fx = (p_1 x | cF_1 dx, cF_2 dx), \\
F_1 x = (p_2 x | aF_1 bx, FFfx), \\
F_2 x = (p_3 x | aF_2 bx, gx).
\end{cases}$$

The same reduction leads to an infinite hierarchy of MR-schemes of bounded complexity.

**Theorem 6.2:** Let $m$ be an asymptotically mimeoinvariant complexity measure, $\equiv_r$ be some reasonable equivalence relation on $\mathcal{E}$. Then there is $c \geq 0$ such that for any cs-tree $T$ of $m$-complexity $k$ there exists a MR-scheme $E_k$ in $\mathcal{E}^m_{k+c}$ which is not $\equiv_r$-equivalent to any MR-scheme in any class $\mathcal{E}_m^l$ with $l < k$.

**Proof:** Consider a cs-tree $T=(\Delta, l)$ such that $\bar{\Delta} > 1$ and $m(T) = k$. Let $\Sigma_T$ be the set of all terminal labels of nodes in $\Delta$. We add four new terminal symbols $a, b, c, d$ in $\Sigma - \Sigma_T$ to $\Sigma_T$ and set $\Sigma_1 = \Sigma_T \cup \{a, b, c, d\}$. With each nonbottom node $v$ of $T$ we associate two nonterminals $A_v, B_v$ in $W$ in such a way that $\{A_v, B_v\} \cap \{A_u, B_u\} = \emptyset$ for $v \neq u$ and set $W_1 = \{A_v, B_v | v \in \Delta\}$. To each nonbottom node $v$ of $T$ such that $i(v) = \{v_1, \ldots, v_n\}$ and $v_1 \prec v_2 \prec \ldots \prec v_n$ we relate the system of productions $P(v)$:

$$A_v \rightarrow cB_v d,$$
$$B_v \rightarrow aB_v b,$$
$$B_v \rightarrow aX_1 X_2 \ldots X_n b,$$
where:

\[ X_i = \begin{cases} 
A_{v_i} & \text{if } l(v_i) \text{ is in } W, \\
(l(v_i)) & \text{if } l(v_i) \text{ is in } \Sigma_T. 
\end{cases} \]

Thus we obtain the \( cf \)-grammar \( G_T = (\Sigma_1, W_1, A_\lambda, P) \), where \( P = \bigcup_{v \in \Delta} P(v) \). In the first place we note that for each \( cs \)-tree \( T' \) in \( S(G_T) \) \( T \leq^s T' \) holds. Since \( m \) is asymptotically mimeoinvariant there is \( c \geq 0 \) such that \( m(T') \leq m(T) + c = k + c \) for all \( T' \) in \( S(G_T) \) and hence \( m_{G_T} \leq k + c \). Secondly we observe that \( G_T \) is in right-normal form. Thus by proposition 6.1 there is a \( MR \)-scheme \( E_T \) such that \( S(E_T) = S(G_T) \) and hence \( E_T \) is in \( \mathcal{E}^m_{k+c} \). In the paper [8] (corollary 3 from theorem 1) we prove that if a \( cf \)-grammar \( G \) is equivalent to \( G_T \) then there is no such \( l < k \) that \( m_G < l \). Assume that there is a \( MR \)-scheme \( E \) such that \( E \equiv_r E_T \) and \( E \) is in \( \mathcal{E}^m_l \) for some \( l < k \). By proposition 6.2 and the axiom \( B^4 \) in the definition 5.4 there is a \( cf \)-grammar \( G_E \) such that \( m_{G_E} \leq m_E \) and \( L(G_E) = TL(E) \). Hence we conclude that \( m_{G_E} \leq l < k \). But \( L(G_T) = TL(E_T) = TL(E) = L(G_E) \), a contradiction.

Q.E.D.

**Corollary 6.3:** Let \( m \) be an asymptotically mimeoinvariant complexity measure and \( \equiv_r \) be a reasonable equivalence relation on \( \mathcal{E} \). Then there is an infinite sequence of nonnegative integers \( n_1 < n_2 < \ldots < n_k < \ldots \) and \( MR \)-schemes \( E_1, E_2, \ldots, E_k, \ldots \) such that for all \( k > 1 \) \( E_k \) is in \( \mathcal{E}^m_{n_k} \) and for no \( l \leq n_{k-1} \) and no \( E \) in \( \mathcal{E}^m_l \) \( E \equiv_r E_k \).

**7. Infinite Hierarchies of \( MR \)-Schemes of Nonreducible Complexities**

Our main objective is to exhibit conditions under which for a mimeoinvariant complexity measure \( m \) and for a reasonable equivalence relation \( \equiv_r \) there is an infinite hierarchy of \( MR \)-schemes \( \{ E_{f_j} \} \) of \( r \)-nonreducible \( m \)-complexities \( f_1 > f_2 > f_3 > \ldots \). This will give hierarchy of complexity classes \( \mathcal{E}^m_{f_1} \equiv \mathcal{E}^m_{f_2} \equiv \mathcal{E}^m_{f_3} \equiv \ldots \) such that for no \( i, j, i < j, \mathcal{E}^m_{f_i} \Rightarrow \mathcal{E}^m_{f_j} \). A similar hierarchy \( \{ G_{f_i} \} \) of \( cf \)-grammars of nonreducible complexities is described in [6] (theorem 9.5 and its corollaries) and in [8] (theorem 2 and its corollaries). Simple reductions of the preceding section are unfit however for reconstruction of \( \{ G_{f_i} \} \) into \( \{ E_{f_j} \} \). The reason is that the grammars \( G_{f_i} \) are not in right-normal form and even worse: the traditional reductions of \( G_{f_i} \) to right-normal form increases their complexity to the maximal. Thus we must strengthen the results of [6, 8] and expose an infinite hierarchy of \( cf \)-grammars in right-normal form of nonreducible complexity. To this end we need some notions and notation related to Turing machines.

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NOTATION: Let $\Sigma^T \subset \Sigma$ be an infinite alphabet. We consider the class $\mathcal{M}(\Sigma^T)$ of all one-tape one-head deterministic Turing machines such that for each $M$ in $\mathcal{M}(\Sigma^T)$ the set $K_M$ of states of $M$ and the set $V_M$ of tape symbols of $M$ are both subsets of $\Sigma^T$. We use the standard encoding of situations and computations of Turing machines. A situation of $M$ is a string $Q$ in $V_M^* K_M V_M^*$. If $q_0$ is a start and $q_f$ is a final state of $M$ then situations $Q_1 = q_0 z_1$ and $Q_2 = q_f z_2, z_1 z_2$ in $V_M^*$, are starting and final respectively. The symbol $Q$ with or without indices we reserve for the sequel as a variable over the set of Turing machine situations or their substrings. For situations $Q_1, Q_2$ of $M Q_1 \rightarrow M Q_2$ means that $Q_2$ immediately follows $Q_1$ in a computation of $M$. $n_M(Q)$ denotes the length of the situation $Q'$ such that $Q \rightarrow Q'$. A substring of a situation $Q$ is active if it contains an occurrence of a state and passive if it doesn’t. Let $\$ be a symbol in $\Sigma - \Sigma^T$. A $x$-computation record of $M$ is the string $P_M(x) = \$ Q_1 \$ Q_2 \ldots Q_{n-1} \$ Q_n$, where $Q_1 = q_0 x$ is the starting situation with input string $x$, $Q_n$ is a final situation and $Q_i \rightarrow M Q_{i+1}$ for all $1 \leq i < n$. In the class $\mathcal{M}(\Sigma^T)$ we select the subclass $\mathcal{N}(\Sigma^T)$ of all Turing machines such that:

1. the record function $\lambda x. P_M(x)$ is total;
2. the function $\lambda x. |P_M(x)|$ is nondecreasing with respect to the lengths of strings $x$, i.e. $|x_1| \geq |x_2|$ implies $|P_M(x_1)| \geq |P_M(x_2)|$.

With each Turing machine $M$ we associate the following integervalued function $p_M$ which is in a sense inverse to the record-length function $\lambda x. |P_M(x)|$:

$$p_M(n) = \begin{cases} \text{if } \exists x \{ |P_M(x)| \leq n - |x| \} \\
\text{then max } \{ r | (\exists x) | |x| = r \& 2 |P_M(x)| \leq n - r \} \\
\text{else } 1.\end{cases}$$

Remark: It is easily seen that for each machine $M$ in $\mathcal{N}(\Sigma^T)$ the function $p_M(n)$ is recursive and $p_M(n) \leq n$ for all $n$.

THEOREM 7.1: Let $m$ be a mimeoinvariant complexity measure and $f$ be a semihomogeneous unbounded nondecreasing function $m$-limiting the s-set $S(G)$ of a cf-grammar $G = (\Sigma_1, W_1, I_1, P_1)$. Then for each Turing machine $M$ in $\mathcal{N}(\Sigma^T)$ there is a right-normal form cf-grammar $G_M$ of non-reducible $m$-complexity $\lambda n.f(p_M(n))$.

Proof: To expose the needed cf-grammar it is convenient to describe first the language it generates. To this end we introduce several operations and predicates. Let $m, f, G,$ and $M$ be fixed.
NOTATION: 1. Consider \( L \subseteq \Sigma^* \) and:

\[
U_1, U_2 \subseteq \Sigma^* \times \Sigma^* . \quad U_1 = \{ z_1 z_2 \mid z \text{ in } L, (z_1, z_2) \text{ in } U_1 \},
\]

\[
U_2 = \{ (z_1 z_2, z_2 z_1) \mid (z_1, z_2) \text{ in } U_1, (z_2, z_1) \text{ in } U_2 \},
\]

\[
U_1(0) = U_1, U_1(k+1) = U_1(k) \bullet U_1, \quad U_1^* = \bigcup_{k=0}^{\infty} U_1(k).
\]

2. Let \( Q_1, Q_2 \) be two strings in \( V_M^* \cup V_M^* K_M V_M^* \). Then \( ERR_M(Q_1, Q_2) \) means that either: (a) \( Q_1 \) is active, \( |Q_1| = n_M(Q_1) \), but \( Q_2 \) does not coincide with the \( M \)-situation \( Q \) such that \( Q_1 \rightarrow Q \), or (b) \( Q_1 \) is passive, \( |Q_1| = |Q_2| \), but \( Q_1 \neq Q_2 \).

Now we proceed to the description of the grammar \( G_M \) and the language \( L_M = L(G_M) \).

1. First of all we apply \( \Gamma \) to the grammar \( G = (\Sigma_1, W_1, I_1, P_1) \) the construction outlined above in the proof of the theorem 6.1, relating to it the grammar \( \Gamma[G] = (\Sigma_0, W_0, I_0, P_0) \) with \( \Sigma_0 = \Sigma_1 \cup \{ a, b, c, d \} \). The language \( L(\Gamma[G]) \) we denote by \( L_0 \).

2. Then we introduce the following system of languages and pair languages ( is a symbol in \( \Sigma - (\Sigma^* \cup \{ \$ \}) \)):

\[
L_1 = \{ c \alpha^j y \beta^j d \mid j \geq 0, \ y \in \{ \Lambda \} \cup \{ c \cup \Sigma_1 \} \Sigma_0^* (\{ d \} \cup \Sigma_1) \}.
\]

\[
L_2 = \{ Q^R \$ x \mid Q \text{ is a situation of } M, \ x \in \Sigma_0^*, \ |x| \geq |Q| \} \ \{^6\}.
\]

\[
U_0 = \{ (\Lambda, \star) \}.
\]

\[
U_1 = \{ (Q^R, \$, \$ Q) \mid Q \text{ is a final situation of } M \}.
\]

\[
U_2 = \{ (Q^R, \$, \$ Q) \mid Q \text{ is a situation of } M \}.
\]

\[
U_3 = \{ (\Lambda, \$, \$ Q) \mid Q \text{ is a situation of } M \}.
\]

\[
U_4 = \{ (Q^R_2, \$, \$ Q_1, \$, \$ Q_3) \mid Q_1, Q_2, Q_3 \}
\]

are situations of \( M, \ |Q_1, Q_2| < |Q_3| + n_M(Q_3) \}. \]

\[
U_5 = \{ (Q^R_2, \$, \$ Q_1, \$, \$ Q_3) \mid Q_1, Q_2, Q_3 \}
\]

are situations of \( M, \ |Q_2| + n_M(Q_1) < 2 |Q_3| \}. \]

\[
U_6 = \{ (Q^R_2, Q_{21}, \$, \$ Q_{11}, \$, \$ Q_{31}, Q_{32}) \mid Q_1, Q_{21}, Q_{22}, Q_{31}, Q_{32} \}
\]

are situations of \( M, \ |Q_{31}| \geq |Q_{1, Q_{21}}|/3 - 1 \). \]

\[
U_7 = \{ (Q^R_2, Q_{12}, Q_{11}, \$, \$ Q_{31}, Q_{32}) \mid Q_{11}, Q_{12}, Q_{2}, Q_{31}, Q_{32} \}
\]

are situations of \( M, \ |Q_{32}| \geq |Q_{2, Q_{12}}|/3 - 2 \), \( ERR_M(Q_{32}, Q_{22}) \). \]

\[
U_N = \{ (Q^R, \$, \$ Q) \mid Q \text{ is a starting situation of } M \}.
\]

\((^6) \ z^R \) denotes the reversal of a string \( z : [A]^R = \Lambda, [wa]^R = a[w]^R. \)

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Now the language $L_M$ is defined as $L_M = \bigcup_{i=1}^{6} L_{Mi}$, where:

$\begin{align*}
L_{M_1} &= (U_0^* \cdot U_1 \cdot U_2^* \cdot U_8)^* L_0, \\
L_{M_2} &= (U_0^* \cdot U_1 \cdot U_2 \cdot U_3 \cdot U_1 \cdot U_2^*)^* L_1, \\
L_{M_3} &= (U_0^* \cdot U_2 \cdot U_5 \cdot U_3 \cdot U_2^*)^* L_1, \\
L_{M_4} &= (U_0^* \cdot U_2 \cdot U_3 \cdot U_6 \cdot U_2^*)^* L_1, \\
L_{M_5} &= (U_0^* \cdot U_2 \cdot U_7 \cdot U_5 \cdot U_2^*)^* L_1, \\
L_{M_6} &= (U_0^* \cdot U_2 \cdot U_3)^* L_2.
\end{align*}$

4. So that to specify the needed $cf$-grammar $G_M$ let us notice that:

(a) $L_2 = U_9 \cdot \{ \$ \}$, where $U_9 = \{(Q^R, x) | Q$ is a situation of $M$, $x$ in $\Sigma_0^+, |x| \geq |Q| \}$;

(b) $L_1 = (U_{10} \cdot U_{11})^* R$, where $U_{10} = \{(c, d)\}$, $U_{11} = \{(a, b)\}$, $R = \{A\} \cup (\{c\} \cup \Sigma_1) \Sigma_0^* (\{d\} \cup \Sigma_1)$ is regular;

(c) for each pair language $U_{ij}, 0 \leq i \leq 11$, and each pair $(z, w)$ in it $w \neq \Lambda$, so, there is a linear function $g_j$ for each $0 \leq i \leq 11$ such that for all $(v, u)$ in $U_j |v| \leq g_j(|u|)$;

(d) for each regular language there is a right-linear $cf$-grammar generating it.

From (a)-(d) it follows directly that there are unambiguous right-normal form linear $cf$-grammars $G_{M_0}, G_{M_2}, G_{M_3}, G_{M_4}, G_{M_5}, G_{M_6}$ generating respectively the languages:

$L_{M_0} = (U_0^* \cdot U_1 \cdot U_2^* \cdot U_8)^* \{ I_0 \}, \quad L_{M_2}, L_{M_3}, L_{M_4}, L_{M_5}, L_{M_6}.$

Let $\overline{W}_j$ and $\overline{P}_j$ be respectively nonterminal alphabets and production sets of the grammars $G_M$ for $j \in \{0, 2, 3, 4, 5, 6\}$. We may assume without loss of generality that:

1. All grammars $G_{M_0}, G_{M_2}, G_{M_3}, G_{M_4}, G_{M_5}, G_{M_6}$ have a common axiom $I$;

2. The alphabets $W_0, \overline{W}_0 - \{ I \}, \overline{W}_2 - \{ I \}, \overline{W}_3 - \{ I \}, \overline{W}_4 - \{ I \}, \overline{W}_5 - \{ I \}, \overline{W}_6 - \{ I \}$ are pairwise disjoint.

Let us denote by $\Sigma_1^M$ the alphabet $K_M \cup V_M \cup \{ \$, \$ \} \cup \Sigma_0$ and by $G_{M_1}$ the $cf$-grammar $(\Sigma_1^M, \overline{W}_1, I, \overline{P}_1)$, where $\overline{W}_1 = W_0 \cup \overline{W}_0$ and $\overline{P}_1 = P_0 \cup \overline{P}_0$.

Finally let us set $G_M = (\Sigma_1^M, W^M, I, P^M)$, where $W^M = \bigcup_{i=1}^{6} \overline{W}_j$ and $P^M = \bigcup_{i=1}^{6} \overline{P}_j$.

It is easily seen that $L(G_{M_1}) = L_{M_1}$ and $L(G_M) = L_M$. By the construction all the $cf$-grammars we have described, including $G_M$, are in right-normal form. It remains to prove that $G_M$ is a $cf$-grammar of nonreducible $m$-complexity $\lambda n.f(p_M(n))$. 

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UPPER BOUND: \( m_{G_M} \leq f(p_M) \). To establish this bound we will show that there is a \( c > 0 \) such that \( m_{G_M}(z) \leq cf(p_M(z)) \) for all \( z \) in \( L_M \). Since the complexity measure \( m \) is mimeoinvariant there evidently is an integer \( c_m > 0 \) such that for each linear \( s \)-tree \( T \) \( m(T) < c_m \). This implies that for each \( cs \)-tree \( T \) in \( S(G_M), j \) in \( \{0, 2, 3, 4, 5, 6\} \), \( m(T) < c_m \), thus \( m_{G_M}(z) < c_m \) for all \( z \) in \( \bigcup_{j=2}^{6} L_{M_j} \). It suffice to establish the upper bound for all \( z \) in \( L_{M_1} \).

Now, let us take a string \( z \) in \( L_{M_1} \). \( z \) may be represented in form:
\[
z = Q_n^R \, \delta \, Q_{n-1}^R \, \delta \cdots \delta \, Q_1^R \, \delta \, Q_2 \, \delta \, Q_3 \, \delta \cdots \delta \, Q_{n-1} \, \delta \, Q_n \, \delta^k,
\]
for some \( k \geq 0, n > 1 \), starting situation \( Q_1 \) of \( M \), final situation \( Q_n \) of \( M \), \( 1 < j < n \), and \( x \) in \( L_0 \). Three alternatives arise.

1. The string \( \delta Q_1 \, \delta Q_2 \, \delta \cdots \delta Q_n \) is not a computation record of \( M \). In such a case there is a \( i, 1 \leq i < n \), such that \( Q_i \neq Q_{i+1} \). Let \( i_0 \) be the least such \( i \). The computation error \( Q_{i_0} \delta Q_{i_0+1} \) may be of the following four kinds.

\( a \) \( |Q_{i_0+1}| > n_M(Q_{i_0}) \). In this case we find that:
\[
Q_{i_0-1}^R \, \delta \, Q_2^R \, \delta \, Q_1^R \, \delta \, Q_2 \, \delta \, Q_3 \, \delta \cdots \delta \, Q_{i_0-1} \, \delta \, Q_{i_0},
\]
belongs to \( (U_3 \bullet U_2^*) \, \square \, L_1 \), \( ([\delta Q_{i_0} \, \delta Q_{i_0+1}]^R, \, \delta Q_{i_0+1}) \) belongs to \( U_5 \) because:
\[
2 |Q_{i_0+1}| > |Q_{i_0}+| + n_M(Q_{i_0}), \, ([\delta Q_{i_0+2} \, \delta \cdots \delta Q_n]^R, \, \delta Q_{i_0+2} \cdots \delta Q_n \, \delta^k)
\]
is in \( U_0^* \, \bullet \, U_2^* \), and \( x \) is in \( L_1 \) because \( L_0 \subseteq L_1 \). Thus \( z \) belongs to \( L_{M_3} \), there is a linear \( cs \)-tree \( T \) in \( S(G_M) \) with the yield \( z \) and therefore \( m_{G_M}(z) < c_m \).

\( b \) \( |Q_{i_0+1}| < n_M(Q_{i_0}) \). In this situation we see that:
\[
Q_{i_0-1}^R \, \delta \, Q_2^R \, \delta \, Q_1 \, \delta \, Q_2 \, \delta \, Q_3 \, \delta \cdots \delta \, Q_{i_0-1},
\]
belongs to \( U_2^* \, \square \, L_1 \), \( ([\delta Q_{i_0} \, \delta Q_{i_0+1}]^R, \, \delta Q_{i_0}) \) is in \( U_4 \) because:
\[
|Q_{i_0}| + n_M(Q_{i_0}) > |Q_{i_0} \, \delta Q_{i_0+1}|,
\]
\( (\Lambda, \, \delta Q_{i_0+1}) \) is in \( U_3 \), \( ([\delta Q_{i_0+2} \, \delta \cdots \delta Q_n]^R, \, \delta Q_{i_0+2} \, \cdots \delta Q_n \, \delta^k) \) belongs to \( U_0^* \, \bullet \, U_2^* \), and \( x \) belongs to \( L_1 \). Thus \( z \) is in \( L_{M_2} \) and therefore \( m_{G_M}(z) < c_m \).

The bounds established in \( a \) and \( b \) show that in the rest we may assume without loss of generality that \( |Q_{i_0+1}| = n_M(Q_{i_0}) \). We need some additional notions and notation for the analysis to follow.

**NOTATION:** Let \( Q = z_1 \alpha \beta \gamma z_2 \) be the representation of a situation of \( M \) such that \( |z_1| = |z_2|, |\beta| \leq 1 \), and \( \alpha, \gamma \) are in \( V_M \cup K_M \). A central partition of \( Q \) is the unique
partition \( Q = l_M(Q) r_M(Q) \) such that:

(a) if \( z_j \alpha \beta \) is active then \( l_M(Q) = z_1 \alpha \beta \gamma \) and \( r_M(Q) = z_2 \);
(b) if \( \gamma z_2 \) is active then \( l_M(Q) = z_1 \) and \( r_M(Q) = \alpha \beta \gamma z_2 \).

2. Let \( Q_1, Q_2 \) be two situations of \( M \) such that \( |Q_2| = n_M(Q_1) \). We call a partition \( Q_2 = Q_2' Q_2'' \) of \( Q_2 \) \( Q_1 \)-derivative if either \( l_M(Q_1) \) is active and \( |Q_2'| = n_M(l_M(Q_1)) \), or \( r_M(Q_1) \) is active and \( |Q_2''| = n_M(r_M(Q_1)) \).

Let us return to the proof.

Let \( Q = Q_b \), \( Q'' \) be the central partition of \( Q_b \) and \( Q_{b+1} = Q_b' Q_{b+1}'' \) be the \( Q_b \)-derivative partition of \( Q_{b+1} \). There are four additional cases.

c) \( Q_{b} \) is active and \( Q'_b \neq Q_{b+1}' \). First of all we have \( |Q_{b+1}| = n_M(Q_b) \) and \( |Q''_{b+1}| = |Q''_b| \). From this follows:

\[
|Q''_{b+1}| \leq |Q''_b| + |Q_b| + 1 \leq |Q''_b| + 2|Q''_b| + 3 + 1 = 3|Q''_{b+1}| + 4.
\]

Thus:

\[
|Q''_{b+1}| \geq \left\lceil \frac{|Q_{b+1} Q''_b| - 4}{3} \right\rceil - 2.
\]

Besides this we have \( ERR_M(Q'_b, Q_{b+1}') \). That is why in this case:

\[
Q_{b-1}' \cdots Q_2' \subseteq Q_1' \subseteq Q_2 \cdots \subseteq Q_{b+1}' \subseteq Q_n' \quad \text{is in} \quad (U_3 \bullet U_2') \subseteq L_1,
\]

\[
([Q'_b, Q'_b', \ldots, Q'_n], [Q_{b+1}'', Q_{b+1}'', \ldots, Q_{b+1}'', \ldots, Q_n'']) \quad \text{is in} \quad U_7.
\]

and:

\[
([Q_{b+2}''', \ldots, Q_n''], [Q_{b+3}''', \ldots, Q_n''', \ldots, Q_n''''), \ldots] \quad \text{is in} \quad U_0 \bullet U_2'.
\]

Therefore \( z \) is in \( L_{M5} \).

d) \( Q'_b \) is passive and \( Q_{b+1}' \neq Q_{b+1}' \). Since \( Q'_b \) is passive we have \( |Q'_{b+1}| < |Q''_{b+1}| \), \( |Q_b'| \leq |Q_{b+1}'| + 1 \) and therefore \( |Q_b + 1 Q''_b| \leq 3|Q''_{b+1}| \). Hence

\[
|Q''_{b+1}| > \left\lceil \frac{|Q_{b+1} Q''_b|}{3} \right\rceil - 2.
\]

Of course, \( ERR_M(Q'_b, Q_{b+1}') \) is true, so, as in the preceding case \(([Q'_b Q''_b, Q_{b+1}''], [Q_{b+1}'', Q_{b+1}''])\) is in \( U_7 \) and \( z \) falls into \( L_{M5} \) again.

e) \( Q'_b \) is active and \( Q''_b \neq Q_{b+1}''. \) In this case \( |Q''_b| \leq |Q_b| + 3 \) and

\[
|Q_{b+1}'| = |Q'_b|, \quad \text{hence:}
\]

\[
|Q_b, Q_{b+1}'| \leq 3|Q_b'| + 3 \quad \text{and} \quad |Q'_b| \geq \left\lceil \frac{|Q_b Q'_b| + 3}{3} \right\rceil - 1.
\]
As $ERR_M(Q''_i, Q''_{i+1})$ is true we infer that:

$$\left(\{Q_{i_0}, Q_{i_0+1} Q''_{i_0+1}\}, Q_{i_0} Q''_{i_0}\right)$$

belongs to $U_6$. Therefore:

$$Q_{i_0+1} R Q_{i_0} \ldots R Q_1 R x R Q_1 \ldots R Q_{i_0}$$

is in $(U_6 \cup U_2) \sqcup L_1$,

$$Q_{i_0+1} R \ldots R Q_1 R x R Q_1 \ldots R Q_{i_0+1}$$

is in $(U_3 \cup U_6 \cup U_2) \sqcup L_1$,

and $z$ is in $L_{M_4}$.

(f) The last alternative is: $Q''_i$ is passive and $Q''_{i+1} \neq Q''_i$. Here we have $|Q''_{i_0}| < |Q''_{i_0+1}|$ and $|Q''_{i_0+1}| \leq |Q''_{i_0}| + 1$. Hence:

$$|Q_{i_0}, Q''_{i_0+1}| < 3 |Q_{i_0}| + 1 \quad \text{and} \quad |Q_{i_0}| > \left[\frac{|Q_{i_0}, Q''_{i_0+1}|}{3}\right] - 1.$$

As $|Q''_{i_0}| = |Q''_{i_0+1}|$ and $Q''_{i_0+1} \neq Q''_i$, $ERR_M(Q''_i, Q''_{i_0+1})$ is true and $\left(\{Q_{i_0}, Q_{i_0+1} Q''_{i_0+1}\}, Q_{i_0} Q''_i\right)$ belongs to $U_6$. Therefore just as in the preceding case $z$ falls into $L_{M_4}$.

Thus in all the cases (c) to (f) $m_{G_M}(z) < c_m$. This shows that $m_{G_M}(z) < c_m$ for all $z$ in $L_{M_1}$ of the form $z = P x P \star k$, where $P$ is not a computation record of $M$. Now let us take the second alternative.

2. $Q_1 \ldots Q_n$ is a computation record of $M$ but $|x| \geq |Q_1|$. Then we find that $Q_x R x$ belongs to $L_2$, $Q_x R x R Q_1$ belongs to $U_3 \sqcup L_2$, and $z$ falls into $L_{M_6}$. So in this case as in all the preceding $m_{G_M}(z) < c_m$.

The last alternative is:

3. $z = Q_n R \ldots R Q_{n-1} R \ldots R Q_1 R x R Q_1 \ldots R Q_n \star k$, $x$ in $L_0$, $|x| < |Q_1|$, and $\{Q_1, \ldots, Q_n\}$ is a $y$-computation record of $M$ for some $y$. Such a string $z$ is evidently in $L_{M_1} \sqcup \bigcup_{i=2}^6 L_{M_0}$. Hence there is the only one derivation tree $T$ in $S(G_{M_1})$ such that $t(T) = z$ and therefore $m_{G_M}(z) = m(T)$. By the definition of $G_M$ this tree $T$ can be represented in the form $T = \text{com}(T_0, v, T_1)$, where $T_0$ is the linear derivation tree in $S(G_{M_0})$ such that $t(T_0) = Q_n \ldots Q_1 R I_0 R Q_1 \ldots R Q_n \star k$, $v$ is the bottom node of $T_0$ such that $l(v) = I_0$, and $T_1$ is the derivation tree in $S(\Gamma[G])$ such that $t(T_1) = x$.

Now let us notice that since $|x| \leq |y|$ we have $|P_M(x)| \leq |P_M(y)|$. This implies that $|P_M(x) x P_M(x)| \leq |z| - k$ which in turn implies $|x| \leq p_M(|z| - k) \leq p_M(|z|)$.

Since $T_1$ is a tree in $S(\Gamma[G])$ there is a tree $T_2$ in $S(G)$ such that $t(T_2) = u, u$ in $L(G)$, and $T_2 \subseteq T_1$. First of all this implies that $|u| \leq |x|$ and hence $|u| \leq p_M(|z|)$. Secondly, $f$ is $m$-limiting the s-set $S(G)$. From this it follows that
$m(T_2) \leq c_f f(|u|)$ for a constant $c_f > 0$ independent of $T_2$. Third, because $m$ is mimeoinvariant $m(T_1) \leq c_A m(T_2)$ is true for $c_A > 0$ independent of $T_1$, $T_2$. Finally we remark that $T = \text{com}(T_0, v, T_1)$ implies $T_1 \leq {}^* T$ and therefore $m(T) \leq c_A m(T_1)$. Grouping together all these inequalities and bearing in mind that $f$ is nondecreasing we infer:

$$m_{G_M}(z) = m(T) \leq c_A m(T_1) \leq c_A^2 m(T_2) \leq c_f c_A^2 f(|u|) \leq c_f c_A^2 f(p_M(|z|)),$$

which gives the needed upper bound.

**LOWER BOUND:** To establish this bound it is enough to specify for each $cf$-grammar $G$ such that $L(G) = L_M$ an infinite sequence $(z_i | i \in \mathbb{N})$ in $L_M$ such that for an integer $c > 0$ and for all $i$ in $\mathbb{N}$ $cm_G(z_i) \geq f(p_M(|z_i|))$.

Let $G$ be a $cf$-grammar such that $L(G) = L_M$. The following lemma which is the main technical means of our proof of the lower bound assigns to each $cf$-grammar $G$ a parameter $n(G)$.

**Lemma 7.1:** For each $cf$-grammar $G$ with infinite language $L(G)$ there is an integer $n(G) > 0$ such that for any $x$ in $L(G)$ and any its complete derivation $D = (I = X_1, \ldots, X_m = x)$ if $x = x_1 z x_2$ and $|z| > n(G)$ then the substring $z$ can be decomposed into three parts $z = z_1 uz_2$ so that $|u| \neq 0$ and the derivation $D$ is representable either in form:

$$I \Rightarrow^{*} x_1 z_1 A y \Rightarrow^{*} x_1 z_1 u A u_1 y \Rightarrow^{*} x_1 z_1 u w u_1 y = x,$$

or in form:

$$I \Rightarrow^{*} y A z_2 x_2 \Rightarrow^{*} y u_1 A u z_2 x_2 \Rightarrow^{*} y u_1 u z_2 x_2 = x.$$

[This result is due to A. V. Gladkii (see for example [13]). Some later it was reproven in a stronger form by W. Ogden [14].]

To specify the sequence $(z_i | i \in \mathbb{N})$ we need some notions and notation.

**Notation:** With each tree $T$ in $S(G)$ we will associate an infinite sub-language of $L_M$ whose elements we will call $T$-terms. A $T$-term will be defined by induction on full subtrees $T(v)$ of $T$:

1. let $v$ be a bottom node of $T$ with $l(v) = X$. Then $X$ is an elementary $T(v)$-term;

2. let $v$ be a node of $T$ such that $i(v) = \{v_1, \ldots, v_k\}$, $v_1 < v_2 < \ldots < v_k$ and the $T(v_i)$-terms $\theta_i$ have been defined so far for all $1 \leq i \leq k$. Then for all $j \geq 0$ the string $\theta^{(j)}(v) = c a^j \theta_1 \ldots \theta_k b^j d$ is a $T(v)$-term, $j$ is a degree of this term, and $\theta_1, \ldots, \theta_k$ are its subterms;

3. each term $\theta$ is a subterm of itself and each subterm of a subterm of $\theta$ is a subterm of $\theta$;

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(4) If \( v_0 \) is the root of \( T \) then \( \theta^{(j)}(T, v_0) \) is a \( T \)-term for each \( j \geq 0 \).

Now we can specify the sequence \((z_j|i \in N)\). Let \( T_1, T_2, T_3, \ldots \) be a fundamental sequence of \( f \) in \( S(G) \). Let us associate with each \( T_j \) the \( T_j \)-term \( \theta_j \) such that the degrees of all its nonelementary subterms are equal to \( n_1 = n(G) + 1 \). Let \( e \) be the first symbol in \( V_M \). Then we set:

\[
z_i = [P_M(e')]^r_\theta \, P_M(e') \otimes^{[\theta_i]},
\]

where \( r \) is the width \((^7)\) of \( G \) and \( r_i = (r + 1) |\theta_i| \).

Let us note that the set \( \{ r_i | i \in N \} \) and hence the set \( \{ z_i | i \in N \} \) are infinite. Besides this \( p_M(\{ z_i \}) = r_i \) for each \( i \). Therefore it is enough to show that there is an integer \( c > 0 \) such that for all \( i, cm_G(z_i) \geq f(r_i) \). The proof of this last statement is rather tedious and lengthy. Its essence is the following proposition.

**Proposition 7.1:** Let \( T \) in \( S(G) \) be a cs-tree such that \( t(T) = z_i \), for some \( i \). Then \( T \) can be represented in form \( T = \text{com}(T_0, v, T(v)) \) so that \( T_i \leq T(v) \).

The proof of this proposition is omitted here. However, it may be found in [6] (a part of the proof of theorem 9.5) and in [8] (a part of the proof of theorem 2).

So let \( T \) be the simplest tree in \( S(G) \) such that \( t(T) = z_i \). Then \( m_G(z_i) = m(T) \). Since \( m \) is mimeoinvariant and by proposition 7.1 \( d_B m(T) \geq m(T_i) \) for a \( d_B > 0 \) independent of \( T \) and \( T_i \). As \( (T_i | i \in N) \) is a fundamental sequence of \( f \) in \( S(G) \) there is a \( e_f > 0 \) independent of \( i \) such that \( e_f m(T_i) > f(|T_i|) \). It is easy to see that \( |\theta_i| < 2(v + 1)(2n_1 + 3)|T_i| \), which implies that \( r_i \leq 2(v + 1)^2(2n_1 + 3)|T_i| \). Since \( f \) is a semihomogeneous function there is a \( c > 0 \) independent of \( i \) such that \( cf(|T_i|) \geq f(r_i) \). Finally we have

\[
d_B e_f cm_G(z_i) = d_B e_f cm(T) \geq e_f cm(T_i) \geq cf(|T_i|) \geq f(r_i)
\]

for all \( i \),

Q.E.D.

**Corollary 7.1:** Let \( m \) be a mimeoinvariant complexity measure and \( f \) be a nondecreasing unbounded semihomogeneous recursive function \( m \)-limiting the \( s \)-set \( S(G) \) of a \( c \)-free grammar \( G \). Then for each unbounded and nondecreasing with respect to both arguments recursive function \( \lambda m, n, h(m, n) \) there are an unbounded nondecreasing recursive function \( \varphi \) and a \( c \)-grammar \( G \) in right-normal form of nonreducible \( m \)-complexity \( \varphi \) such that for all but finitely many \( n, \varphi(n) \leq h(n, f(n)) \).

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[^7]: The width of a \( c \)-free grammar \( G \) is the least integer \( v \) such that \( S(G) \leq S^*(\Sigma, W, v) \).
Corollary 7.2: Let $m$ be a mimeoinvariant complexity measure, $\equiv_r$ be a reasonable equivalence relation on $\mathcal{E}$, and $f$ be a nondecreasing unbounded semihomogeneous recursive function $m$-limiting the $s$-set $S(E)$ of a MR-scheme $E$. Then for each unbounded and nondecreasing with respect to both arguments recursive function $\lambda m, n. h(m, n)$ there are an unbounded nondecreasing recursive function $\varphi$ and a MR-scheme $E_\varphi$ of $r$-nonreducible $m$-complexity $\varphi$ such that for all but finitely many $n$, $\varphi(n) \leq h(n, f(n))$.

Corollary 7.3: Let $m$ be a mimeoinvariant complexity measure and $\equiv_r$ be a reasonable equivalence relation on $\mathcal{E}$. If there exist a MR-scheme $E$ and an unbounded nondecreasing semihomogeneous function $f$ $m$-limiting $S(E)$ then there is an infinite sequence of unbounded nondecreasing functions $f_1 > f_2 > f_3 > \ldots$ such that $f = f_1$ and for no $i < j$, $\mathcal{E}^m_{f_i} \Rightarrow \mathcal{E}^m_{f_j}$.

Corollary 7.4: Let $m$ be a mimeoinvariant complexity measure and $\equiv_r$ be a reasonable equivalence relation on $\mathcal{E}$. If there exist a MR-scheme $E$ and an unbounded nondecreasing semihomogeneous recursive function $f$ $m$-limiting $S(E)$ then for no nondecreasing unbounded recursive function $g$, $\mathcal{E}^m_g \Rightarrow \mathcal{E}^m_{\text{Const.}}$.

The proof of the corollary 7.1 may be found in [6] (corollary 1 from the theorem 9.3) and in [7] (corollary 1 from the theorem 3). Corollaries 7.2-7.4 follow from it directly.

8. All Mimeoinvariant Complexity Measures Provide Infinite Classifications of MR-Schemes

In sections 6, 7 we considered some simple conditions sufficient for the existence of individual MR-schemes or infinite hierarchies of MR-schemes of nonreducible complexities. It is a pity but we cannot guarantee that these conditions hold for all mimeoinvariant complexity measures. So in this section classes of MR-schemes are compared in terms of set theoretical inclusion, and not in terms of translatability. In this much weaker sense we will show that all mimeoinvariant complexity measures provide nondegenerate classifications of MR-schemes. To this end we will simplify the construction of the theorem 7.1 so as to infer that for each mimeoinvariant complexity measure $m$ there is an infinite hierarchy of cf-grammars in right-normal form of different $m$-complexities (these grammars however not always being of nonreducible $m$-complexities).

Theorem 8.1: For each mimeoinvariant complexity measure $m$ there is an unbounded nondecreasing recursive function $f$ such that for each Turing machine $M$ in $\mathcal{N}(\Sigma^*)$ there is a right-normal form cf-grammar $G_{M_m}$ such that $m_{G_{M_m}} \preceq f(p_M)$.
Proof: Our mimeoinvariant complexity measure $m$ is nondegenerate by definition. So there is an unambiguous MR-scheme $E$ with unbounded complexity function. We apply to $E$ the proposition 6.2 so that to obtain the MR-scheme $\hat{E}$ with the following properties:

1. $G(\hat{E})$ is a right-normal form cf-grammar;
2. $\hat{E}$ is unambiguous (because $E$ is unambiguous);
3. $m_\hat{E}$ is recursive (because every nonbottom node of every tree $T$ in $S(E)$ has a width no less than 2, and therefore there are only finitely many trees in $S(\hat{E})$ with $n$ bottom nodes for all $n$);
4. $m_\hat{E}$ is unbounded (because $m$ is mimeoinvariant and hence $m_E \preceq m_\hat{E}$).

Next we show that the complexity function $m_\hat{E}$ is $m$-limiting $S(\hat{E})$. First of all we infer from unambiguity of $S(\hat{E})$ that:

$$m_{s(\hat{E})}(n) = \max \{ 0, m(T) \mid T \in S(\hat{E}), |T| \leq n \} \text{ for all } n.$$ 

This means that for every tree $T$ in $S(\hat{E})$, $m(T) \leq m_{s(\hat{E})}(|T|)$. Secondly we choose for each $n$ a tree $T_n$ in $S(E)$ (if any) such that:

$$|T_n| \leq n \quad \text{and} \quad m(T_n) = \max \{ m(T) \mid T \in S(\hat{E}), |T| \leq n \}.$$ 

Since $m_{s(\hat{E})}$ is unbounded and because of the abovementioned width property of $S(\hat{E})$ the set $\{|T_n| \mid n > 0\}$ is infinite. Finally we note that $m(T_n) = m_{s(\hat{E})}(|T_n|)$ for all (but finitely many) $n$.

This argument shows that there are a cf-grammar $G_0 = (\Sigma_0, W_0, I_0, P_0)$ in right-normal form and an unbounded nondecreasing recursive function $f$ $m$-limiting the s-set $S(G_0)$. We apply to $G_0$ the following construction.

Let $M$ be a Turing machine in $\mathcal{N}(\Sigma^*)$. We associate with $M$ and $G_0$ the pair languages $U_L - U_R$ from the proof of the theorem 7.1 and the languages:

$$L_3 = \{ Q^R x \mid Q \text{ is a situation of } M, x \text{ is in } \Sigma_0^*, 1 + |x| > |Q| \},$$

$$L_4 = \{ Q^R x Q_2 \mid Q_1, Q_2 \text{ are situations of } M, x \text{ is in } \Sigma_0^*, 2 |Q_2| > |Q_1| + |x| + 1 \},$$

and set $L_M = \bigcup_{i=1}^{7} L_{M^i}$, where:

$$L_{M^i} = (U_1 \circ U_2 \circ U_3 \circ U_4 \circ U_5 \circ U_6 \circ U_7 \circ U_8) \circ L(G_0),$$

$$L_{M^2} = (U_2 \circ U_3 \circ U_4 \circ U_5 \circ U_6 \circ U_7 \circ U_8) \circ \Sigma_0^+,$$

$$L_{M^3} = (U_2 \circ U_3 \circ U_4 \circ U_5 \circ U_6 \circ U_7 \circ U_8) \circ \Sigma_0^+,$$

$$L_{M^4} = (U_2 \circ U_3 \circ U_4 \circ U_5 \circ U_6 \circ U_7 \circ U_8) \circ \Sigma_0^+,$$

$$L_{M^5} = (U_2 \circ U_3 \circ U_4 \circ U_5 \circ U_6 \circ U_7 \circ U_8) \circ \Sigma_0^+,$$

$$L_{M^6} = (U_2 \circ U_3 \circ U_4 \circ U_5 \circ U_6 \circ U_7 \circ U_8) \circ L_3,$$

$$L_{M^7} = U_2 \circ L_4.$$
There are linear cf-grammars in right-normal form $G_{Mn0}$, $G_{Mn2}$, $G_{Mn3}$, $G_{Mn4}$, $G_{Mn5}$, $G_{Mn6}$, $G_{Mn7}$ generating respectively the languages $L_{Mn0}=\{I_0\}$, $L_{Mn2}$, $L_{Mn3}$, $L_{Mn4}$, $L_{Mn5}$, $L_{Mn6}$, $L_{Mn7}$. We assume that these grammars share the axiom $I$ and that for any two of them $I$ is their single common nonterminal. Besides this we denote by $\Sigma_{0}$ the alphabet $K_M \cup V_M \cup \Sigma_0 \cup \{\$\}$ and by $G_{Mn1}$ the cf-grammar $(\Sigma_{0}, W'_0, I, P'_0)$, where $W'_0 = W_0 \cup W_0$, $P'_0 = P_0 \cup P_0$, $W_0$ is the nonterminal alphabet and $P_0$ is the production set of $G_{Mn0}$.

Finally we set $G_{Mn} = (\Sigma_{0}, W_1, I, P)$, where $W_1$ is the union of nonterminal alphabets and $P$ is the union of production sets of the grammars $G_{Mnj}$, $1 \leq j \leq 7$. Of course, $L(G_{Mn}) = L_{Mn}$ and $L(G_{Mn1}) = L_{Mn1}$.

**UPPERBOUND:** $m_{G_{Mn}} \leq f(p_M)$. The proof of this inequality is very close to the proof of the corresponding inequality in the theorem 7.1 and is left to the reader.

**LOWER BOUND:** $m_{G_{Mn}} \geq f(p_M)$. The proof of this statement is straightforward. Indeed, since $f$ is $m$-limiting $S(G_0)$ we find there a fundamental sequence $(T_i^I | i > 0)$. Let us denote by $x_i$ the string $t(T_i^I)$ and set $z_i = [P_M(e^{\mid x_i\mid})]^R x_i P_M(e^{\mid x_i\mid})$ for each $i$, where $e$ is the first symbol of $V_M$.

The string $[P_M(e^{\mid x_i\mid})]^R I_0 P_M(e^{\mid x_i\mid})$ is the yield of a single tree $T_i^M$ in $S(G_{Mn0})$ for each $i$. We denote by $T_i$ the tree $	ext{com}(T_i^M, v_i, T_i^I)$, where $v_i$ is the single bottom node of $T_i^M$ labelled by $I_0$. It is obvious that $T_i$ is in $S(G_{Mn1})$, $t(T_i) = z_i$, and $T_i$ is the single tree in $S(G_{Mn})$ with the yield $z_i$. This means that for each $i$, $m_{G_{Mn}}(z_i) = m(T_i)$. Since $T_i^I \leq T_i$ and $m$ is mimeoinvariant there is an integer $d_{gb} > 0$ (not dependent on $i$) such that $m(T_i^I) \leq d_{gb} m(T_i)$. Further, $(T_i^I | i > 0)$ is a fundamental sequence for $f$ in $S(G_0)$, so there is an integer $c_f > 0$ one for all $i$ such that $c_f m(T_i^I) \geq f(\mid x_i\mid)$. Finally, for all $i$, $\mid x_i \mid = p_M(\mid z_i\mid)$. Summarizing these inequalities we obtain for all $i$:

$$c_b c_f m_{G_{Mn}}(\mid z_i\mid) \geq c_b c_f m_{G_{Mn}}(z_i) = c_b c_f m(T_i) \geq c_f m(T_i^I) \geq f(\mid x_i\mid) = f(p_M(\mid z_i\mid)).$$

Q.E.D.

**COROLLARY 8.1:** For any mimeoinvariant complexity measure $m$ there is an infinite sequence of unbounded nondecreasing recursive functions $f_1 > f_2 > f_3 > \ldots$ such that for all $j > 0$, $\delta_{f_j}^m - \delta_{f_{j+1}}^m \neq \emptyset$.

**COROLLARY 8.2:** If $m$ is a mimeoinvariant complexity measure then for any nondecreasing unbounded recursive function $\phi$, $\delta_{\phi}^m - \delta_{\text{Const.}}^m \neq \emptyset$. 

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9. COMPLEXITY OF UNAMBIGUOUS MR-SCHEMES

In this little section we show that under most reasonable conditions complexity of unambiguous MR-schemes is of extremal nature. We discuss first a formalization of an informal concept of extremal complexity.

**Definition 9.1:** Let \( m \) be a complexity measure and \( f \) be a function limiting it. Then we say that a MR-scheme \( E \) is of maximal \( m \)-complexity if for no \( g < f \), \( E \) is in \( \mathcal{E}^m_g \).

The following proposition shows that this definition is sensible at least for mimeoinvariant complexity measures.

**Proposition 9.1:** Let \( m \) be a complexity measure and \( f \) be a function limiting \( m \). Then:

(a) if \( E \) is a MR-scheme of maximal \( m \)-complexity then it is not in \( \mathcal{E}^m_{\text{const}} \);

(b) if \( m \) is mimeoinvariant and \( f \) is semihomogeneous then there exist MR-schemes of maximal \( m \)-complexity.

*Proof:* (a) Since the range of \( m \) is infinite \( f \) is unbounded; hence \( m_E \) is unbounded too.

(b) Let \( T_1, T_2, T_3, \ldots \) be a fundamental sequence of \( f \). Since \( m \) is mimeoinvariant we may assume without loss of generality that in every tree \( T_i \) in this sequence each nonbottom node is of width no less than 2 (we will refer to this condition as width condition).

Let \( k_0 \) be a number such that \( \{ T_1, T_2, T_3, \ldots \} \subseteq \mathcal{F}^c(\Sigma, W, k_0) \). Consider the MR-scheme:

\[
E_{k_0} : F x = (px | cx, a F bx, \ldots, a F^{k_0} bx),
\]

where \( p \) is in \( \mathcal{P}_{k_0+1} \), and \( a, b, c \) are basic function symbols. \( E_{k_0} \) is unambiguous and has the following property: for each \( i > 0 \) there is a tree \( T_i^* \) in \( S(E_{k_0}) \) such that \( T_i \leq^s T_i^* \) while \( |T_i^*| < 3 |T_i| \) (this upper bound follows directly from the width condition). Since \( E_{k_0} \) is unambiguous we have \( m_{E_{k_0}}(|T_i^*|) \geq m_{E_{k_0}}(T_i^*) = m(T_i^*) \), by the axiom \( B \) in the definition 5.4 there is a \( d > 0 \) (one for all \( i \)) such that \( d m(T_i^*) \geq m(T_i) \). As \( T_i \) is a member of the fundamental sequence \( cm(T_i) \geq f(|T_i|) \) (c is independent of \( i \)). Finally the linear inequality \( |T_i^*| < 3 |T_i| \) and semihomogeneity of \( f \) imply that there is a \( b > 0 \) such that \( bf(|T_i|) \geq f(|T_i^*|) \) for all \( i \). Hence \( bc d m_{E_{k_0}}(T_i^*) \geq f(|T_i^*|) \) for all \( i \).

Q.E.D.

**Remark:** For density and branching we have \( \mu_{E_i} \asymp \log n, b_{E_i} \asymp n \). Since \( \mu \) and \( b \) are both mimeoinvariant and the functions \( \log n \) and \( \lambda n \) are both
semihomogeneous $E_2$ is a MR-scheme of maximal density and maximal branching.

**Theorem 9.1:** Let $m$ be a mimeoinvariant complexity measure $m$-limited by a semihomogeneous function. Then every unambiguous MR-scheme is either of maximal $m$-complexity or of bounded density (i.e. falls into $\mathcal{E}_{\text{Const.}}^m$).

**Proof:** The proposition 6.2 guarantees that for each MR-scheme $E$ there is a MK-scheme $\hat{E}$, unambiguous if $E$ is unambiguous, such that $m_E \asymp m_{\hat{E}}$ and $S(E) = S(G(\hat{E}))$. This reduces our theorem to the following theorem proven in [6, 9]:

If a mimeoinvariant complexity measure $m$ is $m$-limited by a semihomogeneous function then every unambiguous cf-grammar is either of maximal $m$-complexity or of bounded density.

Q.E.D.

**Remark:** From results of [11, 12] it follows that $\mathcal{E}_{\text{Const.}}^m$ coincides with the class of all quasirational $(\mathcal{Q})$ MR-schemes.

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**REFERENCES**


(*) A definition of quasirational schemes may be found in [4].


