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TOPOLOGIES ON FREE MONOIDS INDUCED BY CLOSURE OPERATORS OF A SPECIAL TYPE (*)

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Abstract. — *For* $\mathscr{L} \subseteqq \mathfrak{P}(\Sigma^*)$ *the language operator* $\mathrm{Anf}_{\mathscr{L}}(A)$ *is defined by* $\{ z \mid z \diagdown A \in \mathscr{L} \}$. *If* $\mathscr{L}$ *fulfills some properties then* $\alpha_{\mathscr{L}}(A) = A \cup \mathrm{Anf}_{\mathscr{L}}(A)$ *is a closure operator and hence topologies can be defined. These topologies are characterized by different points of view.*

Furthermore some topological properties are discussed in terms of properties of $\mathscr{L}$.

Résumé. — *Pour une famille de langages* $\mathscr{L} \subseteqq \mathfrak{P}(\Sigma^*)$ *l'opérateur* $\mathrm{Anf}_{\mathscr{L}}(A)$ *est défini par* $\{ z \mid z \diagdown A \in \mathscr{L} \}$. *Si* $\mathscr{L}$ *satisfait certaines conditions, l'application* $\alpha_{\mathscr{L}}(A) = A \cup \mathrm{Anf}_{\mathscr{L}}(A)$ *est une fermeture ce qui permet de définir des topologies. Ces topologies peuvent être caractérisées sous différents points de vue.*

En plus, certaines propriétés topologiques seront considérées dépendant des propriétés de $\mathscr{L}$.

1. INTRODUCTION AND PRELIMINARIES

There are several papers giving a connection between topology and the theory of formal languages [1, 2, 5, 8, 11].

For example, in the classical paper of Chomsky and Schützenberger [4], convergent sequences of languages are considered. For this purpose a topology over the set of formal languages is necessary. In the present paper topologies over the free monoid Σ^* are considered. However, there are methods to lift topologies over X to topologies over the powerset $\mathfrak{P}(X)$; a special one is described here.

The methods to obtain these topologies are closely related to [7] and [2]; the last paper contains an extensive motivation to make such considerations.

As announced, there are studied topological spaces $(\Sigma^*, \mathscr{O})$ in connection with the concept of the operators $\mathrm{Anf}_{\mathscr{L}}(A) = \{ z \mid z \diagdown A \in \mathscr{L} \}$ presented in [7] $(\mathscr{L} \subseteqq \mathfrak{P}(\Sigma^*))$.

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In chapter 2 those sets $\mathscr{L}$ are characterized which lead to closure operators. This is seen by different points of view.

In chapter 3 some examples are presented.

In chapter 4 properties of topological spaces are seen as properties of $\mathscr{L}$, especially some separation properties.

In chapter 5 some remarks are made concerning continuity, generalizations of closure operators and topologies on the set of formal languages.

Now the essential definitions needed here are given. (All topological conceptions are to be found in [3].)

Let Σ denotes a finite alphabet, Σ^* the set of all words over Σ, ε the empty word, $\Sigma^+ = \Sigma^* - \{\varepsilon\}$, x^R the mirror image of $x \in \Sigma^*$,

$$x \setminus L = \{ z \mid xz \in L \},$$

$$\text{Init}(L) = \{ x \mid \text{there is a } z \text{ such that } xz \in L \}.$$

$\mathfrak{P}(\Sigma^*)$ means the powerset of Σ^*, $\mathfrak{P}_0(\Sigma^*) = \mathfrak{P}(\Sigma^*) - \{\emptyset\}$.

A function $\alpha : \mathfrak{P}(X) \to \mathfrak{P}(X)$ is called a closure operator iff it fulfills the axioms (A1)-(A4):

$$\alpha(\emptyset) = \emptyset; \tag{A1}$$

$$\alpha(A \cup B) = \alpha(A) \cup \alpha(B); \tag{A2}$$

$$A \subseteq \alpha(A); \tag{A3}$$

$$\alpha(\alpha(A)) = \alpha(A). \tag{A4}$$

[A, B stand for arbitrary elements of $\mathfrak{P}(X)$; here only $X = \Sigma^*$ is treated.]

It is well-known that it is equivalent to speak about the set of closed sets or of the corresponding closure operator with properties (A1)-(A4).

2. GENERAL PROPERTIES

Throughout this paper Σ denotes a fixed alphabet.

DÉFINITION 2.1: For $\mathscr{L} \subseteq \mathfrak{P}(\Sigma^*)$ let

$$\alpha(\mathscr{L}) = \mathscr{L} \cup \{ A \mid \varepsilon \in A \},$$

$$\partial(\mathscr{L}) = \mathscr{L} - \{ A \mid \varepsilon \in A \}$$

and

$$\alpha_{\mathscr{L}} : \mathfrak{P}(\Sigma^*) \to \mathfrak{P}(\Sigma^*)$$

the function defined by

$$\alpha_{\mathscr{L}}(A)=A \cup \operatorname{Anf}_{\mathscr{L}}(A).$$

LEMMA 2.2: *For all A, $\mathscr{L}$:*

$$\operatorname{Anf}_{\alpha(\mathscr{L})}(A)=\alpha_{\mathscr{L}}(A)=A \times \operatorname{Anf}_{\partial(\mathscr{L})}(A)=\alpha_{\partial(\mathscr{L})}(A)$$

($\times$ denotes disjoint union).

Proof: $x \in \operatorname{Anf}_{\alpha(\mathscr{L})}(A)$ iff $x \setminus A \in \mathscr{L} \cup \{A \mid \varepsilon \in A\}$ which is equivalent to $x \in \operatorname{Anf}_{\mathscr{L}}(A) \cup A=\alpha_{\mathscr{L}}(A)$. $x \in \alpha_{\mathscr{L}}(A)$ means $x \in A$ or $x \setminus A \in \mathscr{L}$. If $x \notin A$ then $\varepsilon \notin x \setminus A$ and thus $x \setminus A \in \partial(\mathscr{L})$. If $x \in A$ then $\varepsilon \in x \setminus A$ and so $x \setminus A \notin \partial(\mathscr{L})$, which verifies the second equality; the third one is clear by definition.

This shows that it depends only on $\partial(\mathscr{L})$ whether or not $\alpha_{\mathscr{L}}$ is a closure operator. $\operatorname{Anf}_{\mathscr{L}}(A)=\alpha_{\mathscr{L}'}(A)$ for all A is equivalent with $\mathscr{L}=\alpha(\mathscr{L}')$: First, $\operatorname{Anf}_{\mathscr{L}}(A)=\operatorname{Anf}_{\mathscr{L}'}(A) \cup A$ yields $A \subseteq \operatorname{Anf}_{\mathscr{L}}(A)$ for all A, thus $\alpha(\mathscr{L})=\mathscr{L}$. From $\alpha_{\mathscr{L}}(A)=\alpha_{\mathscr{L}'}(A)$ for all A it can be deduced that $\alpha(\mathscr{L})=\alpha(\mathscr{L}')$ in a similar way as in theorem 4.7 of [7]. Now assume $\mathscr{L}=\alpha(\mathscr{L}')$. Then $\mathscr{L}=\alpha(\mathscr{L})=\alpha(\mathscr{L}')$ and thus $\operatorname{Anf}_{\mathscr{L}}(A)=\alpha_{\mathscr{L}}(A)=\alpha_{\mathscr{L}'}(A)$ for all A. Thus $\operatorname{Anf}_{\mathscr{L}}$ is a closure operator iff $\mathscr{L}=\alpha(\mathscr{L})$ and $\alpha_{\mathscr{L}}$ is a closure operator.

Now the sets $\mathscr{L}$ for which $\operatorname{Anf}_{\mathscr{L}}$ is a closure operator are characterized.

THEOREM 2.3: *Let $\mathscr{L}=\alpha(\mathscr{L})$. Then $\operatorname{Anf}_{\mathscr{L}}$ is a closure operator iff the following axioms are valid:*

$$\varnothing \notin \mathscr{L}; \tag{T1}$$

$$\text{if } A \in \mathscr{L},\ A \subseteq B \text{ then } B \in \mathscr{L}; \tag{T2}$$

$$\text{if } A \cup B \in \mathscr{L} \text{ then } A \in \mathscr{L} \text{ or } B \in \mathscr{L}; \tag{T3}$$

$$A \in \mathscr{L} \text{ iff } \operatorname{Anf}_{\mathscr{L}}(A) \in \mathscr{L}. \tag{T4}$$

Proof: Since $\operatorname{Anf}_{\mathscr{L}}(\varnothing)=\varnothing$ iff $\varnothing \notin \mathscr{L}$ (A1) is equivalent to (T1). By [7] (T2) and (T3) are equivalent to (A2). (A3) is clear because of the assumption $\mathscr{L}=\alpha(\mathscr{L})$. By [7] (T4) is equivalent to (A4).

Characterizing those sets $\mathscr{L}$ for which $\alpha_{\mathscr{L}}=\alpha_{\partial(\mathscr{L})}$ is a closure operator one obtains weaker axioms for $\partial(\mathscr{L})$.

THEOREM 2.4: *$\alpha_{\mathscr{L}}$ is a closure operator iff the following axioms are valid:*

$$\varnothing \notin \partial(\mathscr{L}); \tag{T1'}$$

$$\text{if } A \in \partial(\mathscr{L}),\ A \subseteq B,\ \varepsilon \notin B \text{ then } B \in \partial(\mathscr{L}); \tag{T2'}$$

$$\text{if } A \cup B \in \partial(\mathscr{L}) \text{ then } A \in \partial(\mathscr{L}) \text{ or } B \in \partial(\mathscr{L}); \tag{T3'}$$

$$\text{if } \operatorname{Anf}_{\partial(\mathscr{L})}(A) \in \partial(\mathscr{L}),\ \varepsilon \notin A \text{ then } A \in \partial(\mathscr{L}). \tag{T4'}$$

Proof: It is sufficient to show that (T1)-(T4) hold for $\alpha(\mathscr{L})$ iff (T1')-(T4') hold.

For (T1) and (T1') this is clear.

Let (T2) be true for $\alpha(\mathscr{L})$ and $A \in \partial(\mathscr{L})$, $A \subseteq B$, $\varepsilon \notin B$. Then by (T2) $B \in \alpha(\mathscr{L})$ and $\varepsilon \notin B$, thus $B \in \partial(\mathscr{L})$.

Let (T2') be true and $A \in \alpha(\mathscr{L})$, $A \subseteq B$. If $\varepsilon \in B$ then $B \in \alpha(\mathscr{L})$. If $\varepsilon \notin B$ then $\varepsilon \notin A$ and by (T2') $B \in \partial(\mathscr{L}) \subseteq \alpha(\mathscr{L})$.

The proof for (T3) and (T3') is analoguous.

Let (T4) be true for $\alpha(\mathscr{L})$ and $\mathrm{Anf}_{\partial(\mathscr{L})}(A) \in \partial(\mathscr{L})$, $\varepsilon \notin A$. By (T2') is $A \cup \mathrm{Anf}_{\partial(\mathscr{L})}(A) = \mathrm{Anf}_{\alpha(\mathscr{L})}(A) \in \partial(\mathscr{L}) \subseteq \alpha(\mathscr{L})$ and by (T4) is $A \in \alpha(\mathscr{L})$. Since $\varepsilon \notin A$, $A \in \partial(\mathscr{L})$ holds.

Let conversely (T4') be true. If $A \in \alpha(\mathscr{L})$ then $\varepsilon \in \mathrm{Anf}_{\alpha(\mathscr{L})}(A)$ and so $\mathrm{Anf}_{\alpha(\mathscr{L})}(A) \in \alpha(\mathscr{L})$. Let $\mathrm{Anf}_{\alpha(\mathscr{L})}(A) \in \alpha(\mathscr{L})$. If $\varepsilon \in A$ or $\varepsilon \in \mathrm{Anf}_{\partial(\mathscr{L})}(A)$ then $A \in \alpha(\mathscr{L})$. If

$$\varepsilon \notin A \cup \mathrm{Anf}_{\partial(\mathscr{L})}(A) = \mathrm{Anf}_{\alpha(\mathscr{L})}(A) \in \partial(\mathscr{L})$$

then by (T3') $A \in \partial(\mathscr{L})$ or $\mathrm{Anf}_{\partial(\mathscr{L})}(A) \in \partial(\mathscr{L})$. In the first case there is nothing to show. In the second one (T4') can be used.

The next goal of this paper is to characterize those topologies on Σ^* which are induced by closure operators of the form $\alpha_{\mathscr{L}}$ from different points of view.

DEFINITION 2.5: Let $\alpha_{\mathscr{L}}$ be a closure operator. The corresponding topological space is denoted by $X_{\mathscr{L}} = (\Sigma^*, \mathscr{O}_{\mathscr{L}})$, where $\mathscr{O}_{\mathscr{L}}$ denotes the family of open sets.

First those operators f: $\mathfrak{P}(\Sigma^*) \to \mathfrak{P}(\Sigma^*)$ are characterized which can be represented as $\mathrm{Anf}_{\mathscr{L}}$.

DEFINITION 2.6: An operator f is called leftquotient-permutable (lq-permutable) if for all x, A:

$$f(x \setminus A) = x \setminus f(A).$$

LEMMA 2.7: *An operator f is representable as* $\mathrm{Anf}_{\mathscr{L}}$ *iff f is lq-permutable.*

Proof: In [7] it is shown that each $\mathrm{Anf}_{\mathscr{L}}$ is lq-permutable.

Conversely let f be lq-permutable and define $\mathscr{L}_f = \{A \mid \varepsilon \in f(A)\}$. Then $x \in f(A)$ iff $\varepsilon \in x \setminus f(A)$ which is equivalent to $x \setminus A \in \mathscr{L}_f$ and this means $x \in \mathrm{Anf}_{\mathscr{L}_f}(A)$.

REMARK: Since $\mathrm{Anf}_{\mathscr{L}_1} \neq \mathrm{Anf}_{\mathscr{L}_2}$ for $\mathscr{L}_1 \neq \mathscr{L}_2$ (*see* [7]) the set $\mathscr{L}_f$ of lemma 2.7 is unique.

For fixed $\mathscr{L}$ the relation $x \in \mathrm{Anf}_{\mathscr{L}}(A)$ depends only on $x \setminus A$, i. e. if $x \setminus A = y \setminus B$ then $x \in \mathrm{Anf}_{\mathscr{L}}(A)$ iff $y \in \mathrm{Anf}_{\mathscr{L}}(B)$.

DEFINITION 2.8: An operator f is leftquotient-dependend (lq-dependend) if whenever $x \setminus A = y \setminus B$ then $x \in f(A)$ iff $y \in f(B)$.

The lq-dependence is characteristic for the operators $\mathrm{Anf}_{\mathscr{L}}$:

LEMMA 2.9: *The following two properties of f are equivalent:*

(i) *f is lq-dependend;*

(ii) *f is lq-permutable.*

Proof: Let (i) hold. Since $w \setminus (x \setminus A) = xw \setminus A$, $w \in f(x \setminus A)$ iff $xw \in f(A)$ which is equivalent to $w \in x \setminus f(A)$.

Let (ii) hold and $x \setminus A = y \setminus B$. Then $f(x \setminus A) = f(y \setminus B)$. That means $\varepsilon \in f(x \setminus A) = x \setminus f(A)$ iff $\varepsilon \in f(y \setminus B) = y \setminus f(B)$ and so $x \in f(A)$ iff $y \in f(B)$.

COROLLARY 2.10: *An operator f is representable as* $\mathrm{Anf}_{\mathscr{L}}$ *iff it is lq-dependend.*

As a consequence the following theorem is obtained:

THEOREM 2.11: *Let $X = (\Sigma^*, \mathcal{O})$ be a topological space. The following statements are equivalent:*

(i) *$X = X_{\mathscr{L}}$ for some $\mathscr{L}$;*

(ii) *the closure operator f of X is lq-permutable;*

(iii) *the closure operator f of X is lq-dependend.*

Proof: If $X = X_{\mathscr{L}}$ then $f = \alpha_{\mathscr{L}} = \mathrm{Anf}_{\alpha(\mathscr{L})}$ is both lq-permutable and lq-dependend. (ii) and (iii) are equivalent due to lemma 2.9. If f is lq-permutable then $f = \mathrm{Anf}_{\mathscr{L}}$ for some $\mathscr{L}$ and since f is a closure operator $X = X_{\mathscr{L}}$ holds.

Next the topological spaces $X_{\mathscr{L}}$ will be characterized in terms of their open sets.

DEFINITION 2.12: $\mathscr{L} \subseteq \mathfrak{P}(\Sigma^*)$ is called left-stable if whenever $A \in \mathscr{L}$ and $x \in \Sigma^*$, $x \setminus A \in \mathscr{L}$ and $xA \in \mathscr{L}$.

LEMMA 2.13: *$A \in \mathcal{O}_{\mathscr{L}}$ iff for all $x \in A$ $(x \setminus A)^c \notin \mathscr{L}$. ($A^c$ denotes the complement of A, i. e. $\Sigma^* - A$.).*

Proof: $A \in \mathcal{O}_{\mathscr{L}}$ iff A^c is closed, i. e. $\alpha_{\mathscr{L}}(A^c) = A^c \cup \mathrm{Anf}_{\mathscr{L}}(A^c) = A^c$. That means $\mathrm{Anf}_{\mathscr{L}}(A^c) \subseteq A^c$ or equivalently $A \subseteq [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c$. Thus $A \in \mathcal{O}_{\mathscr{L}}$ iff $A \subseteq [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c$. If $A \subseteq [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c$ and $x \in A$ then $x \setminus A^c = (x \setminus A)^c \notin \mathscr{L}$. If conversely $A \nsubseteq [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c$, then there is a $x \in A$ such that $x \in \mathrm{Anf}_{\mathscr{L}}(A^c)$, i.e. $x \setminus A^c \in \mathscr{L}$.

LEMMA 2.14: *$\mathcal{O}_{\mathscr{L}}$ is left-stable.*

Proof: Let be $A \in \mathcal{O}_{\mathscr{L}}$ and $x \in \Sigma^*$. Then $\mathrm{Anf}_{\mathscr{L}}(A^c) \subseteq A^c$, thus $x \setminus \mathrm{Anf}_{\mathscr{L}}(A^c) = \mathrm{Anf}_{\mathscr{L}}(x \setminus A^c) \subseteq x \setminus A^c$ and so $x \setminus A^c$ is closed. Therefore $x \setminus A \in \mathcal{O}_{\mathscr{L}}$. To show that $xA \in \mathcal{O}_{\mathscr{L}}$ assume the contrary. If $xA \notin \mathcal{O}_{\mathscr{L}}$ then by lemma 2.13 there is a $xy \in xA$ such that $(xy \setminus xA)^c \in \mathscr{L}$. Since $xy \setminus xA = y \setminus A$ a contradiction is obtained.

LEMMA 2.15: *Let $X=(\Sigma^*, \mathcal{O})$ be a topological space where $\mathcal{O}$ is left-stable. Then $X=X_{\mathcal{L}}$ for some $\mathcal{L}$.*

Proof: Let $\mathcal{L}=\mathfrak{P}(\Sigma^*)-\{A \mid$ there is a $0\in\mathcal{O}$ such that $\varepsilon\in 0$ and $A\subseteqq O^c\}$.

First (T1′)-(T4′) will be verified for $\partial(\mathcal{L})$:

Since $\mathcal{O}\neq\emptyset$ and $\emptyset\subseteqq O^c$, (T1′) holds.

Let be $A\in\partial(\mathcal{L})$, $A\subseteqq B$, $\varepsilon\notin B$ and assume that $B\notin\partial(\mathcal{L})$. Then $B\notin\mathcal{L}$, i. e. there is a $0\in\mathcal{O}$ such that $\varepsilon\in 0$ and $B\subseteqq O_1^c$. Hence $A\subseteqq O^c$ and $A\notin\partial(\mathcal{L})$. Thus (T2′) is shown by contradiction.

To show (T3′) let be $A\cup B\in\partial(\mathcal{L})$ and suppose that $A\notin\partial(\mathcal{L})$ and $B\notin\partial(\mathcal{L})$. Then there are $O_1, O_2\in\mathcal{O}$, $\varepsilon\in O_1, O_2$ such that $A\subseteqq O_1^c$ and $B\subseteqq O_2^c$. Therefore $A\cup B\subseteqq(O_1\cap O_2)^c$ and $O_1\cap O_2$ is open and contains ε. So $A\cup B\notin\partial(\mathcal{L})$ would hold.

Now (T4′) will be shown: Let be $\mathrm{Anf}_{\partial(\mathcal{L})}(A)\in\partial(\mathcal{L})$ and assume that $A\notin\partial(\mathcal{L})$. Then $A\subseteqq O^c$ where $0\in\mathcal{O}$ and $\varepsilon\in 0$. Since $\mathcal{O}$ is left-stable, for all $x\in 0$ the following holds: $x\setminus 0\in\mathcal{O}$ and $\varepsilon\in x\setminus 0$ and $x\setminus A\subseteqq(x\setminus 0)^c$. That means that for all $x\in 0$, $x\setminus A\notin\partial(\mathcal{L})$ or equivalently $x\notin\mathrm{Anf}_{\partial(\mathcal{L})}(A)$. Hence $0\subseteqq[\mathrm{Anf}_{\partial(\mathcal{L})}(A)]^c$, i. e. $\mathrm{Anf}_{\partial(\mathcal{L})}(A)\notin\partial(\mathcal{L})$, which is a contradiction.

Hence $\alpha_{\mathcal{L}}$ is a closure operator and it remains to show that $\mathcal{O}_{\mathcal{L}}=\mathcal{O}$.

Let be $A\in\mathcal{O}$. Then for all $x\in A$, $\varepsilon\in x\setminus A$ and $x\setminus A\in\mathcal{O}$ and therefore $(x\setminus A)^c\notin\mathcal{L}$ for all $x\in A$ which means by lemma 2.13 that $A\in\mathcal{O}_{\mathcal{L}}$.

Let conversely be $A\in\mathcal{O}_{\mathcal{L}}$. Then for all $x\in A$ $(x\setminus A)^c\notin\mathcal{L}$ hence for all $x\in A$ there is a $O_x\in\mathcal{O}$ such that $\varepsilon\in O_x$ and $(x\setminus A)^c\subseteqq O_x^c$ i. e. $O_x\subseteqq(x\setminus A)$. Since $\mathcal{O}$ is left-stable $xO_x\in\mathcal{O}$ and $xO_x\subseteqq x(x\setminus A)$ for all $x\in A$. Now

$$A=\bigcup_{x\in A}x\subseteqq\bigcup_{x\in A}xO_x\subseteqq\bigcup_{x\in A}x(x\setminus A)=A$$

and therefore $A=\bigcup_{x\in A}xO_x$ is in $\mathcal{O}$.

Combining the above results leads to:

THEOREM 2.16: *Let $X=(\Sigma^*, \mathcal{O})$ be a topological space. Then $X=X_{\mathcal{L}}$ for some $\mathcal{L}$ iff $\mathcal{O}$ is left-stable.*

3. EXAMPLES

LEMMA 3.1: *$\mathcal{O}_{\mathcal{L}_1}=\mathcal{O}_{\mathcal{L}_2}$ iff $\partial(\mathcal{L}_1)=\partial(\mathcal{L}_2)$.*

Proof: Let be $\mathcal{L}_i'=\mathfrak{P}(\Sigma^*)-\{A \mid$ there is a $0\in\mathcal{O}_{\mathcal{L}_i}$ such that $\varepsilon\in 0$ and $A\subseteqq O^c\}$. Then by lemma 2.15 $\partial(\mathcal{L}_i)=\partial(\mathcal{L}_i')$ $(i=1, 2)$ holds. Hence $\mathcal{O}_{\mathcal{L}_1}=\mathcal{O}_{\mathcal{L}_2}$ iff $\mathcal{O}_{\mathcal{L}_1'}=\mathcal{O}_{\mathcal{L}_2'}$ iff $\partial(\mathcal{L}_1')=\partial(\mathcal{L}_2')$ iff $\partial(\mathcal{L}_1)=\partial(\mathcal{L}_2)$.

THEOREM 3.2: *The following statements are equivalent:*

(i) *$\mathcal{O}_{\mathcal{L}}$ is the discrete topology;*

(ii) $\partial(\mathcal{L})=\emptyset$;

(iii) $\mathcal{O}_{\mathcal{L}}$ *contains a nonempty finite set.*

Proof: $\partial(\mathcal{L})=\emptyset$ iff for all A $\mathrm{Anf}_{\partial(\mathcal{L})}(A)=\emptyset$ i. e. A closed. [It should be noted that $\mathcal{O}_{\mathcal{L}_1}=\mathcal{O}_{\mathcal{L}_2}$ iff $\partial(\mathcal{L}_1)=\partial(\mathcal{L}_2)$.]

That (i) implies (iii) is clear. Let conversely (iii) hold. Then there is a $0\in\mathcal{O}_{\mathcal{L}}$ and 0 is finite and nonempty. Let $z\in 0$ be of maximal length. Then $\{z\}=0\cap z\Sigma^*\in\mathcal{O}_{\mathcal{L}}$, hence $z\setminus\{z\}=\{\varepsilon\}\in\mathcal{O}_{\mathcal{L}}$, which means that Σ^+ is closed, i. e. $\varepsilon\notin \mathrm{Anf}_{\partial(\mathcal{L})}(\Sigma^+)$. So $\varepsilon\setminus\Sigma^+=\Sigma^+\notin\mathcal{L}$ and by (T2′) $\partial(\mathcal{L})=\emptyset$.

For $x, y\in\Sigma^*$ let the order relation $\leqq$ be defined by $x\leqq y$ iff $y=xz$ for some z and $\mathcal{O}_{\leqq}$ the right topology.

THEOREM 3.3: $\mathcal{O}_{\leqq}=\mathcal{O}_{\mathfrak{P}_0(\Sigma^*)}$.

Proof: $\mathfrak{B}=\{x\Sigma^*\mid x\in\Sigma^*\}$ is a base for the topology $\mathcal{O}_{\leqq}$ [3]. It will be shown that $\mathfrak{B}$ is a base for $\mathcal{O}_{\mathfrak{P}_0(\Sigma^*)}$. Let be $0\in\mathcal{O}_{\mathfrak{P}_0(\Sigma^*)}$. Then by lemma 2.13 $(x\setminus 0)^c\notin\mathfrak{P}_0(\Sigma^*)$ for all $x\in 0$. This means that $x\setminus 0=\Sigma^*$ for all $x\in 0$. Thus $0=\bigcup_{x\in 0}x\Sigma^*$.

REMARK: (i) The corresponding closure operator is Init. (ii) The open sets are exactly those sets of the form $L\Sigma^*$ for arbitrary $L\subseteqq\Sigma^*$.

Next let $\mathcal{L}=\alpha(\mathcal{U})$ where $\mathcal{U}$ is the family of infinite sets (over Σ^*). Then it can easily be verified that $\mathrm{Anf}_{\mathcal{L}}$ is a closure operator and for all A: $\mathrm{Anf}_{\mathcal{L}}(A)=A\cup\mathrm{Anf}(A)$.

[In [7] $\mathrm{Anf}(A)$ is defined to be $\{w\mid$ there are infinitely many z such that $wz\in A\}$.]

THEOREM 3.4: $\mathfrak{B}=\{xA\mid x\in\Sigma^*,\ A^c$ *finite*$\}$ *is a base for* $\mathcal{O}_{\mathcal{U}}$.

Proof: Let be $0\in\mathcal{O}_{\mathcal{U}}$. Then by lemma 2.13 $(x\setminus 0)^c\notin\mathcal{U}$ or equivalently $x\setminus O^c$ is finite for all $x\in 0$. Since $0=\bigcup_{x\in 0}x(x\setminus 0)$ the statement holds.

REMARK: Each set LA where $L\subseteqq\Sigma^*$, A^c finite is open, but there are open sets 0 which are not of this form (e. g. let $\Sigma=\{a, b\}$ and $0=\{a^n bw\mid w\neq a^n\}$).

Now 3 other examples are presented in short. Let $a\in\Sigma$ and

$$\mathcal{L}_a=\{A\mid \text{there is a } ax\in A\},$$
$$\mathcal{L}^a=\{A\mid \text{there is a } xa\in A\},$$
$$\mathcal{L}^{\,a}_{\ }=\{A\mid \text{there is a } xay\in A\}.$$

Then it is easily shown that $X_{\mathcal{L}_a}$, $X_{\mathcal{L}^a}$, $X_{\mathcal{L}_{a}}$ are topological spaces; they can be seen as right topologies with respect to the partial orders $\leqq_a$, $\leqq^a$, $\leqq_a$ defined by

$$x\leqq_a y \quad \text{iff } x=y \quad \text{or} \quad y=xaz;$$

$$x \leqq^a y \quad \text{iff } x=y \quad \text{or} \quad y=xza;$$
$$x \leqq a y \quad \text{iff } x=y \quad \text{or} \quad y=xz_1 a z_2.$$

4. CONNECTIONS OF TOPOLOGICAL PROPERTIES AND PROPERTIES OF $\mathscr{L}$

In this chapter some investigations are made upon the topological spaces $X_{\mathscr{L}}$.

LEMMA 4.1: *$\mathcal{O}_{\mathscr{L}_2}$ is coarser than $\mathcal{O}_{\mathscr{L}_1}$ iff $\partial(\mathscr{L}_1) \subseteqq \partial(\mathscr{L}_2)$.*

Proof: Let $\mathscr{L}'_i = \mathfrak{P}(\Sigma^*) - \{ A \mid$ there is a $0 \in \mathcal{O}_{\mathscr{L}_i}$ such that $\varepsilon \in 0$ and $A \subseteqq O^c \}$. Then by Lemma 2.15 $\partial(\mathscr{L}'_i) = \partial(\mathscr{L}_i)$ $(i=1, 2)$ holds. Hence $\mathcal{O}_{\mathscr{L}_2} \subseteqq \mathcal{O}_{\mathscr{L}_1}$ iff $\mathscr{L}'_1 \subseteqq \mathscr{L}'_2$ iff $\partial(\mathscr{L}'_1) \subseteqq \partial(\mathscr{L}'_2)$ iff $\partial(\mathscr{L}_1) \subseteqq \partial(\mathscr{L}_2)$. (Compare lemma 3.1.)

LEMMA 4.2: *Let $\mathcal{O}_{\mathscr{L}_i}$, $i \in I$ be topologies. Then there are families $\mathscr{L}_\cap$ and $\mathscr{L}_\cup$ such that $\mathcal{O}_{\mathscr{L}_\cap} = \bigcap_{i \in I} \mathcal{O}_{\mathscr{L}_i}$ and $\bigcup_{i \in I} \mathcal{O}_{\mathscr{L}_i}$ is a subbase for $\mathcal{O}_{\mathscr{L}_\cup}$.*

Proof: By theorem 2.16 it suffices to show that $\bigcap_{i \in I} \mathcal{O}_{\mathscr{L}_i}$ as well as the family of arbitrary unions of finite intersections of $\bigcup_{i \in I} \mathcal{O}_{\mathscr{L}_i}$ are left-stable. But this is trivial because of the fact that concatenation of a single word (division by a single word) can be distributed over arbitrary unions (intersections).

THEOREM 4.3: *The family of the topologies $\mathcal{O}_{\mathscr{L}}$ is a complete lattice where $\mathcal{O}_{\mathfrak{P}_0(\Sigma^*)}$ is the 0-element and $\mathcal{O}_{\emptyset}$ is the 1-element.*

Proof: The statement follows immediately from lemmas 4.1 and 4.2.

THEOREM 4.4: *Each $X_{\mathscr{L}}$ is a T_0-space (Kolmogoroff-space).*

Proof: Let be $x, y \in \Sigma^*$, $x \neq y$. Without loss of generality let $y \nleqq x$ hold. Then $y \notin \operatorname{Init}(\{x\})$ and because of

$$\alpha_{\mathscr{L}}(\{x\}) = \{x\} \cup \operatorname{Anf}_{\mathscr{L}}(\{x\}) \subseteqq \operatorname{Init}(\{x\})$$

is $[\alpha_{\mathscr{L}}(\{x\})]^c$ an open set containing y but not x.

THEOREM 4.5: *The following two statements are equivalent:*

(i) *$X_{\mathscr{L}}$ is a T_1-space (i. e. each set $\{x\}$ is closed);*

(ii) *$\partial(\mathscr{L})$ contains no set of cardinality 1.*

Proof: Let (i) hold. Would be $\{z\} \in \partial(\mathscr{L})$ then $\varepsilon \in \alpha_{\mathscr{L}}(\{z\})$ and $\{z\}$ would not be closed.

If conversely (i) does not hold, then there is a x such that $\alpha_{\mathscr{L}}(\{x\})$ contains a $y \neq x$. Therefore $y \setminus \{x\} \in \partial(\mathscr{L})$ and since $\emptyset \notin \partial(\mathscr{L})$ $y \setminus \{x\}$ contains exactly one element.

Next those families $\mathscr{L}$ will be characterized for which $X_{\mathscr{L}}$ is a T_2-space (Hausdorff-space).

LEMMA 4.6: *$X_{\mathscr{L}}$ is a T_2-space iff for each $z \neq \varepsilon$ there is an open set 0 such that $\varepsilon \in 0$ and $0 \cap z\,0 = \emptyset$.*

Proof. — Let $X_{\mathscr{L}}$ be a T_2-space and $z \neq \varepsilon$. Then there are open sets O_1, O_2 such that $\varepsilon \in O_1$, $z \in O_2$ and $O_1 \cap O_2 = \emptyset$. Let be $0 = O_1 \cap z \setminus (O_2 \cap z\Sigma^*)$. Since 0 is obtained by applying allowed operations to open sets, 0 is open. Furthermore $\varepsilon \in 0$. Since $0 \subseteq O_1$ and $z\,O \subseteq O_2$, $0 \cap z\,0 = \emptyset$ holds.

To show the converse let be $x, y \in \Sigma^*$, $x \neq y$. If neither $x \leq y$ nor $y \leq x$ holds then $x\Sigma^*$ and $y\Sigma^*$ are suitable open sets which separate x and y. Now let be $x \leq y$ i. e. $y = xz$, where $z \neq \varepsilon$. By assumptions there is an open set 0 such that $\varepsilon \in 0$ and $0 \cap z\,0 = \emptyset$. Therefore $x\,O$ and $y\,O = x(z\,O)$ are open and disjoint.

THEOREM 4.7: *The following statements are equivalent:*

(i) *$X_{\mathscr{L}}$ is a T_2-space;*

(ii) *for all $z \neq \varepsilon$ there is a set A such that*

$$\varepsilon \in A, \qquad z\,A \notin \mathscr{L} \qquad \text{and} \qquad A^c \notin \mathscr{L}.$$

Proof: Let (ii) hold. By lemma 4.6 it is sufficient to show that for all $z \neq \varepsilon$ z and ε can be separated by open sets.

Let be $z \neq \varepsilon$. Then there is a A such that $\varepsilon \in A$, $z\,A \notin \mathscr{L}$ and $A^c \notin \mathscr{L}$. From lemma 3.1, $\partial(\mathscr{L}) = \partial(\mathscr{L}')$, where $\mathscr{L}'$ is the set defined in lemma 2.15. For any set B which does not contain ε, $B \in \mathscr{L}$ iff $B \in \partial(\mathscr{L})$, hence $B \in \mathscr{L}$ iff $B \in \mathscr{L}'$. The sets $z\,A$ and A^c do not contain ε, thus $z\,A \notin \mathscr{L}'$ and $A^c \notin \mathscr{L}'$. This means by lemma 2.15 that there is an open set 0 containing ε such that $z\,A \subseteq O^c$. Therefore $A \subseteq z \setminus O^c$ or equivalently $z \setminus 0 \subseteq A^c$. Would be $z \setminus 0 \in \mathscr{L}$ then by (T2′) $A^c \in \mathscr{L}$ would hold, which is not possible. Hence $z \setminus 0 \notin \mathscr{L}$ which means that z is not in the closure of 0 and so in the interior of O^c. Thus 0 and the interior of O^c separate ε and z.

Let conversely (i) hold. Let be $z \neq \varepsilon$. Then by lemma 4.6 there is an open set 0 containing ε, such that $0 \cap z\,0 = \emptyset$. Set $A = z \setminus O^c$. Since $z \notin 0$, $z \in O^c$ and so $\varepsilon \in A$ holds. Would be $z\,A \in \mathscr{L}$ then, because of (T2′), $O^c \in \mathscr{L}$ and so $\varepsilon \in \alpha_{\mathscr{L}}(O^c) = O^c$ which is a contradiction. Hence $z\,A \notin \mathscr{L}$. Would be $A^c = z \setminus 0 \in \mathscr{L}$ then $z \in \alpha_{\mathscr{L}}(0)$ which means that z is not in the interior of O^c which is a contradiction to $0 \cap z\,0 = \emptyset$. Hence $A^c \notin \mathscr{L}$.

COROLLARY 4.8: *$X_{\mathfrak{P}_0(\Sigma^*)}$ and $X_{\mathscr{U}}$ are not T_2-spaces.*

Proof: Since $\mathcal{O}_{\mathfrak{P}_0(\Sigma^*)}$ is coarser than $\mathcal{O}_{\mathscr{U}}$ it is sufficient to show that $X_{\mathscr{U}}$ is not a T_2-space.

Since for all $z \neq \varepsilon$, $A \subseteqq \Sigma^*$, $zA \notin \mathscr{U}$ implies A is finite and thus $A^c \in \mathscr{U}$, (ii) in theorem 4.7 cannot be true for $\mathscr{U}$.

Because of corollary 4.8, $X_{\ \ 0(\Sigma^*)}$ and $X_{\mathscr{U}}$ are not compact. But the following theorem holds:

THEOREM 4.9: *$X_{\mathfrak{P}_0}(\Sigma^*)$ and $X_{\mathscr{U}}$ are quasicompact.*

Proof: Let be Anf $(A) = \{ w \mid w \setminus A$ is infinite$\}$ as in chapter 3.

It is sufficient to show that $\mathcal{O}_{\mathscr{U}}$ has the finite intersection property.

Let A_i $(i \in I)$ be closed sets such that $\bigcap_{i \in I} A_i = \emptyset$. Would all A_i be infinite then $\varepsilon \in \text{Anf}\ (A_i) \subseteqq A_i$ would hold and so $\varepsilon \in \bigcap_{i \in I} A_i$. Therefore there is a finite $A_{i_0} = \{ x_1, \ldots, x_n \}$. For each j there is a A_{i_j} such that $x_j \notin A_{i_j}$. Hence $A_{i_0} \cap A_{i_1} \cap \ldots \cap A_{i_n} = \emptyset$.

5. MISCELLANEOUS

In this chapter some additional remarks are made.

THEOREM 5.1: *The mapping $\psi_z : (\Sigma^*, \mathcal{O}_{\mathscr{L}}) \to (\Sigma^*, \mathcal{O}_{\mathscr{L}})$ defined by $\psi_z(x) = zx$ is always continuous.*

Proof: Assume A to be a closed set. From $u \in \text{Anf}_{\mathscr{L}}\ (z \setminus A)$ it follows $u \setminus (z \setminus A) = zu \setminus A \in \mathscr{L}$, hence $zu \in \text{Anf}_{\mathscr{L}}(A) \subseteqq A$, thus $u \in z \setminus A$ as required.

Remark that an analoguous statement for $\rho_z(x) = xz$ is not true in general. For $\mathscr{L} = \mathfrak{P}_0(\Sigma^*)$ $(\mathscr{L} = \mathscr{U})$, $z = b$ counterexamples are obtained by taking $\{ \varepsilon, a, ab \}$ $(\{ a \} \{ b \}^* \cup \{ \varepsilon \})$.

From this it follows that the mapping $(x, y) \to xy$ is not continuous in general; the same is true for $x \to x^R$.

To speak about convergence seems to be not very interesting if one takes corollary 4.8 in account.

Now some statements are made in order to generalize the concept of a closure operator.

A closure operator α on a partial ordering is defined by the axioms (A3), (A4) and

$$\text{If} \quad x \leqq y \quad \text{then} \quad \alpha(x) \leqq \alpha(y). \tag{A2'}$$

Since (A2′) is weaker than (A2) this is a generalization.

THEOREM 5.2: *Let* $\mathscr{L}=\alpha(\mathscr{L})$. *Then* $\mathrm{Anf}_{\mathscr{L}}$ *is a closure operator in the above sense iff* (T2) *and* (T4) *are valid.*

Proof: Contained in the proof of theorem 2.3.

More general is the following concept of Lapscher [6].

α is called a closure operator iff (L2) and (L3) are valid:

$$\text{If} \quad x \leqq y \quad \text{then} \quad \alpha(x) \leqq \alpha(y), \tag{L2}$$

$$s_x = \sup(x, \alpha(x)), \tag{L3}$$

and

$$i_x = \inf(x, \alpha(x)) \quad \text{exist and} \quad \alpha(s_x) = \alpha(i_x).$$

[In the case $\mathfrak{P}(\Sigma^*, \subseteq)$ this reads $\alpha(A \cup \alpha(A)) = \alpha(A \cap \alpha(A))$.]

Now some properties are given guaranteeing that some mappings are closure operators in the sense of Lapscher.

$$\text{If} \quad A, \ B \in \mathscr{L} \quad \text{then} \quad A \cap B \in \mathscr{L}, \tag{T5}$$

$$\text{If} \quad A, \ B \in \mathscr{L} \quad \text{then} \quad A^c \notin \mathscr{L}. \tag{T6}$$

From [7] it can be deduced that then

$$\mathrm{Anf}_{\mathscr{L}}(A \cap B) = \mathrm{Anf}_{\mathscr{L}}(A) \cap \mathrm{Anf}_{\mathscr{L}}(B)$$

and

$$\mathrm{Anf}_{\mathscr{L}}(A) \cap \mathrm{Anf}_{\mathscr{L}}(A^c) = \emptyset.$$

Remark that $\mathscr{L} \neq \emptyset$, (T1), (T2), (T5) means that $\mathscr{L}$ is a filter.

Assume in the sequel that (T1)-(T6) are valid.

DEFINITION 5.3:

$$\varphi_{\mathscr{L}}(A) = (\mathrm{Anf}_{\mathscr{L}}(A) \cup A) \cap (\mathrm{Anf}_{\mathscr{L}}(A^c))^c = \mathrm{Anf}_{\mathscr{L}}(A) \cup (A \cap (\mathrm{Anf}_{\mathscr{L}}(A^c))^c).$$

LEMMA 5.4: $\varphi_{\mathscr{L}}(A^c) = (\varphi_{\mathscr{L}}(A))^c$.

Proof:

$$(\varphi_{\mathscr{L}}(A))^c = [(\mathrm{Anf}_{\mathscr{L}}(A) \cup A) \cap (\mathrm{Anf}_{\mathscr{L}}(A^c))^c]^c$$
$$= ((\mathrm{Anf}_{\mathscr{L}}(A))^c \cap A^c) \cup \mathrm{Anf}_{\mathscr{L}}(A^c) = \varphi_{\mathscr{L}}(A^c).$$

THEOREM 5.5: $\varphi_{\mathscr{L}}$ *is a closure operator in the sense of Lapscher.*

Proof: (L2): If $A \subseteq B$ then $\alpha_{\mathscr{L}}(A) \subseteq \alpha_{\mathscr{L}}(B)$ and $\mathrm{Anf}_{\mathscr{L}}(A^c) \supseteq \mathrm{Anf}_{\mathscr{L}}(B^c)$.

(L3):

$$\varphi_{\mathscr{L}}(A) \cup A = A \cup \mathrm{Anf}_{\mathscr{L}}(A) \cup (A \cap (\mathrm{Anf}_{\mathscr{L}}(A^c))^c) = \alpha_{\mathscr{L}}(A).$$

$$\varphi_{\mathscr{L}}(A) \cap A = ((\varphi_{\mathscr{L}}(A))^c \cup A^c)^c = (\varphi_{\mathscr{L}}(A^c) \cup A^c)^c = (\alpha_{\mathscr{L}}(A^c))^c.$$

Hence

$$\varphi_{\mathscr{L}}(A \cup \varphi_{\mathscr{L}}(A)) = \varphi_{\mathscr{L}}(\alpha_{\mathscr{L}}(A)) = \varphi_{\mathscr{L}}(A \cup \mathrm{Anf}_{\mathscr{L}}(A))$$
$$= [\mathrm{Anf}_{\mathscr{L}}(A \cup \mathrm{Anf}_{\mathscr{L}}(A)) \cup A \cup \mathrm{Anf}_{\mathscr{L}}(A)]$$
$$\cap [\mathrm{Anf}_{\mathscr{L}}(A \cup \mathrm{Anf}_{\mathscr{L}}(A))^c]^c = [\mathrm{Anf}_{\mathscr{L}}(A) \cup \mathrm{Anf}_{\mathscr{L}}(\mathrm{Anf}_{\mathscr{L}}(A)) \cup A \cup \mathrm{Anf}_{\mathscr{L}}(A)]$$
$$\cap [\mathrm{Anf}_{\mathscr{L}}(A^c) \cap \mathrm{Anf}_{\mathscr{L}}((\mathrm{Anf}_{\mathscr{L}}(A))^c)]^c = \alpha_{\mathscr{L}}(A) \cap [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c = \varphi_{\mathscr{L}}(A),$$

since from

$$\mathrm{Anf}_{\mathscr{L}}(X) \cap \mathrm{Anf}_{\mathscr{L}}(X^c) = \emptyset$$

it follows $\mathrm{Anf}_{\mathscr{L}}(X) \subseteqq \mathrm{Anf}_{\mathscr{L}}(X^c)^c$, and thus

$$\mathrm{Anf}_{\mathscr{L}}((\mathrm{Anf}_{\mathscr{L}}(A))^c) \supseteqq \mathrm{Anf}_{\mathscr{L}}(\mathrm{Anf}_{\mathscr{L}}(A^c)) = \mathrm{Anf}_{\mathscr{L}}(A^c).$$

$$\varphi_{\mathscr{L}}(A \cap \varphi_{\mathscr{L}}(A)) = \varphi_{\mathscr{L}}((\alpha_{\mathscr{L}}(A^c))^c) = (\varphi_{\mathscr{L}}(\alpha_{\mathscr{L}}(A^c))^c$$
$$= (\varphi_{\mathscr{L}}(A^c))^c = \varphi_{\mathscr{L}}(A^{cc}) = \varphi_{\mathscr{L}}(A).$$

Remark: One can say that $\varphi_{\mathscr{L}}(A)$ "approximates" A, since the difference is "not too much" and $\varphi_{\mathscr{L}}(A)$ is a "simpler set", in the sense of the following theorem.

THEOREM 5.6: *Assume that $\mathscr{L}$ fulfills additionally*

$$\textit{If} \quad L \in \mathscr{L}, \quad z \in \Sigma^* \qquad \textit{then} \quad z \diagdown L \in \mathscr{L}. \tag{T7}$$

Let $x \sim_L y$ be the classical right congruence defined by

$$xz \in L \qquad \textit{iff} \quad yz \in L \quad \textit{for all } z$$

and $x \,\theta_L\, y$ be defined by $x \sim_L y$ or $\{z \mid xz \in L \text{ and } yz \in L\} \in \mathscr{L}$ or

$$\{z \mid xz \notin L \text{ and } yz \notin L\} \in \mathscr{L},$$

then:

$$x \sim_{\varphi_{\mathscr{L}}(L)} y \qquad \textit{iff} \quad x \,\theta_L\, y.$$

Proof: The proof is not hard but long and therefore omitted.

Remark: A similar argument shows

$$x \sim_{\alpha_{\mathscr{L}}(L)} y \qquad \text{iff} \quad x \sim_L y \quad \text{or} \quad \{z \mid xz \in L \text{ and } yz \in L\} \in \mathscr{L}.$$

The last remark deals with the possibility to define topologies on $\mathfrak{P}(X)$ (*see* [3]).

(This seems to be not quite uninteresting since languages are somehow more interesting than words from some points of view.)

A concrete possibility is presented.

Assume $(X, \mathcal{O})$ to be a topological space and let the bar denotes its closure operator. Define

$$\alpha: \ \mathfrak{P}(\mathfrak{P}(X)) \to \mathfrak{P}(\mathfrak{P}(X))$$

by

$$\{A_\lambda \mid \lambda \in \Lambda\} \to \bigcup_{\lambda \in \Lambda} \{B \mid B \subseteqq \overline{A_\lambda}\}.$$

It is easy to verify the axioms (A1) – (A4).

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