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A NOTE ON GRAPH COLORING

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Abstract. — *A result concerning edge colorings in graphs is extended to the case of vertex colorings. Let $S_1, \dots S_k$ be a coloring of the vertices of G and let s_i be the cardinality of S_i. It is shown that there always exists a k-coloring with*

$$|s_j - s_i| \leqslant (l-2) \min(s_i, s_j) + 1 \quad \textit{for any} \quad i, j.$$

where l is such that no vertex belongs to more than l maximal cliques.

A multigraph consists of a finite nonempty set X of vertices and a set U of edges. A *k-edge-coloring* is a partition of U into subsets $H_1, H_2, \dots, H_k$ such that no two edges in the same H_k are adjacent. Let h_i be the cardinality of $H_i (i = 1, \dots, k)$, We will say that the sequence $(h_1, h_2, \dots, h_k)$ where $h_1 \geqslant h_2 \geqslant \dots \geqslant h_k$ is *color-feasible* in G.

The following proposition appears in [1] and [2] :

Proposition 1 : If $(h_1, h_2, \dots, h_k)$ is color-feasible in G, then any sequence $(h_1, h_2, \dots, h_k)$ with :

a) $$h'_1 \geqslant \dots \geqslant h'_k$$

b) $$\sum_{i=1}^{l} h'_i \leqslant \sum_{i=1}^{l} h_i \qquad l = 1, \dots, k-1$$

c) $$\sum_{i=1}^{k} h'_i = \sum_{i=1}^{k} h_i$$

is color-feasible in G.

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Let the *chromatic index* $q(G)$ of G be the smallest k for which G has a k-edge-coloring. As a consequence of proposition 1 we have :

Proposition 2 : For any $k \geqslant q(G)$, G has a k-edge coloring

$$H_1, \ldots, H_k \text{ with } |h_i - h_j| \leqslant 1, \qquad i, j = 1, \ldots, k$$

In this note we will extend these results to the more general case of vertex colorings.

A *k-coloring* of G is a partition of its vertices into subsets $S_1, S_2, \ldots, S_k$ of nonadjacent vertices.

(Note that whenever we are dealing with vertex colorings, we just have to consider *simple* graphs, i.e. graphs without multiple edges.)

The *chromatic number* $\gamma(G)$ of G is the smallest k for which G has a k-coloring.

A *clique* K in G is a subset of vertices such that any two vertices in K are adjacent in G. A clique K is *maximal* if there is no clique K' in G which strictly contains K.

Given a subset A of X, $\langle A \rangle$ will denote the subgraph spanned by A : its edges are those edges of G with both endpoints in A. The *degree* of a vertex x in G is the number of edges in G which are adjacent to x.

Let $S_1, S_2, \ldots, S_k$ define a k-coloring of G ; s_i will denote the cardinality of S_i. We assume that no vertex in G belongs to more than l maximal cliques ($l \geqslant 2$). If $l = 1$, each connected component G' of G is a clique.

Proposition 3 : The degrees in the subgraph $\langle S_i \cup S_j \rangle$ are at most l for any i, j.

Proof : Assume a vertex x in S_i is adjacent to $p > l$ vertices $x_1, x_2, \ldots, x_p$ in S_j ; any two of these vertices are nonadjacent, so they cannot belong to the same clique. Hence the maximal cliques K_i containing x and x_i are distinct ($i = 1, \ldots, p$) which is a contradiction.

Proposition 4 : Let $S'_i \subset S_i, S'_j \subset S_j$ define a connected component $G' = \langle S'_i \cup S'_j \rangle$ of $\langle S_i \cup S_j \rangle$; then $|s'_j - s'_i| \leqslant (l-2) \min(s'_i, s'_j) + 1$.

Proof : Suppose $s'_i = p$ and $s'_j > (l-1)p + 1$; since G' is bipartite, it has at most $l \cdot p$ edges (no degree exceeds l) ; however G' has more than $p + (l-1)p + 1 = l \cdot p + 1$ vertices, hence it cannot be connected, so $s'_j \leqslant (l-1)p + 1$ and the proposition follows.

Proposition 5 : Given a k-coloring $S_1, S_2, ..., S_k$ of a graph G where no vertex belongs to more than l maximal cliques, any two subsets S_i, S_j with $s_j > (l-1)s_i + 1$ may be replaced by two subsets $\bar{S}_i, \bar{S}_j$ satisfying

$$|\bar{s}_j - \bar{s}_i| \leqslant (l-2) \min(\bar{s}_i, \bar{s}_j) + 1.$$

Proof : Let $s_j = s_i + K$ with $K > (l-2)s_i + 1$; then $G' = \langle S_i \cup S_j \rangle$ is not connected and there is a connected component $\langle S'_i \cup S'_j \rangle$ of G' with $s_j' = s_i' + K'$ where $0 < K' \leqslant (l-2)s'_i + 1 \leqslant (l-2)s_i + 1 < K$.

By interchanging the vertices of S'_i and S'_j we obtain two subsets $\bar{S}_i$ and $\bar{S}_j$ of nonadjacent vertices.

They satisfy :

$$s_i = s_j - K < \bar{s}_j = s_j - K' < s_j$$

$$s_i < s_i + K' = s_i < s_i + K = s_j$$

Hence $|\bar{s}_j - \bar{s}_i| < K = |s_j - s_i|$.

If $|\bar{s}_j - \bar{s}_i| > (l-2) \min(\bar{s}_i, \bar{s}_j) + 1 > K$ the interchange procedure may be reiterated and finally we will obtain two subsets $\bar{S}_i, \bar{S}_j$ satisfying

$$|\bar{s}_j - \bar{s}_i| \leqslant (l-2) \min(\bar{s}_i, \bar{s}_j) + 1.$$

Quite similarly to the case of edge-colorings, we say that a sequence $(s_1, s_2, ..., s_k)$ with $s_1 \geqslant s_2 \geqslant ... \geqslant s_k$ is color-feasible in G if there exists a k-coloring $S_1, S_2, ..., S_k$ of G where S_i has cardinality $s_i (i = 1, ..., k)$.

According to Proposition 5, let $S = (s_1, s_2, ..., s_k)$ be color-feasible in G ; if $S' = (s'_1, s'_2, ..., s'_k)$ is any sequence obtained from S by interchanges between subsets S_i, S_j with $|s_j - s_i| > (l-2) \min(s_i, s_j) + 1$, then S' is also color-feasible.

In particular, by making successive interchanges, we obtain :

Proposition 6 : Let G be a graph with chromatic number $\gamma(G)$ and where no vertex belongs to more than l maximal cliques ; then for any $k \geqslant \gamma(G)$, there exists a color-feasible sequence $(s_1, s_2, ..., s_k)$ with $s_1 \leqslant (l-1)s_k + 1$.

We conclude this note with a few remarks :

REMARK 1 : Proposition 6 should be related to a theorem of Hajnal and Szemerédi [3] : For any graph G with maximum degree h, there exists a color-feasible sequence $(s_1, s_2, ..., s_{h+1})$, with $s_1 \leqslant s_{h+1} + 1$.

In other words, if $h + 1$ colors are to be used for the vertices of G, then it is always possible to find an $(h+1)$ — coloring where all cardinalities of the $S'_i s$ are within 1.

However if less than $h + 1$ colors may be used, then it is not always possible to do so. As an exemple consider graph G_1 with 4 vertices u, v, w, x and 3 edges (u, v), (u, w), (u, x) ; the only way of coloring its vertices with $2 < h + 1 = 4$ colors is $S_1 = \{ v, w, x \}$., $S_2 = \{ u \}$ and so we have $s_1 - s_2 = 3 - 1 = (l - 2)s_2 + 1 = 1 + 1 > 1$ since u belongs to $l = 3$ maximal cliques.

REMARK 2 : It is well known that an edge coloring problem in G may be reduced to a vertex coloring problem in a graph G' whose vertices are the edges of G : any two adjacent edges in G' are represented by adjacent vertices in G' and there exists in G' a family F of cliques such that :

a) each pair of adjacent vertices belongs to exactly one clique of F ;

b) each vertex belongs to at most 2 cliques of F (F contains all maximal cliques of G' which are not normal triangles (4, p. 390]).

Thus any subset S of vertices in G' with $|S \cap K| \leqslant 1$ for any clique K of F represents a subset of nonadjacent edges in G.

It is thus possible to consider that the only « maximal » cliques of G' are those in F ; so $l = 2$ and it follows from Proposition 5 that interchanges can be made between S_i and S_j whenever $|s_j - s_i| > 1$. This means of course that Propositions 1 and 2 are valid.

REMARK 3 : One could think of deducing the result of Hajnal and Szemerédi from Proposition 6 in the following way : if for any graph G with maximum degree h it is possible to introduce some edges in such a way that

a) the maximum degree is still h

b) each vertex belongs to at most 2 maximum cliques of the new graph, then obviously (since $\gamma(G) \leqslant h + 1$) it is possible to find an $(h + 1)$-coloring $S_1, \ldots, S_k$ with $s_1 - s_k \leqslant 1$.

Unfortunately, this is not true as is shown by considering graph G_2 with vertices $x_1, x_2, x_3, y_1, y_2, y_3$ and edges $(x_i, y_j) i, j = 1, 2, 3$; each vertex belongs to 3 maximal cliques and the introduction of any supplementary edge increases the maximum degree.

REMARK 4 : Finally Proposition 6 may be formulated in terms of hypergraphs (notions which are not defined here can be found in [4]) ; we want to color the edges of a hypergraph H in such a way that no 2 edges E_i, E_j with $E_i \cap E_j \neq \emptyset$ are of the same color. Now l is the rank of H i.e.

$$r(H) = \max_i |E_i| = l$$

and let $q(H)$ be the minimum number of colors required to color the edges of H ; then for any $k \geqslant q(H)$ there exists a k-edge-coloring $S_1, \ldots, S_k$ of H with $|s_i - s_j| \leqslant (r(H) - 2) \min(s_i, s_j) + 1$.

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