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VALUED VECTOR SPACES AND ABELIAN p-GROUPS

by Laszlo FUCHS

Let A be an abelian p-group, written additively, where p denotes a fixed prime. As usual, we define $pA = \{pa ; a \in A\}$; and in general, $p^\sigma A$ (for ordinals σ) is defined transfinitely by

$$p^{\sigma+1} A = p(p^\sigma A) \quad \text{and} \quad p^\rho A = \bigcap_{\sigma<\rho} p^\sigma A$$

for limit ordinals ρ . If A has no divisible (i. e. injective) subgroups $\neq 0$ (which can be assumed without loss of generality), then $p^\tau A = 0$ for some ordinal τ . The height $h(a)$ of $a \in A$ is defined by setting $h(a) = \sigma$ if $a \in p^\sigma A \setminus p^{\sigma+1} A$ and $h(0) = \infty$. Now we concentrate on the socle

$$A[p] = \{a \in A ; \ pa = 0\}$$

of A which is viewed as a vector space over $\mathbb{Z}/(p)$, furnished with the height as "valuation". In this way, we obtain what is called a valued vector space.

H. PRÜFER (in 1923) was the first to discover the relevance of the socle. In 1955, B. CHARLES has investigated the socles of p-groups without elements of infinite heights [1]. Later K. HONDA [10], P. HILL [8] and P. HILL-C. MEGIBBEN [9] studied various aspects of socles. In order to make use of the socles in the structure theory of abelian p-groups, in 1975, I investigated systematically valued vector spaces, using certain ideas from non-archimedean Banach spaces and totally ordered vector spaces. There are a few interesting developments, and my present goal is to give a short survey of the most essential results in the study of socles via valued vector spaces.

PART I. Results on valued vector spaces.

Here we collect the basic definitions and results needed for applications in abelian p-groups. We restrict ourselves to the special case when the values are ordinal numbers and the underlying scalar field is trivially valued, though most of the results extend easily to more general cases.

1. Basic definitions.

Let Γ denote the class of ordinals and let Φ ~~be an arbitrary~~ field. Vector spaces V only over Φ will be considered.

A valuation v of V is a function

$$v : V \longrightarrow \Gamma \cup \{\infty\}$$

(the symbol ∞ is regarded as being larger than any ordinal) satisfying the following conditions :

(i) $v(a) = \infty$ if, and only if, $a = 0$,

(ii) $v(\alpha a) = v(a)$ for all $a \in V$ and non zero $\alpha \in \Phi$,

(iii) $v(a + b) \geqslant \min(v(a) , v(b))$ for all $a , b \in V$.

A valued vector space is a pair (V , v) where V is a vector space and v is a valuation of V . These are the objects of the category $\mathcal{V}$ of valued vector spaces, the morphisms

$$\mu : (V , v) \longrightarrow (V' , v')$$

are those Φ-linear maps $\mu : V \longrightarrow V'$ which satisfy

$$v'(\mu a) \geqslant v(a) \quad \text{for all} \quad a \in V .$$

Two valued vector spaces are isometric if they are isomorphic in $\mathcal{V}$, i. e. there is a vector space isomorphism between them that preserves values.

For a subspace U of V , the restriction of v to U makes U into a valued vector space, $(U , v|U)$ is a subobject of (V , v) . A $\mathcal{V}$-map $\mu : (W , w) \to (V , v)$ is an injection (or embedding) if μ is an isometry of (W , w) with a subobject of (V , v) . This has to be distinguished from the $\mathcal{V}$-monomorphisms which are those $\mathcal{V}$-maps which are one-to-one on W .

Let U be a subspace of V . The quotient space V/U can be equipped with the valuation

$$v(a + U) = \sup_{u \in U} v(a + u) .$$

It is straightforward to check that $\mathcal{V}$ is an additive category with kemels and cokemels, limits and colimits. It is, however, not abelian (not all monic $\mathcal{V}$-maps are kemels).

It is useful to describe explicitly the categorical coproduct and product of a set of valued vector spaces (V_i , v_i) , $i \in I$. Their coproduct (V , v) is the vector space direct sum $V = \bigoplus V_i$ where the value of an $a = \sum a_i \in V$ is defined as $\min_i v_i(a_i)$. Their product (V^* , v^*) is the cartesian product $V^* = \prod V_i$ where the value of $a = (\ldots , a_i , \ldots) \in V^*$ is again $\min_i v_i(a_i)$. We write

$$(V , v) = \coprod (V_i , v_i) , \text{ and } (V^* , v^*) = \prod (V_i , v_i) .$$

An element a of a valued vector space (V , v) is said to be orthogonal to a subspace U of V if

$$v(a) \geqslant v(a + u) , \text{ for all } u \in U .$$

A subspace W of V is orthogonal to U , if every $a \in W$ is orthogonal to U . The orthogonality of subspaces is a symmetric relation, and U , W are orthogonal subspaces of V , if $U + W$ is canonically isometric to the coproduct $U \amalg W$ of U and W .

A topology-like structure can be introduced in V by declaring a subspace U of V s-dense (spherically dense) whenever 0 is the only vector in V orthogonal to U. On the other hand, U is s-closed in V, if U is not contained properly in any subspace of V as an s-dense subspace. Or, equivalently, every coset of V mod U contains an element orthogonal to U. For every subspace U of V, there exists a (not necessarily unique) subspace U^* of V such that U is s-dense in U^*, and U^* is s-closed in V. Such a U^* is an s-closure of U in V. For details we refer to [4].

By the support of V is meant the set $\{v(a) \; ; \; 0 \neq a \in V\}$, while the lenght of V is defined as $\sup \{v(a) + 1 \; ; \; 0 \neq a \in V\}$. With every ordinal σ, there are associated the subspaces

$$V^\sigma = \{a \in V \; ; \; v(a) \geqslant \sigma\} \text{ , and } V_\sigma = \{a \in V \; ; \; v(a) > \sigma\} .$$

Clearly, $V^\sigma \supseteq V_\sigma$, for every ordinal σ.

2. Free valued vector spaces.

By a valued set (X , g) is meant a set X along with a function $g : X \longrightarrow \Gamma$. A free valued vector space (F , f) on the valued set (X , g) is a valued vector space with an inclusion map $i : X \longrightarrow F$ preserving values such that : given any function $h : X \longrightarrow V$ into a valued vector space (V , v) satisfying $v(h(x)) \geqslant g(x)$ for all $x \in X$, there exists a unique $\mathcal{V}$-morphism

$$\mu : (F , f) \longrightarrow (V , v) \text{ such that } \mu i = h .$$

It is easy to see that on any valued set (X , g) there is a valued vector space (F , f), unique up to isometry, namely.

$$F = \coprod_{x \in X} \Phi x \text{ , where } v(x) = g(x) \text{ for } x \in X .$$

The functor that associates (F , f) with (X , g) is the left adjoint of the forgetful functor from $\mathcal{V}$ into the category of valued sets.

There are enough free objects in $\mathcal{V}$.

THEOREM 1. – *Every valued vector space is isometric to a quotient* F/K *of some free valued vector space* F. *Moreover* K *can be chosen so as to be* s-*closed in* F.

The proof is standard, see e. g. [4].

There is an important functor that associates with every valued vector space V a free valued vector space B. This functor is the grading functor $gr : V \longrightarrow B$, and B is isometric to a basic subspace of V. By a basic subspace is meant a free s-dense subspace ; such ones do exist and are unique up to isometry (see [4]). It is convenient to write and to use as a standard notation

$$B = \coprod_\sigma B(\sigma)$$

where $B(\sigma)$ is the coproduct of those 1-dimensional subspaces (in some representation of B as a free valued vector space) where each vector $\neq 0$ has value σ (we say : $B(\sigma)$ is σ-homogeneous).

We have the Conrad-Fleischer embedding theorem [4].

THEOREM 2. - If V is a valued vector space and $B = \coprod_\sigma B(\sigma)$ is a basic subspace of V, then there is an injection

$$V \longrightarrow \prod_\sigma B(\sigma)$$

which is the identity on B.

Let us mention a few results of interest on free valued vector spaces ; for details we refer to [4].

THEOREM 3. - Finite valued subspaces of valued vector spaces are free summands.

THEOREM 4. - A valued vector space with countable support is free if, and only if, it is the union of a countable ascending chain of finite-valued vector spaces.

THEOREM 5. - Countable dimensional valued vector spaces are free.

THEOREM 6. - Countable-valued subspaces of free valued vector spaces are free.

3. Injective valued vector spaces.

Adopting the costumary definition, we call a valued vector space T injective, if for every injection $\psi : U \longrightarrow V$ and any γ-map $\xi : U \longrightarrow T$ there exists a γ-map $\eta : V \longrightarrow T$ such that $\eta\psi = \xi$.

Homogeneous vector spaces are always injective. A product of valued vector spaces is injective exactly, if all components are injective. Hence $\prod B(\sigma)$ in theorem 2 is injective, and we see that every valued vector space is contained in an injective valued vector space. For the discussion of injective hulls, the following concept is needed.

A valued vector space V is said to be s-complete if it is not s-dense in any larger valued vector space. A proof given by INGLETON [11] (in terms of spheres) applies to show :

THEOREM 7. - A valued vector space is injective if, and only if, it is s-complete.

If T is any injective valued vector space containing a valued vector space V, then an s-closure V^* of V in T turns out to be injective, and it is easy to see that it has to be a minimal injective containing V. This is the injective hull of V ; it is unique up to isometry. More over we have the following theorem.

THEOREM 8. - Every valued vector space V has an injective hull $\hat{V}$, unique up to isometry. V is s-dense in $\hat{V}$. If $B = \coprod_\sigma B(\sigma)$ is a basic subspace of V, then we have the isometry

$$\hat{V} \cong \prod_{\sigma} B(\sigma) \ .$$

There is a subspace of the injective hull $\hat{V}$ which we shall need in the sequel. Let λ = lenght of V be a limit ordinal, and define a subspace W of $\hat{V}$ via

$$W/V = (\hat{V}/V)^{\lambda} \ ,$$

i. e. W/V is the largest λ-homogeneous subspace of $\hat{V}/V$. This W can be characterized as follows.

PROPOSITION 1. - W <u>is determined, up to isometry over</u> V, <u>by the following properties</u>

(a) V <u>is</u> s-<u>dense in</u> W,

(b) W/V <u>is</u> λ-<u>homogeneous</u>,

(c) W <u>is maximal with respect to the properties</u> (a) <u>and</u> (b).

It will be called the augmentation of V and will be denoted as $W = V^{[\lambda]}$. Notice that we have defined it for λ = lengh of V only.

It should be pointed out that $V^{[\lambda]}$ can also be obtained as an inverse limit. In fact, for $\sigma < \rho < \lambda$ let

$$\pi_{\sigma\rho} : V/V^{\rho} \longrightarrow V/V^{\sigma}$$

be the canonical map $a + V\rho \longmapsto a + V^{\sigma}$. For the arising inverse system of valued vector spaces we can prove the following theorem.

THEOREM 9. - $V^{[\lambda]} \cong \text{proj lim } \{V/V^{\sigma} \ ; \ \pi_{\sigma\rho}\}$.

4. Subspaces of free valued vector spaces.

A valued vector space S is called <u>subfree</u> if it is an s-dense subspace of some free valued vector space F. Any basic subspace of S is basic in F, so it is isometric to F. Consequently, F is unique up to isometry.

Subfree valued vector spaces are not necessarily free, as is shown by the following example.

<u>Example</u>. - Let $F = \coprod \Phi a_{\sigma}$ be a free valued vector space where $v(a_{\sigma}) = \sigma$, and σ runs over all ordinals less than some limit ordinal λ not cofinal with ω. Define

$$S = \sum_{\rho,\sigma<\lambda} \Phi(a_{\sigma} - a_{\rho}) \ ,$$

i. e. the subspace spanned by all $a_{\sigma} - a_{\rho}$ with $\rho, \sigma < \lambda$. It is immediate that $S \neq F$, and that S and any a_{σ} span F. Thus F/S is 1-dimensional of value λ. Since F has no vector of value λ, S has to be s-dense in F, i. e. S is subfree. By way of contradiction, assume S is free, say $S = \coprod \Phi b_{\sigma}$ (it has to be isometric to F as its basic subspace). If $x \in F \setminus S$, then for every $\sigma < \lambda$, there is a $c_{\sigma} \in S$ such that $v(x - c_{\sigma}) > \sigma$. The coordinates of c_{σ} and c_{ρ}

$(\sigma < \rho)$ are the same in every Φb_γ with $\gamma < \sigma$, so $x \notin S$ implies that there is a sequence $\rho_1 < \dots < \rho_n < \dots$ of ordinals such that $c_{\rho_{n+1}}$ has a non-zero coordinate in some $\Phi b_{\gamma_{n+1}}$ where $c_{\rho_1}, \dots, c_{\rho_n}$ have 0 coordinates. From cof $\lambda \neq \omega$ we infer that there is a $\rho_0 < \lambda$ with $\sup \rho_n < \rho_0$. Then c_{ρ_0} has the same non-zero coordinate as c_{ρ_n} in Φb_{γ_n}, a contradiction to $c_{\rho_0} \in S$. Hence S is sub-free, but not free.

There are known several criteria under which a subspace of a free valued vector space is again free. Here we mention only two results of this kind : theorem 10 and 12.

THEOREM 10. - Let S be a subfree valued vector space, i. e. s-dense in some free F. If the support of F/S contains only ordinals cofinal with ω, then S is free.

The proof of this result is rather lengthy, see [5]. To obtain a corollary to theorem 10, the following result is needed.

THEOREM 11. - Let S be an s-dense subspace in a free valued vector space F. There is a subspace U of F such that

(i) $S \subseteq U \subseteq F$,

(ii) all non-zero vectors in U/S have values $\neq$ cof ω,

(iii) all non-zero vectors in F/U have values cof ω.

COROLLARY 1. - For any subfree valued vector space S there exists a free valued vector space F in which S is s-dense and all the values of vectors in F/S are not cofinal with ω.

In fact, by theorem 10 and 11, F can be replaced by U in the definition of sub-free spaces.

We say that a valued vector space V satisfies the countability condition if $X \subset$ support of V and $\lambda = \sup X$ implies $\lambda = \sup Y$ for some countable subset Y of X. This condition suffices to prove the following theorem.

THEOREM 12. - An s-closed subspace of a free valued vector space with the countability condition is likewise free.

Notice that here "s-closed" can be omitted in view of theorem 10.

PART II. Valued vector spaces in p-groups.

We wish to apply the general results on valued vector spaces to the theory of p-groups. As pointed out in the introduction, the socles are in a naturel way valued vector spaces. The study of the socles, however, needs a more general perspective, and our main objective is now to study what we call slices and their relations to the groups.

5. Slices of abelian groups.

In this section, λ will always denote a limit ordinal.

By the λ-slice of an abelian group A, we mean the socle of the quotient $A/p^\lambda A$ as a valued vector space over $\mathbb{Z}/(p)$, equipped with the height as valuation :

$$S_\lambda(A) = (A/p^\lambda A)[p] .$$

Since $p^\lambda A$ is a so-called "nice" subgroup, the heights in $S_\lambda(A)$ can be computed not only via $A/p^\lambda A$, but also as the supremum (= maximum in this case) of heights of elements of A in the coset. In the definition, it is irrelevant that λ is a limit ordinal, but for non-limit ordinals the slices do not seem to be of interest in our present subject.

Let $\delta : A \longrightarrow A/p^\lambda A$ denote the canonical map. We define

$$U_\lambda(A) = \delta(A[p]) = (A[p] + p^\lambda A)/p^\lambda A .$$

Our main concern is now the relation between $U_\lambda(A)$ and $S_\lambda(A)$. Though we are interested in p-groups, now we do not assume A is one : it can be an arbitrary abelian group. The following basic fact is straight foward to prove the following theorem.

THEOREM 13. - For any abelian group A, any prime p, and any limit ordinal λ, the following is true

(1) $$U_\lambda(A) \subseteq S_\lambda(A) \subseteq U_\lambda(A)^{[\lambda]} ,$$

(2) $$S_\lambda(A)/U_\lambda(A) \cong p^\lambda A/p^{\lambda+1} A .$$

This result yields an essential information on how the subgroup $p^\lambda A$ and the quotient $A/p^\lambda A$ are glued together to form the group A. For a fuller information, a finer analysis of (2) is required. This is based on a filtration induced by A.

It is costumary to denote, for a subgroup G of A, by $p^{-1} G$ the set of all $a \in A$ with $pa \in G$, i. e. the set of elements contained in the cosets $(A/G)[p]$. For every integer $n \geq 1$, we write

$$U_\lambda^n(A) = [\{p^{-1}(p^\lambda A) \cap A[p^n]\} + p^\lambda A]/p^\lambda A .$$

With increasing n, these form an increasing chain of subspaces of $S_\lambda(A)$ (i. e. filtration)

$$U_\lambda(A) = U_\lambda^1(A) \subseteq \dots \subseteq U_\lambda^n(A) \subseteq \dots \subseteq U_\lambda^\omega(A) \subseteq S_\lambda(A)$$

where

$$U_\lambda^\omega(A) = \cup_n U_\lambda^n(A) = [\{p^{-1}(p^\lambda A) \cap T_p\} + p^\lambda A]/p^\lambda A ,$$

here T_p stands for the p-component of the torsion part of A. It is readily observed that $U_\lambda^n(A)\backslash U_\lambda^{n-1}(A)$ consists of all cosets in $S_\lambda(A)$ which can be represented by elements of order p^n of A, but not by elements of order $\leqslant p^{n-1}$, and $S_\lambda(A)\backslash U_\lambda^\omega(A)$ consists of those which have no representatives of finite order in A. For matter of convenience, we set $U_\lambda^{\omega+1} = S_\lambda$.

For every $n \geqslant 1$, we can select a basis $\{\bar{a}_{n_i}\}$ $(i \in I_n)$ of the quotient vector space $U_\lambda^{n+1}(A)/U_\lambda^n(A)$ (including $n = \omega$), and pich a representative $a_{n_i} \in \bar{a}_{n_i}$ of smallest order in its coset mod $p^\lambda A$. Then the following holds.

LEMMA 1. - <u>The elements</u> $b_{n_i} = pa_{n_i}$, <u>taken for all</u> $n \leqslant \omega$ <u>and all</u> $i \in I_n$, <u>form a</u> p-<u>basic subgroup of</u> $p^\lambda A$.

For the theory of p-basic subgroups, we refer to [3]. They are useful invariants of arbitrary abelian groups. With the aid of the information contained in this lemma, the construction of A from $p^\lambda A$ and $A/p^\lambda A$ can be performed in a more explicit fashion, though in general it does not determine A.

6. The socles of p-groups.

One of the basic questions is to characterize the socles of abelian p-groups A as valued vector spaces. (Needless to say, $A[p]$ is viewed as a vector space over $\mathbb{Z}/(p)$ with the height as valuation.) We consider this problem in order to understand better the structure of p-groups in general.

Let V be a valued vector space over the field $\Phi = \mathbb{Z}/(p)$ and let $B = \coprod_\sigma B(\sigma)$ denote a basic subspace of V. Here each $B(\sigma)$ is σ-homogeneous free and σ runs over a set of ordinals. From results in [4], one can easily verify that there is a decomposition

$$V = C(\lambda) \amalg V^\lambda \text{ , for every } \lambda$$

where we can assume that

$$B(\sigma) \subseteq C(\lambda) \quad \text{for every } \sigma < \lambda \text{ ,}$$

while we always have $B(\sigma) \subseteq V^\lambda$ for $\sigma \geqslant \lambda$. Here $C(\lambda)$ is unique up to isometry only.

From theorem 13 and lemma 1, it follows easily the lemma 2.

LEMMA 2. - <u>For</u> $V = A[p]$, <u>we have that</u> $C(\lambda)$ <u>has lenght</u> λ <u>and</u>

(1) $$\dim C(\lambda)^{[\lambda]}/C(\lambda) \geqslant \sum_{\lambda \leqslant \sigma < \lambda+\omega} \dim B(\sigma)$$

<u>for every limit ordinal</u> λ <u>not exceeding the length of</u> V.

This lemma provides us with a necessary condition for V to be isometric to $A[p]$ for some A. For valued vector spaces of countable length, this turns out to be sufficient as well. For uncountable length, the precise condition is not yet known. The difficulty lies in the peculiar behavior of abelian groups when inverse limits are taken over ordinals tending to a limit ordinal $\neq$ cof ω.

THEOREM 14. - A valued vector space V of countable length τ is isometric to the socle of a p-group A if, and only if, it satisfies (1) for every limit ordinal $\lambda \leq \tau$.

The proof makes use of heavy machinery of abelian group theory, in particular, of certain existence theorems. Here we can only state the fundamental lemmas on which the proof is based.

Recall that a subgroup G of a p-group T is called isotype if

(2) $$p^\sigma G = G \cap p^\sigma T \text{ , for every ordinal } \sigma .$$

G is p^λ-pure in T if the exact sequence $0 \longrightarrow G \longrightarrow T \longrightarrow T/G \longrightarrow 0$ represents an element of $p^\lambda \operatorname{Ext}(T/G , G)$. If G is p^λ-pure in T, then by [12], (2) holds for all $\sigma < \lambda$.

LEMMA 3. - Let G be a p-group of limit length λ. Then there is a p-group T of the same length such that

(i) $T[p] = V$ where V is any vector space between $G[p]$ and $G[p]^{[\lambda]}$,
(ii) G is p^λ-pure in T,
(iii) $T = G + p^\sigma T$, for every $\sigma < \lambda$.

LEMMA 4. - Let T be a p-group of limit length λ, and V a subsocle of T that is dense in the λ-topology of $T[p]$. Then V supports a p^λ-pure subgroup G of T such that T/G is divisible.

These combined with a modified form of the existence theorem [3 (t. 2, theorem 105 105-3, p. 210)] for p-groups, lead to theorem 14.

7. The socles of totally projective p-groups.

There is an important class of p-groups for which a satisfactory structure theorem is known : this is the class of totally projective p-groups.

A totally projective p-group can be characterized in various ways. By the original definition of Nunke [14], a reduced p-group A is totally projective if

$$p^\sigma \operatorname{Ext}(A/p^\sigma A , C) = 0 \text{ , for every } \sigma \text{ and every group } C .$$

HILL [7] proved that within the class of totally projective p-groups, the groups are characterized by their Ulm-Kaplansky invariants, i. e. by the basic subspaces of their socles.

The obvious question presents itself : which valued vector spaces are isometric to socles of totally projective p-groups ?

First, the most important fact should be stated.

THEOREM 15. - The socles of totally projective p-groups are subfree valued vector spaces.

The proof is by transfinite induction on the length.

So far, we do not have a satisfactory characterization of the socles of totally projectives. Instead of discussing cumbersome conditions, we note that their socles are completely determined by the basic subspaces. So the problem can be split into two parts. First, when a free valued vector space can be a basic subspace of the socle of a totally projective p-group, secondly, how to construct the space from its basic subspace. The existence theorem on totally projectives by Hill [7] and Crawley-Hales [2] yields an immediate answer to the first question.

THEOREM 16. - <u>A free valued vector space</u> $B = \coprod_{\sigma<\tau} B(\sigma)$ <u>of length</u> τ <u>is the basic subspace of a totally projective</u> p-<u>group exactly if</u>

$$\sum_{\rho\geq\sigma+\omega} \dim B(\rho) \leq \sum_{n<\omega} \dim B(\sigma + n)$$

<u>for every</u> σ <u>with</u> $\sigma + \omega < \tau$.

The second question we raised will be discussed elsewhere.

The totally projective p-groups of countable length are known to be precisely the direct sums of countable p-groups [14]. Their socles and all their λ-slices are free valued vector spaces, as it follows at once from theorem 5. Moreover, there is a useful characterization of these groups in terms of their slices.

THEOREM 17. - <u>An abelian</u> p-<u>group of countable length is a direct sum of countable</u> p-<u>groups if, and only if, its socle and all its</u> λ-<u>slices are free valued vector spaces</u>.

In fact, this result is equivalent to a theorem of Megibben [13].

The last theorem is a typical example for results we are driving at. Once a satisfactory characterization of socles of totally projectives is obtained, a similar result is expected for totally projective p-groups.

8. Classifications in terms of socles.

There are several classes of abelian p-groups within which the individual groups are distinguished by their socles.

1° As noticed in 7, the totally projective p-groups are distinguishable by their basic subspaces. So they can be distinguished by their socles as well.

We should add two remarks. First, the p-groups whose socles are free valued vector spaces of lengths $\leq \omega$ are completely determined by their socles. In fact, they are the direct sums of cyclic p-groups. Secondly, as pointed out by HILL and MEGIBBEN [9], no p-group of length $> \omega_1$ can have a free valued vector space as a socle.

2° It is not difficult to see that the torsion-complete p-groups are the only abelian groups whose socles are injective valued vector spaces (this follows from theorem 13). Thus an injective space is the socle of a p-group if, and only if, it is of length $\leq \omega$.

The torsion-complete p-groups are distinguishable by their socles. Moreover, HILL [8] proved a remarkable generalization of this result to direct sums of torsion-complete p-groups.

3° A class of p-groups which is close to the direct sums of cyclic p-groups, but which displays quite a variety in structure is the class of the so-called $p^{\omega+1}$-projective p-groups. Their simplest definition is that they are extensions of direct sums of groups of order p by direct sums of cyclic p-groups. J. IRWIN and the author [6] have proved that two $p^{\omega+1}$-projective p-groups are isomorphic if, and only if, their socles are isometric. In other words, these groups can be classified with the aid of their socles only.

4° A dual class consists of extensions of direct sums of groups of order p by a torsion-complete p-group. As shown by RICHMAN [16], two groups in this class are isomorphic exactly if their socles are isometric.

These results can be generalized if slices with filtrations are considered.

L. SALCE and the author are investigating the almost totally projective p-groups A (defined by

$$p^{\sigma} \operatorname{Ext}(\underset{\sim}{Z}(p^{\infty}) , A/p^{\sigma} A) = 0 \text{ , for every } \sigma \text{),}$$

by using the socles and the λ-slices in order to get structure theorems for them. It turns out that, for countable length, the relation between the socle, the λ-slices and the filtration discussed in 5, yields sufficient information to distinguish between the groups in the mentioned class.

We believe that the application of valued vector spaces to abelian p-groups is still at an early stage, but the results obtained so far are encouraging and support our conviction that a great deal can be learned about the structure of p-groups by studying their slices.

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