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Orthodox bands of modules


<http://www.numdam.org/item?id=GEA_1975-1976__1__A12_0>
ORTHODOX BANDS OF MODULES

par Francis PASTIJN

Summary. - In this paper, we shall consider orthodox bands of commutative groups, together with a ring of endomorphisms. We shall generalize the concept of a left module by introducing orthodox bands of left modules; we shall also deal with linear mappings, the transpose of a linear mapping and with the dual of an orthodox band of left modules.

We shall use the notations and terminology of [1](chap 2, § 1) and [2].

1. Definition.

Let $(R, +, \cdot)$ be a ring with zero element $0$ and identity $1$. Let $S$ be a semigroup and $R \times S \rightarrow S$, $(\alpha, x) \rightarrow \alpha x$ a mapping satisfying the following conditions:

(i) $\alpha(xy) = (\alpha x)(\alpha y)$ for every $\alpha \in R$ and every $x, y \in S$,

(ii) $(\alpha + \beta)x = (\alpha x)(\beta x)$ for every $\alpha, \beta \in R$ and every $x \in S$,

(iii) $(\alpha \cdot \beta)x = \alpha(\beta x)$ for every $\alpha, \beta \in R$ and every $x \in S$,

(iv) $1x = x$ for every $x \in S$.

The so-defined structure will be called an orthodox band of left $R$-modules. Next theorem justifies our terminology.

2. Theorem 1. - Let $R$, $S$ and mapping $R \times S \rightarrow S$ be as in 1. Then $S$ is an orthodox band of commutative groups, and the maximal subgroups of $S$ are left invariant by the elements of $R$.

Proof. - Let $x$ be any element of $S$, and $\alpha$ any element of $R$; we then have

$$(0x)(0x) = (0 + 0)x = Ox,$$

$$(\alpha x)(0x) = (\alpha + 0)x = \alpha x = (0 + \alpha)x = (0x)(\alpha x),$$

$$(\alpha x)(-\alpha x) = (\alpha - \alpha)x = 0x = (-\alpha + \alpha)x = ((-\alpha x))(\alpha x).$$

This implies that for any $\alpha \in R$ and any $x \in S$, $\alpha x$ belongs to the maximal subgroup of $S$ with identity $0x$, the inverse of $\alpha x$ in this maximal subgroup must be $(-\alpha)x$. More specifically $1x = x$ belongs to the maximal subgroup of $S$ with identity $0x$, and its inverse in this maximal subgroup must be $(-1)x$. We conclude that $S$ must be a completely regular semigroup and that all maximal subgroup of $S$ are left invariant by the elements of $R$.

For every $x, y \in S$ we have

$$(xy)(xy) = (1 + 1)(xy) = ((1 + 1)x)((1 + 1)y) = x^2 y^2.$$
Let $e, f$ be any idempotents of $S$, then the foregoing implies that 
\[(ef)^2 = e^2 f^2 = ef, \text{ hence } E_S = \{x \in S ; \quad x^2 = x\}\]
must be a subsemigroup of $S$. Let $x$ and $y$ belong to a same maximal subgroup of $S$, then the foregoing implies 
\[xy = (((-1)x)x^2 y^2((-1)x)xyy((-1)y)) = yx,\]
hence $S$ is a union of commutative groups. We conclude that $S$ is an orthodox union of commutative groups [3].

Let $e$ and $f$ be any idempotent of $S$, and $x \in H_e$, $y \in H_f$. We put $(-1)x = x'$ and $(-1)y = y'$, then 
\[ef = (ef)^2 = (1 + 1)(ef) = (1 + 1)(x(x'f)) = x^2(x'f)^2 = x^2 x'fx'f = (xf)(x'f)\]
and analogously 
\[ef = (x'f)(xf) .\]

Since $ef$, $x'f$ and $xf$ are elements of rectangular group $D_{ef}$ [3], the foregoing implies that $xf$ and $x'f$ are mutually inverse elements of maximal subgroup $H_{ef}$. Dually, $ey$ and $ey'$ are mutually inverse elements of maximal subgroup $H_{ef}$.

Since $(xy)y' = xf$ and $(xf)y = xy$ we have $xy \in xf$, hence $xy \in ef$. Analogously, since $x'(xy) = ey$ and $x(ey) = xy$ we have $xy \in ey$, hence $xy \in ef$. We conclude that $xy \in ef$. Green's relation $\sim$ must then be a congruence on $S$. Thus $S$ is an orthodox band of commutative groups [3].

3. Remark.

Let $S$ be an orthodox band of commutative groups. Then, by Yamada's theorem ([3] and [11]), there exists a band $E$, and a semilattice of commutative groups $Q$, both having the same structure semilattice $Y$, such that $S$ is the spined product of $Q$ and $E$ over $Y : S = Q \times Y E$. Let $Q = \bigcup_{\mu \in Y} G_\mu$ and $E = \bigcup_{\mu \in Y} E_\mu$, then $S$ consists of ordered pairs $(x_\mu, e_\mu), \mu \in Y, x_\mu \in G_\mu, e_\mu \in E_\mu$; multiplication is defined by 
\[(x_\mu, e_\mu)(y_\mu, f_\mu) = (x_\mu y_\mu, e_\mu f_\mu)\]
for any $\lambda, \mu \in Y, x_\lambda \in G_\lambda, y_\mu \in G_\mu, e_\lambda \in E_\lambda, f_\mu \in E_\mu$. The identity element of $G_\mu$, $\mu \in Y$ will be denoted by $1_\mu$.

The following result will generalize a theorem of [4] about semilattices of left modules. In patching up next theorem and theorem 1, we actually get a characterization for orthodox bands of commutative groups.

4. THEOREM 2. - Let $S$ be any orthodox band of commutative groups, and let $Z$ be the ring of integers. Let $e$ be any idempotent of $S$, and $x$ and $x'$ mutually inverse elements of maximal subgroup $H_e$. Define mapping $Z \times S \to S$, 
\[(k, x) \mapsto kx\] by
Then $S$ is an orthodox band of left $\mathbb{Z}$-modules.

Proof. - Conditions (i), (ii), (iii) and (iv) of 1 are checked by some easy calculations.

5. Definitions and remarks.

Let $S$ be an orthodox band of left $R$-modules, and $\tau$ a congruence on semigroup $S$. The natural homomorphism of $S$ onto $S/\tau$, will be denoted by $\tau^h$. $\tau$ will be called $R$-stable if, and only if, $x \tau y$ implies $(\alpha x) \tau (\alpha y)$ for every $x, y \in S$ and every $\alpha \in R$. We can define a mapping $R \times (S/\tau) \rightarrow S/\tau$, $(\alpha, \bar{x}) \mapsto \alpha x = \bar{\alpha}$. $S/\tau$ will then be an orthodox band of left $R$-modules.

Let $S$ and $T$ be orthodox bands of left $R$-modules. Mapping $\phi : S \rightarrow T$ will be called $R$-linear if, and only if,

(i) $\phi(xy) = (\phi(x))(\phi(y))$ for every $x, y \in S$

(ii) $\phi(\alpha x) = \alpha \bar{\phi}(x)$ for every $x \in S$ and every $\alpha \in R$.

$\phi(S)$ will then be an orthodox band of left $R$-modules.

Subset $A$ of $S$ will be called $R$-stable if, and only if, $\alpha x \in A$ for every $x \in A$ and every $\alpha \in R$. If $\phi$ is an $R$-linear mapping of $S$ into $T$, $\phi(S)$ will be an $R$-stable subsemigroup of $T$, and the kernel of $\phi$ will be an $R$-stable subsemigroup of $S$. Any $R$-stable subsemigroup of an orthodox band of left $R$-modules must of course be an orthodox band of left $R$-modules. If $\tau$ is an $R$-stable congruence on $S$, the union of all $\tau$-classes containing an idempotent will be an $R$-stable subsemigroup of $S$.

Mapping $\bar{\phi} : S \rightarrow T$ will be $R$-linear if, and only if, $\bar{\phi}^{-1} \phi$ is an $R$-stable congruence on $S$. Equivalence relation $\tau$ on $S$ is an $R$-stable congruence if, and only if, $\tau^h$ is an $R$-linear mapping.

Mapping $\bar{\phi} : S \rightarrow E$, $x \mapsto \bar{\phi}(x)$ is an $R$-linear mapping of $S$ onto the band consisting of all idempotents of $S$; $\bar{\phi}^{-1} \phi$ is then the $R$-stable congruence $\%$.

Let $S$ be the spined product of semilattice of commutative groups $Q$ and band $E$. We shall use the same notations as in 3. $Q$ is the greatest inverse semigroup homomorphic image of $S$, and the mapping $\Delta : S \rightarrow Q$, $(x, e) \mapsto x_\kappa$ is a homomorphism of $S$ onto $Q$. We shall put $\Delta^{-1} \Delta = \sigma$. This congruence $\sigma$ is the minimal inverse semigroup congruence on $S$, and we will show that $\sigma$ is $R$-stable. Let $\Gamma$ be the greatest group homomorphic image of $Q$, and $\Gamma : Q \rightarrow G$, $x_\kappa \mapsto \bar{x}_\kappa$ be a homomorphism of $Q$ onto $G$, $\Gamma^{-1} \Gamma$ being the minimal group congruence on $Q$. If $x$ and $y$ are any elements of $Q$, then $x \Gamma^{-1} \Gamma y$, if, and only if, there exists a $\kappa \in \tilde{Y}$, $\kappa \leq \lambda \wedge \mu$, such that $x_\lambda \kappa^{-1} = y_\mu \kappa$. We shall
put \((\Gamma \Delta)^{-1}(\Gamma \Delta) = \rho\); this congruence \(\rho\) is the minimal group congruence on \(S\), and we will show that \(\rho\) is \(R\)-stable.

6. **Theorem 3.** - The minimal inverse semigroup congruence on an orthodox band of left \(R\)-modules is \(R\)-stable.

**Proof.** - Let \(x_{\kappa}\) be any element of \(Q\), and let us take any two elements \((x_{\kappa}, e_{\kappa})\) and \((x_{\kappa}, f_{\kappa})\) in \(\Delta^{-1} \Delta x\). Let \(\alpha\) be any element of \(R\). Since \(\Delta\) is an \(R\)-stable congruence on \(S\), \(\alpha(x_{\kappa}, e_{\kappa})\) belongs to the \(\Delta\)-class \(G_{\kappa} \times e_{\kappa}\) of \(S\) containing \((x_{\kappa}, e_{\kappa})\), hence,

\[
\alpha(x_{\kappa}, e_{\kappa}) = (y_{\kappa}, e_{\kappa}) \text{ for some } y_{\kappa} \in G_{\kappa}.
\]

Analogously,

\[
\alpha(x_{\kappa}, f_{\kappa}) = (z_{\kappa}, f_{\kappa}) \text{ for some } z_{\kappa} \in G_{\kappa}.
\]

Let \((l_{\kappa}, g_{\kappa})\) be \(\Delta\)-related with \((l_{\kappa}, e_{\kappa})\) and \(\Delta\)-related with \((l_{\kappa}, f_{\kappa})\), and let \((l_{\kappa}, h_{\kappa})\) be \(\Delta\)-related with \((l_{\kappa}, e_{\kappa})\) and \(\Delta\)-related with \((l_{\kappa}, f_{\kappa})\).

Since, by the restriction of \(R \times S \rightarrow S\) to \(R \times (G_{\kappa} \times e_{\kappa})\), and \(R \times (G_{\kappa} \times h_{\kappa})\) respectively, \(G_{\kappa} \times e_{\kappa}\) and \(G_{\kappa} \times h_{\kappa}\) become \(R\)-modules, we must have

\[
\alpha(l_{\kappa}, g_{\kappa}) = (l_{\kappa}, g_{\kappa}) \text{ and } \alpha(l_{\kappa}, h_{\kappa}) = (l_{\kappa}, h_{\kappa}).
\]

Furthermore, we have

\[
(z_{\kappa}, e_{\kappa}) = (l_{\kappa}, h_{\kappa})(z_{\kappa}, f_{\kappa})(l_{\kappa}, e_{\kappa})
= (\alpha(l_{\kappa}, h_{\kappa}))((\alpha(x_{\kappa}, f_{\kappa}))((\alpha(l_{\kappa}, g_{\kappa}))
= \alpha((l_{\kappa}, h_{\kappa})(x_{\kappa}, f_{\kappa})(l_{\kappa}, g_{\kappa}))
= \alpha(x_{\kappa}, e_{\kappa}) = (y_{\kappa}, e_{\kappa}),
\]

hence \(z_{\kappa} = y_{\kappa}\), and \(\Delta(\alpha(x_{\kappa}, e_{\kappa})) = \Delta(\alpha(x_{\kappa}, f_{\kappa}))\).

7. **Corollary 1.** - By mapping \(R \times Q \rightarrow Q\), \((\alpha, x_{\kappa}) \rightarrow \alpha_{\kappa} = \Delta(\alpha x_{\kappa}^{-1})\), \(Q\) becomes a semilattice of left \(R\)-modules, and \(\Delta\) an \(R\)-linear mapping of \(S\) onto \(Q\).

8. **Corollary 2.** - Let \(Q\) be any semilattice of left \(R\)-modules, and \(Y\) the structure semilattice of \(Q\), let \(E\) be a band with the same structure semilattice \(Y\), let \(U_{\kappa \in Y} G_{\kappa}\) and \(U_{\kappa \in Y} E_{\kappa}\) be the semilattice decompositions of \(Q\) and \(E\) respectively, let \(S\) be the spined product \(Q \times Y E\) of \(Q\) and \(E\) over \(Y\). By mapping \(R \times S \rightarrow S\), \((\alpha, (x_{\kappa}, e_{\kappa})) \rightarrow (\alpha_{\kappa}, e_{\kappa})\) for every \(\alpha \in R\), and every \(\kappa \in Y\), \(x_{\kappa} \in G_{\kappa}, e_{\kappa} \in E_{\kappa}\), \(S\) become an orthodox band of left \(R\)-modules. Conversely, any orthodox band of left \(R\)-modules can be so constructed.

9. **Corollary 3.** - Let \(S\) be an orthodox normal band of left \(R\)-modules, and let \(S = U_{\kappa \in Y} S_{\kappa}\) be the semilattice decomposition of \(S\). For any \(\lambda, \mu \in Y\), \(\lambda > \mu\), the structure homomorphism \(Y_{\lambda}, \mu\) is an \(R\)-linear mapping of orthodox rectangular band of left \(R\)-modules \(S_{\lambda}\) into orthodox rectangular band of left \(R\)-modules \(S_{\mu}\).
Proof. - In a semilattice of left $R$-modules the structure homomorphisms are $R$-linear [6]. The theorem now follows from corollary 2 and from a result about normal bands [10].

10. Remark.

Structure theorems for semilattices of left $R$-modules [6], together with corollary 2 yield structure theorems for orthodox bands of left $R$-modules.

11. Theorem 4. - The minimal group congruence on an orthodox band of left $R$-modules is $R$-stable.

Proof. - Let $x_\lambda$ be any element of $G$, the greatest group homomorphic image of orthodox band of left $R$-modules $S$. Let us take any two elements $x_\lambda$ and $y_\mu$ in $\Gamma^{-1} \lambda \mu$. There exists $x \in Y$, $x \leq \lambda \wedge \mu$, such that $1_x = 1_x$. Let $\alpha$ be any element of $R$. From

$$(\alpha x)_\mu = (\alpha x)_\mu = (\alpha x)_\mu = (\alpha y)_\mu = (\alpha y)_\mu,$$

and $\alpha x \in G$ and $\alpha y \in G$, we conclude that $\alpha y \in \Gamma^{-1}\Gamma(\alpha x)$, and thus $\alpha x = \alpha y$.

This implies that the minimal group congruence $\Gamma^{-1}\Gamma$ on $Q$ must be $R$-stable. Consequently, the minimal group congruence $\Gamma\Delta$ on $S$ must be $R$-stable.

12. Corollary 4. - By mapping $R \times G \rightarrow G$, $(\alpha, x_\lambda) \mapsto \alpha \tilde{x}_\lambda = \tilde{x}_\lambda$, $G$ becomes a left $R$-module, and the mapping $\Gamma\Delta$ an $R$-linear mapping of $S$ onto $G$.


An orthodox band of right $R$-modules $S$ can be defined in an analogous way as an orthodox band of left $R$-modules. Condition (iii) of $l$ must then be replaced by (iii)' $$(\alpha \cdot \beta)x = \beta(\alpha x)$$ for every $\alpha, \beta \in R$ and every $x \in S$. It will be more convenient to denote mapping $R \times S \rightarrow S$, $$(\alpha, x) \mapsto \alpha x$$ (iii)' becomes

$$(\alpha \cdot \beta) \mapsto (\alpha \cdot \beta) \beta$$ for every $\alpha, \beta \in R$ and every $x \in S$.

If $S$ is at the same time orthodox band of left $R$-modules, and orthodox band of right $R$-modules, then we shall say that $S$ is an orthodox band of $R$-bimodules.

Let $R^\infty = R \cup \{\infty\}$, and define addition in $R^\infty$ as follows. For any $\alpha, \beta \in R$, we put $\alpha + \beta = \gamma$ in $R^\infty$ if, and only if, $\alpha + \beta = \gamma$ in $R$, and

$$\alpha + \infty = \infty + \alpha = \infty.$$ $R^\infty$ will be a group with "zero" $\infty$. We next define mapping $R \times R^\infty \rightarrow R^\infty$ by

$$\alpha + \beta \mapsto \alpha \beta = \gamma$$ if, and only if, $\alpha + \beta = \gamma$ in $R$, and

$$\alpha + \infty \mapsto \alpha \infty = \infty.$$ We also define mapping $R \times R^\infty \rightarrow R^\infty$ by
By these two mappings \( R^\infty \) becomes a semilattice of \( R \)-bimodules, the structure semilattice being the two element semilattice. We shall use \( R^\infty \) later in this paper.

The next theorem generalizes a result of [9].

14. THEOREM 5. - Let \( S \) be an orthodox band of left \( R \)-modules, and \( T \) an orthodox band of right \( R \)-modules. Let \( S_{S, T} \) be the set of all partial mapping of \( S \) into \( T \). Define a multiplication in \( S_{S, T} \) as follows : for every \( \phi, \psi \in S_{S, T} \), \( \text{dom} \ \phi = \text{dom} \ \psi \cap \text{dom} \ \gamma \), and for every \( x \in \text{dom} \ \phi \) we put \( \phi \psi(x) = (\phi x)(\psi x) \). Define mapping \( R \times S_{S, T} \rightarrow S_{S, T} \), \((\alpha, \phi) \mapsto \phi \alpha \) by \( \text{dom}(\phi \alpha) = \text{dom} \ \phi \) and \( (\phi \alpha)x = (\phi x)\alpha \), for every \( x \in \text{dom} \ \phi \). \( S_{S, T} \) will then be an orthodox band of right \( R \)-modules if, and only if, \( T \) is a semilattice of right \( R \)-modules.

Proof. - For any \( \phi, \psi \in S_{S, T} \) and any \( \alpha \in R \) we have
\[
\text{dom}(\phi \psi) \alpha = \text{dom} \ \phi \psi = \text{dom} \ \phi \cap \text{dom} \ \psi = \text{dom} \ \phi \alpha \cap \text{dom} \ \psi \alpha = \text{dom}(\phi \alpha)(\psi \alpha),
\]
and for any \( x \in \text{dom}(\phi \psi) \) we have
\[
((\phi \psi) \alpha)x = ((\phi \psi) x) \alpha = ((\phi x) \alpha)(\psi x) = ((\phi x)\alpha)((\psi x)\alpha) = \phi(\psi \alpha) x = \phi((\psi \alpha)x),
\]
hence \( \phi \psi \alpha = \phi(\psi \alpha) \). For any \( \phi \in S_{S, T} \) and any \( \alpha, \beta \in R \) we have
\[
\text{dom} \ \phi(\alpha + \beta) = \text{dom} \ \phi = \text{dom} \ \phi \alpha \cap \text{dom} \ \phi \beta = \text{dom}(\phi \alpha)(\phi \beta),
\]
and, for any \( x \in \text{dom} \ \phi(\alpha + \beta) \) we have
\[
(\phi(\alpha + \beta))x = (\phi x)(\alpha + \beta) = ((\phi x)\alpha)((\psi x)\beta) = ((\phi x)\alpha)((\psi \beta)x) = (\psi \alpha)(\phi \beta)x,
\]
hence \( \phi(\alpha + \beta) = (\psi \alpha)(\phi \beta) \). Furthermore,
\[
\text{dom} \ \phi(\alpha \circ \beta) = \text{dom} \ \phi = \text{dom} \ \phi \alpha \cap \text{dom} \ \phi \beta = \text{dom}(\phi \alpha)(\phi \beta),
\]
and for any \( x \in \text{dom} \ \phi(\alpha \circ \beta) \) we have
\[
(\phi(\alpha \circ \beta))x = (\phi x)(\alpha \circ \beta) = ((\phi x)\alpha)\beta = ((\phi \alpha)\beta)x,
\]
hence \( \phi(\alpha \circ \beta) = (\phi \alpha)\beta \). Finally, \( \text{dom} \ \phi 1 = \text{dom} \ \phi \), and for any \( x \in \text{dom} \ \phi 1 \) we have
\[
(\phi 1)x = (\phi x)1 = \phi x,
\]
hence \( \phi 1 = \phi \). We conclude that \( S_{S, T} \) is an orthodox band of right \( R \)-modules.

From the definition of the multiplication in \( S_{S, T} \) follows that \( S_{S, T} \) is commutative if, and only if, \( T \) is commutative. From this, follows the last part of the theorem.

15. THEOREM 6. - Let \( S \) be an orthodox band of left \( R \)-modules, \( S' \) the set of \( R \)-linear mappings of \( S \) into \( R \), and \( S^* \) the set of \( R \)-linear mapping of \( S \) into \( R^\infty \). Then \( S' \) is an \( R \)-stable subsemigroup of \( S_{S, R} \) and \( S^* \) is an \( R \)-stable subsemigroup of \( S_{S, R^\infty} \).
Proof. - We show that $S^*$ is an $R$-stable subsemigroup of $S_{S,R}$. The proof of the rest is quite the same. Let $x^*$ and $y^*$ be any elements of $S^*$. Since $R^\omega$ is a semilattice of commutative groups, $x^*y^*$ must be a homomorphism of $S$ into $R^\omega$. For any $x \in S$ and any $x^* \in S^*$ we shall from now put $x^*(x) = \langle x, x^* \rangle$. For any $x \in S$, any $\alpha \in R$ and any $x^*, y^* \in S^*$ we then have

$$\langle \alpha x, x^* y^* \rangle = \langle \alpha x, x^* \rangle + \langle \alpha x, y^* \rangle$$
$$= \alpha \langle x, x^* \rangle + \alpha \langle x, y^* \rangle$$
$$= \alpha (\langle x, x^* \rangle + \langle x, y^* \rangle)$$
$$= \alpha \langle x, x^* y^* \rangle.$$  

We conclude that for any $x^*, y^* \in S^*$, $x^* y^*$ must be an $R$-linear mapping of $S$ into $R^\omega$, hence $x^* y^* \in S^*$. $S^*$ is a subsemigroup of $S_{S,R}$. For any $x, y \in S$, any $x^* \in S^*$ and any $\alpha \in R$ we have

$$\langle xy, x^* \alpha \rangle = \langle xy, x^* \rangle \alpha$$
$$= (\langle x, x^* \rangle + \langle y, x^* \rangle) \alpha$$
$$= \langle x, x^* \rangle \alpha + \langle y, x^* \rangle \alpha$$
$$= \langle x, x^* \rangle \alpha + \langle y, x^* \rangle \alpha,$$

hence $x^* \alpha$ must be a homomorphism of $S$ into $R^\omega$. For any $x \in S$, any $x^* \in S^*$ and any $\alpha, \beta \in R$ we have

$$\langle \beta x, x^* \alpha \rangle = \langle \beta x, x^* \rangle \alpha$$
$$= \beta \langle x, x^* \rangle \alpha$$
$$= \beta \langle x, x^* \alpha \rangle.$$

We conclude that for any $x^* \in S^*$ and any $\alpha \in R$, $x^* \alpha$ must be an $R$-linear mapping of $S$ into $R^\omega$. Consequently $S^*$ must be an $R$-stable subsemigroup of $S_{S,R}$.  

16. COROLLARY 5. - $S^*$ is a semilattice of right $R$-modules. The structure semilattice of $S^*$ is isomorphic with the semilattice of prime ideals of $S$. The mapping $1^*: S \rightarrow R^\omega$, $x \rightarrow 0$ is the identity of $S^*$ and the mapping $0^*: S \rightarrow R^\omega$, $x \rightarrow \infty$ is the zero of $S^*$.  

Proof. - $R^\omega$ is a semilattice of right $R$-modules, hence $S_{S,R}^\omega$ is a semilattice of right $R$-modules. Since $S^*$ is $R$-stable in $S_{S,R}^\omega$, $S^*$ must be a semilattice of right $R$-modules too. 

Let $e^*$ be any idempotent of $S^*$, then

$$V_{e^*} = \{ x \in S ; \langle x, e^* \rangle = \infty \}$$

is a prime ideal of $S$. For any $x \in S \setminus V_{e^*}$

$$\langle x, e^* \rangle \in R \text{ and } \langle x, e^* \rangle = \langle x, e^* \rangle^2 = \langle x, e^* \rangle + \langle x, e^* \rangle,$$

hence $\langle x, e^* \rangle = 0$. Conversely, let $P$ be any prime ideal of $S$, then we can define $e^*_P \in S^*$ by $\langle x, e^*_P \rangle = \infty$ for all $x \in P$, and $\langle x, e^*_P \rangle = 0$ for all $x \in S \setminus P$. Furthermore, if $e^*$ and $f^*$ are any two idempotents of $S^*$, we must have
17. COROLLARY 6. — $S'$ is a right $R$-module which is an $R$-stable subgroup of $S^*$. $S'$ is the maximal submodule of $S^*$ containing the identity $1^*$ of $S^*$.

**Proof.** — All elements of $S'$ are $R$-linear mappings of $S$ into $R$, hence, they can be considered as $R$-linear mappings of $S$ into $R^e$, and consequently $S' \subseteq S^*$. Since $S'$ is $R$-stable in $S_S^R$, and since clearly $S_S^R$ is $R$-stable in $S_S^R$, $S'$ must be $R$-stable in $S_S^R$; from this we imply that $S'$ is $R$-stable in $S^*$. It must be evident that $1^*: S \rightarrow R^e$, $x \mapsto 0$ is the identity of $S'$. Let $x^*$ be any element of $S'$, then $x^*(-1) \in S'$, and for any $x \in S$ we have

$$\langle x, x^*(x^*(-1)) \rangle = \langle x, x^* \rangle + \langle x, x^*(-1) \rangle = \langle x, x^* \rangle + \langle x, x^*(-1) \rangle = 0$$

and analogously

$$\langle x, (x^*(-1))x^* \rangle = 0,$$

hence $x^*(x^*(-1)) = (x^*(-1))x^* = 1^*$. This shows that $x^*$ and $x^*(-1)$ are mutually inverse elements of commutative group $H_{1^*}$, the maximal subgroup of $S^*$ containing $1^*$. For any element $y^* \in H_{1^*}$, we must have $V_{y^*} = \emptyset$, hence any element $y^* \in H_{1^*}$ belongs to $S'$. We can conclude that $H_{1^*} = S'$.

18. THEOREM 7. — Let $S$ be an orthodox band of left $R$-modules and $\tau$ any $R$-stable congruence on $S$. The mapping $\phi: (S/\tau)^* \rightarrow S^*$, $\overline{x^*} \rightarrow \overline{\phi(x)^*}$ defined by

$$\langle x, \phi(y^*) \rangle = \langle \tau^* x, y^* \rangle$$

for every $x \in S$ is an $R$-isomorphism of $(S/\tau)^*$ into $S^*$. Whenever $\tau \subseteq \sigma \subseteq \sigma$, $\sigma$ being the minimal inverse semigroup congruence on $S$, this mapping $\phi$ is a surjective $R$-isomorphism of $(S/\tau)^*$ onto $S^*$.

**Proof.** — Let us suppose that $\overline{x^*}$, $\overline{y^*}$ are any elements of $(S/\tau)^*$, and $x$ any element of $S$. We then have

$$\langle x, \phi(\overline{x^*}, \overline{y^*}) \rangle = \langle \tau^* x, \overline{x^*}, \overline{y^*} \rangle$$

$$= \langle \tau^* x, \overline{x^*} \rangle + \langle \tau^* x, \overline{y^*} \rangle$$

$$= \langle x, \phi(x^*) \rangle + \langle x, \phi(y^*) \rangle$$

$$= \langle x, (\phi(x^*)(\phi(y^*) \rangle,$$

hence $\phi(\overline{x^*}, \overline{y^*}) = (\phi(x^*)\phi(y^*)$. Let us suppose that $\overline{x^*}$ is any element of $(S/\tau)^*$, $\alpha$ any element of $R$ and $x$ any element of $S$, then

$$\langle x, \phi(\overline{x^*}, \alpha) \rangle = \langle \tau^* x, \overline{x^*}, \alpha \rangle$$

$$= \langle \tau^* x, \overline{x^*} \rangle \alpha$$

$$= \langle x, \phi(x^*) \alpha \rangle$$

$$= \langle x, \phi(x^*) \rangle \alpha,$$

hence $\phi(\overline{x^*}, \alpha) = (\phi(x^*) \alpha$. Since $\tau^*$ is an $R$-linear mapping of $S$ onto $S/\tau$,
\( \bar{x}^* \in S^\# \) for any \( \bar{x}^* \in (S/\tau)^* \). We conclude that \( \phi \) is an R-linear mapping of \((S/\tau)^* \) into \( S^* \). Let us now suppose that \( \bar{x}^*, \bar{y}^* \in (S/\tau)^* \), and \( \bar{x}^* = \bar{y}^* \). If for some \( \bar{x} \in S/\tau \), \( \langle \bar{x}, \bar{x}^* \rangle \neq \langle \bar{x}, \bar{y}^* \rangle \), then for any \( x \in (\tau^*)^{-1} x \) we should have

\[ \langle x, \bar{x}^* \rangle = \langle \tau x, \bar{x}^* \rangle = \langle \bar{x}, \bar{x} \rangle \neq \langle \bar{x}, \bar{y}^* \rangle = \langle x, \bar{y}^* \rangle, \]

and this is impossible. We conclude that \( \bar{x}^* = \bar{y}^* \) implies \( \bar{x}^* = \bar{y}^* \), hence \( \phi \) is an isomorphism of \((S/\tau)^* \) into \( S^* \).

It will be sufficient to show that the mapping \( \phi : (S/\sigma)^* \to S^* \), \( x^* \to \bar{x}^* \) defined by \( \langle x, \bar{x}^* \rangle = \langle \tau x, \bar{x}^* \rangle \) for every \( x \in S \), will be an R-isomorphism of \((S/\sigma)^* \) onto \( S^* \). Let \( x^* \) be any element of \( S^* \), and \((x_\mu, e_\mu)\) and \((x_\mu, f_\mu)\) any two \( \sigma \)-related elements of \( S \). Since \((x_\mu, e_\mu)\) and \((x_\mu, f_\mu)\) are \( \sigma \)-related in \( S \), they generate a same principal ideal of \( S \), and thus \( \langle (x_\mu, e_\mu), x^* \rangle = \infty \) if, and only if, \( \langle (x_\mu, f_\mu), x^* \rangle = \infty \). Let us suppose that \((x_\mu, e_\mu)\) and \((x_\mu, f_\mu)\) both belong to \( S \). Let \((i_\mu, g_\mu)\) be \( \sigma \)-related with \((x_\mu, e_\mu)\) and \( \sigma \)-related with \((1_\mu, f_\mu)\), and \((1_\mu, h_\mu)\) \( \sigma \)-related with \((x_\mu, e_\mu)\) and \( \sigma \)-related with \((1_\mu, f_\mu)\); \((i_\mu, g_\mu)\) and \((1_\mu, h_\mu)\) are both \( \sigma \)-related with \((x_\mu, e_\mu)\) and \((x_\mu, f_\mu)\). Hence \((1_\mu, g_\mu), (1_\mu, h_\mu) \in S\setminus x^* \), since these two elements are idempotents of \( S \), and since \( x^* \) is an homomorphism of \( S \) into \( R \), we have

\[ \langle (1_\mu, g_\mu), x^* \rangle = \langle (1_\mu, h_\mu), x^* \rangle = 0. \]

From this follows that

\[ \langle (x_\mu, e_\mu), x^* \rangle = \langle (1_\mu, h_\mu)(x_\mu, f_\mu)(1_\mu, g_\mu), x^* \rangle \]

\[ = \langle (1_\mu, h_\mu), x^* \rangle + \langle (x_\mu, f_\mu), x^* \rangle + \langle (1_\mu, g_\mu), x^* \rangle = \langle (x_\mu, f_\mu), x^* \rangle. \]

In any case \( (x^*)^{-1} x^* \equiv \sigma \). Hence the mapping \( \bar{x}^* \in (S/\sigma)^* \) defined by \( \langle \tau x, \bar{x}^* \rangle = \langle x, x^* \rangle \) for all \( x \in S \) is well-defined, and we shall have \( \bar{x}^* = x^* \).

Thus, in this case \( \phi \) must be surjective.

19. COROLLARY 7. - If \( S \) is an orthodox band of left R-modules, and \( Q \) the greatest inverse homomorphic image of \( S \), then \( S^* \) and \( Q^* \) are R-isomorphic.

20. THEOREM 8. - Let \( S \) be an orthodox band of left R-modules and \( \tau \) any R-stable congruence on \( S \). The mapping \( \psi : (S/\tau)' \to S' \), \( \bar{x}^* \to \psi(\bar{x}^*) \) defined by \( \langle x, \bar{x}^* \rangle = \langle \tau x, \bar{x}^* \rangle \) for any \( x \in S \) is an R-isomorphism of \((S/\tau)' \) into \( S' \).

Whenever \( \tau_S \leq \tau \leq \rho \), \( \rho \) being the minimal group congruence on \( S \), this mapping \( \psi \) is a surjective R-isomorphism of \((S/\tau)' \) onto \( S' \).

Proof. - It is clear that mapping \( \psi \) must be the restriction of mapping \( \phi \) (of theorem 7) to maximal submodule \((S/\tau)' \) of \((S/\tau)^* \), hence \( \psi \) is an R-isomorphism of \((S/\tau)' \) into \( S' \). Since for every \( x \in S \), and every \( \bar{x}^* \in (S/\tau)' \) we must have \( \langle x, \bar{x}^* \rangle \in R \). We conclude \( \bar{x}^* \in S' \) for every \( \bar{x}^* \in (S/\tau)' \), thus, \( \psi \) is an R-isomorphism of \((S/\tau)' \) into \( S' \).

It will be sufficient to show that the mapping \( \psi : (S/\rho)' \to S' \), \( \bar{x}^* \to \psi(\bar{x}^*) \)
defined by \( \langle x, \bar{x}^* \rangle = \langle p^t x, \bar{x}^* \rangle \) for every \( x \in S \) will be an \( R \)-isomorphism of \( (S/\rho)' \) onto \( S' \). Let \( x^* \) be any element of \( S' \). Since \( x^* \) must be a homomorphism of \( S \) into the additive group \( R \), we have \( (x^*)^{-1} x^* = \rho \). Hence the mapping \( \bar{x}^* \in (S/\rho)' \) defined by \( \langle p^t x, \bar{x}^* \rangle = \langle x, x^* \rangle \) for every \( x \in S \) is well-defined, and we shall have \( \bar{x}^* = x^* \). Thus, in this case \( \gamma \) must be surjective.

21. COROLLARY 8. - If \( S \) is an orthodox band of left \( R \)-modules, \( Q \) the greatest inverse homomorphic image of \( S \), and \( G \) the greatest group homomorphic image of \( S \), then \( S' \) and \( Q' \) are both \( R \)-isomorphic with right \( R \)-module \( G' \) which is the dual of left \( R \)-module \( G \).

22. THEOREM 9. - Let \( S \) be an orthodox band of left \( R \)-modules, and
\[
S = \bigcup_{\mu \in Y} S = \bigcup_{\mu \in Y} G \times E_{\mu}, \quad \text{its semilattice decomposition. For any } \lambda \in Y, \text{ mapping}
\[
1^*_{\lambda} : S \rightarrow R^\infty \text{ defined by } \langle x, 1^*_{\lambda} \rangle = 0 \text{ if, and only if, } x \in \bigcup_{\mu \geq \lambda} S_{\mu}, \text{ and}
\langle x, 1^*_{\lambda} \rangle = \infty \text{ otherwise, is an idempotent of } S^*. \quad \text{The maximal submodule } H_{1^*_{\lambda}} \text{ of } S^*
\]
containing \( 1^*_{\lambda} \) is \( R \)-isomorphic with \( \bigcup_{\mu \geq \lambda} S_{\mu}' \) and with right \( R \)-module \( G_{1^*_{\lambda}} \), the
dual of left \( R \)-module \( G_{1^*_{\lambda}} \).

Proof. - For any \( \lambda \in Y, \bigcup_{\mu \geq \lambda} S_{\mu} \) is an \( R \)-stable subsemigroup of \( S \), and \( G_{1^*_{\lambda}} \) will be the greatest group homomorphic image of \( \bigcup_{\mu \geq \lambda} S_{\mu} \). From corollary 8 follows that \( \bigcup_{\mu \geq \lambda} S_{\mu}' \) and \( G_{1^*_{\lambda}} \) are \( R \)-isomorphic right \( R \)-modules. It is easy to show that \( S \setminus \bigcup_{\mu \geq \lambda} S_{\mu} \) is a prime ideal of \( S \). From results in the proof of corollary 5 then follows that \( 1^*_{\lambda} \) must be an idempotent of \( S^* \). We remark that for any \( x^* \in S^*, s^* \in H_{1^*_{\lambda}} \) if, and only if,
\[
V_{x^*} = \{ x \in S; \langle x, x^* \rangle = 0 \} = S \setminus \bigcup_{\mu \geq \lambda} S_{\mu}.
\]
Hence the mapping \( H_{1^*_{\lambda}} \rightarrow \bigcup_{\mu \geq \lambda} S_{\mu}' \), \( x^* \rightarrow x^* \in \bigcup_{\mu \geq \lambda} S_{\mu} \) is an \( R \)-isomorphism of \( H_{1^*_{\lambda}} \) onto \( \bigcup_{\mu \geq \lambda} S_{\mu}' \).

23. COROLLARY 9. - We use the same notations as in 22. Let \( Q \) be the greatest inverse semigroup homomorphic image of \( S \) and \( Q = \bigcup_{\mu \in Y} G_{\mu} \) its semilattice decomposition.

For any \( \lambda, \mu \in Y, \lambda \geq \mu \), let \( t^{*}_{\lambda, \mu} \) be the structure homomorphism of \( Q \), and \( \xi^{*}_{\lambda, \mu} \) its transpose. Then \( 1^*_{\mu} > 1^*_{\lambda} \) in \( S^* \). Let \( \xi^{*}_{\mu, \lambda} : H_{1^*_{\lambda}} \rightarrow H_{1^*_{\lambda}} \) be the struc-
ture homomorphism of \( S^* \). For any \( \lambda \in Y \) the mapping \( \lambda^{*}_{\lambda} : H_{1^*_{\lambda}} \rightarrow G_{1^*_{\lambda}}, \)
\( x^* \rightarrow \lambda^{*}_{\lambda} x^* \), defined by
\[
\langle (x_{\lambda}, e_{\mu}), x^* \rangle = \langle \xi^{*}_{\lambda, \mu}, x_{\mu}, \lambda^{*}_{\lambda} x^* \rangle \quad \text{for all} \quad (x_{\lambda}, e_{\mu}) \in \bigcup_{\mu \geq \lambda} S_{\mu},
\]
is an \( R \)-isomorphism of \( H_{1^*_{\lambda}} \) onto \( G_{1^*_{\lambda}} \), and the following diagram is commutative.
Proof. - The mapping \( U : S \rightarrow G \lambda \), \((x, e) \rightarrow \psi \lambda x\) is an homomorphism of \( U \) onto its greatest group homomorphic image \( G \lambda \). \( \psi \lambda \) must then be an \( R \)-isomorphism of \( H_{1 \lambda}^* \) onto \( G \lambda \) by theorem 8.

Let \( x^* \) be any element of \( H_{1 \mu}^* \), and \( x \) any element of \( G \lambda \). We proceed to show that

\[
\langle x, t^* \rangle_{\psi \lambda, \mu} = \langle x^*, y \rangle_{\psi \lambda, \mu} = \langle x, y \rangle_{\psi \lambda, \mu}
\]

Indeed

\[
\langle x, t^* \rangle_{\psi \lambda, \mu} = \langle x, y \rangle_{\psi \lambda, \mu} = \langle x, e \rangle_{\psi \lambda, \mu} = \langle x, x^* \rangle_{\psi \lambda, \mu} = \langle x, y \rangle_{\psi \lambda, \mu}
\]

for all \( \lambda \geq \mu \), \( \psi \lambda, \mu x = x_{1 \mu} \), \( e \mu \in E \).

24. COROLLARY 10. - We use the same notations as in 22 and 23. Let the structure semilattice of \( S \) be a lattice. Consider \( V = \cup_{\kappa \in \mathcal{K}} G \lambda \), and define multiplication in \( V \) by the following. For any \( x, y \in V \), \( x, y \in G \lambda \), \( y \in G \lambda \), put

\[
x'y = (t^v \psi \lambda, \mu \ y) = (t^v \psi \lambda, \mu \ x)^v
\]

Define mapping \( \mathcal{R} \times V \rightarrow V \), \((\alpha, x^v) \rightarrow x^v \alpha \) in the usual way. Then \( V \) is a semilattice of right \( R \)-modules, and there exists an \( R \)-isomorphism of \( V \) into \( S^* \). If \( Y \) satisfies the minimal condition, \( V \) must be an \( R \)-isomorphic with \( S^* \).


Corollaries 9 and 10 show that \( S^* \) could well be named the dual of \( S \). If \( Y \) is a lattice, the structure semilattice of \( V \) is the \( V \)-semilattice \( Y \). Results of [6] make the connections between structure theorems for \( S \) and structure theorems for \( V \) more explicit.

Theorem 7 is quite analogous with a result in [5] (§ 5) about the character semigroup of a commutative semigroup, and theorem 9, corollary 9 and corollary 10 are in a certain way analogous with results of [7] and [8] (see also [2], chapter 5).

Next theorem generalizes the concept of the transpose of an \( R \)-linear mapping.

26. THEOREM 10. - Let \( S \) and \( T \) be orthodox bands of left \( R \)-modules, and \( \Theta : S \rightarrow T \) an \( R \)-linear mapping. The mapping \( T^\Theta : T^* \rightarrow S^* \), \( t^* \rightarrow T^\Theta t^* \), defined by \( \langle x, T^\Theta t^* \rangle = \langle \Theta x, t^* \rangle \) for all \( x \in S \), must be an \( R \)-linear mapping of \( T^* \) into \( S^* \), and \( T^\Theta(T^*) \) is embeddable in \( (T^* \Theta^{-1} \otimes) = (S^*)^* \).

Proof. - It must be clear that for any \( t^* \in T^* \), we must have \( T^\Theta t^* \in S^* \), since
Let \( t^* \) and \( v^* \) be any elements of \( T^* \), then \( t^*|G \) and \( v^*|G \) are both elements of \( (G^*)^* \), since \( G \) is an \( R \)-stable subsemigroup of \( T \). From the definition of \( T \) we have that \( T \circ t^* = T \circ v^* \) if, and only if, \( v^*|G = t^*|G \). This implies that the mapping \( T \circ (T^*) \rightarrow (G^*)^* \), \( T \circ t^* \rightarrow t^*|G \) is an \( R \)-isomorphism of \( T \circ (T^*) \) into \( (G^*)^* \).

27. COROLLARY 11. - Let \( S, T \) and \( G \) be as in theorem 10. The mapping 
\[ t \circ : T' \rightarrow S', \quad t^* \rightarrow t \circ t^* \], 
defined by \( \langle x, t \circ t^* \rangle = \langle \Theta x, t^* \rangle \) for all \( x \in S \), 
must be an \( R \)-linear mapping of \( T' \) into \( S' \), and \( t \circ (T') \) is embeddable in 
\( (S/\Theta^* \Theta)^* = (G^*)^* \).

28. COROLLARY 12. - We use the same notations as in 26 and 27. Let \( \rho_S \) and \( \rho_T \) be the minimal group congruences on \( S \) and \( T \) respectively. Let \( \psi_S : (S/\rho_S)' \rightarrow S' \), \( \varphi_T = \varphi_T(x^*) \), be the \( R \)-isomorphism defined by \( \langle x, \varphi_T(x^*) \rangle = \langle \psi_S(x), x^* \rangle \) for all \( x \in S \), and \( \varphi_T : (T/\rho_T)' \rightarrow T' \), \( t^* \rightarrow \varphi_T(t^*) \), defined by 
\[ \langle t, \varphi_T(t^*) \rangle = \langle \rho_T(t), t^* \rangle \] for all \( t \in S \).

Then there exists an \( R \)-linear mapping \( \Lambda : (S/\rho_S) \rightarrow (T/\rho_T) \) such that the following diagrams are commutative:

\[ \begin{array}{ccc}
S & \xrightarrow{\Theta} & T \\
\downarrow \rho_S & & \downarrow \rho_T \\
S/\rho_S & \xrightarrow{\Lambda} & T/\rho_T \\
\end{array} \quad \begin{array}{ccc}
S' & \xleftarrow{T \circ} & T' \\
\downarrow \rho_S' & & \downarrow \rho_T' \\
(S/\rho_S)' & \xleftarrow{\Lambda} & (T/\rho_T)' \\
\end{array} \]

Proof. - Since \( \rho_T^* \Theta \) is an \( R \)-linear mapping of \( S \) into left \( R \)-module \( T/\rho_T \), 
(\( \rho_T^* \Theta \))^{-1} \( (\rho_T^* \Theta) \) must be an \( R \)-stable group congruence on \( S \), and, since \( \rho_S \) is the minimal group congruence on \( S \), we must have \( \rho_S \leq (\rho_T^* \Theta)^{-1} (\rho_T^* \Theta) \). This implies that \( \Lambda \) is a well-defined \( R \)-linear mapping of \( S/\rho_S \) into \( T/\rho_T \). \( \Lambda \) is then an \( R \)-linear mapping of \( (T/\rho_T)' \) into \( (S/\rho_S)' \), which is defined by
\[ \langle \rho_T^* x, t^* \rangle = \langle \Lambda \rho_T^* x, \varphi_T(t^*) \rangle \] for all \( x \in S \), and all \( t^* \in (T/\rho_T)' \).

But since \( \Lambda \rho_T^* = \rho_T^* \Theta \), we then have
\[ \langle \rho_T^* x, t^* \rangle = \langle \Lambda \rho_T^* x, \varphi_T(t^*) \rangle \]
\[ = \langle \Theta x, \varphi_T(t^*) \rangle \]
\[ = \langle x, (t \varphi_T(t^*)) \rangle \]
\[ = \langle \rho_T^* x, (\varphi_T^{-1} t \varphi_T) \rangle \]
for all \( x \in S \) and all \( t^* \in (T/\rho_T)' \), hence \( t \Lambda = \varphi_T^{-1} t \varphi_T \).
REFERENCES

   Bourbaki, 6).

[2] CLIFFORD (A. H.) and PRESTON (G. B.). - The algebraic theory of semigroups,
   Vol. 1. - Providence, American mathematical Society, 1961 (Mathematical Sur-
   veys, 7).

[3] CLIFFORD (A. H.). - The structure of orthodox unions of groups, Semigroup Fo-
   rum, t. 4, 1972, p. 283-337.

[4] FIRSOV (J. M.). - Everywhere defined semimodules [in Russian], "Summaries of
   - Gomel, 1975.

[5] HEWITT (E.) and ZUCKERMAN (H. S.). - Finite dimensional convolution algebras,


    Acad., t. 34, 1958, p. 110-112.


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