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ON NOWHERE DENSE ϑ -SETS

by Andrew D. POLLINGTON (*)

In this note, we show that, for every $\vartheta > 1$, there are real numbers k so that $X_k = \{\widehat{k\vartheta^n} ; n = 1, 2, \dots\}$ is nowhere dense. Here $\widehat{x}$ denotes the image of x in $\mathbb{R}/\mathbb{Z}$. This answers a question raised by CHOQUET in [1], and is a straightforward corollary of the following theorem.

THEOREM. - Given $\epsilon > 0$, $\vartheta > 1$ and a sequence of real numbers a_n , there is a sequence ϵ_n of positive real numbers, and a set of real numbers K of Hausdorff dimension at least $\frac{1}{2} - \epsilon$ so that, if $k \in K$, then

$$(1) \quad \|k\vartheta^n - a_n\| > \epsilon_n \text{ for all } n, n \in \mathbb{N}.$$

Proof. - Let $\frac{1}{2} - \epsilon < s < \frac{1}{2}$, and choose r so that

$$(2) \quad \vartheta^r - 4r > \vartheta^{rs}.$$

Put

$$(3) \quad \epsilon_n = \vartheta^{-2^{n+2}r}, \quad t = [\vartheta^r - 1], \quad M = t - 4r.$$

We will construct a nested sequence of sets of intervals $I_0 \subset I_1 \subset \dots$ so that $k \in \bigcap I_j$ satisfies (1). These sets I_n will be chosen so that, if I is an interval of I_{n-1} , then $|I| = \vartheta^{-rn}$. I_{2^n} will be a union of M^{2^n} intervals each containing M^{2^n} intervals of $I_{2^{n+1}}$.

The construction. - Put $I_0 = [0, 1]$.

For each interval I of I_{n-1} , we divide I into t equally spaced subintervals, each of length $\vartheta^{-r} |I|$. We will delete some of these intervals, those remaining will form the set I_n . We will delete the intervals according to a rule depending on n .

Suppose $n = 2^p + u \cdot 2^{p+1}$, so $2^p \parallel n$. If $u = 0$, no intervals are deleted when forming I_n . If $u > 0$, we distinguish two cases :

(a) There is some integer q for which

$$n - 2^{p+1} < 2^q < n.$$

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(b) There is no such q .

We will choose the intervals I so that, if $k \in I_n$, then

$$(4) \quad \|k \sigma^n - a_p\| > \epsilon_p, \quad m = 1, 2, \dots, (n - 2^{p+1})r.$$

Delete J from the choices for intervals of I_n if $J \cap L \neq \emptyset$, where

$$(5) \quad L = \{k; \|k \sigma^n - a_p\| \leq \epsilon_p, \quad (n - 2^{p+2})r + 1 \leq m \leq (n - 2^{p+1})r\}.$$

Case a. - By (3) and (5), for every interval I of I_{2^n} , we delete at most $2^{p+2}r$ intervals contained in I .

Case b. - As above, for every interval I of $I_{n-2^{p+1}}$, we delete at most $2^{p+2}r$ intervals contained in I .

It now only remains to verify the condition concerning the number of intervals in I_{2^n} . Suppose that I is an interval of I_{2^n} . Then I contains at least M^w intervals of $I_{2^{n+w}}$, $0 \leq w \leq 2^n$.

Thus we may choose 2^n intervals of $I_{2^{n+1}}$ in every interval of I_{2^n} . Put $J_n = I_{2^n}$, $n = 1, 2, \dots$. By (2) and a theorem of EGGLESTON [2], the dimension of the set of numbers satisfying (1) is at least s .

COROLLARY. - Given $\epsilon > 1$, the set of real numbers k for which X_k is nowhere dense has Hausdorff dimension at least $\frac{1}{2}$.

Proof. - Apply the theorem with a_n any dense sequence modulo 1.

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