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NUMBERS WITH ORDER ≤ 1

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Let p be a prime integer, let \mathbb{Q}_p be the completion of \mathbb{Q} for the p -adic absolute value $|\cdot|$, let Ω_p be an algebraic closure of \mathbb{Q}_p provided with the unique extension of the absolute value $|\cdot|$, and let \mathbb{C}_p be the (algebraically closed) completion of Ω_p .

In \mathbb{C}_p , we propose to study a notion of "transcendence order $\leq \alpha$ over \mathbb{Q}_p " which looks a little like the transcendence type over \mathbb{Q} ([2], [3], [4]).

Let $\|\cdot\|$ be the Gauss norm on $\mathbb{Q}_p[X]$ defined by

$$\|\sum_{i=0}^n a_i X^i\| = \max_{0 \leq i \leq n} |a_i|.$$

Let K be a transcendence degree 1 extension of \mathbb{Q}_p . Let $|\cdot|$ be an extension to K of the absolute value defined on \mathbb{Q}_p , and let v be the valuation defined by $v(x) = -\log |x|$ where \log is the logarithm function in base p . An element $x \in K$, transcendental over \mathbb{Q}_p , is said to have order $\leq \alpha$ ($\alpha > 0$) if there exists a positive constant C_x such that every polynomial $P(X) \in \mathbb{Q}_p[X]$ satisfies

$$v(P(x)) \leq -\log \|P\| + C_x (\deg P)^\alpha.$$

No $x \in \mathbb{C}_p$ can have order $\leq \alpha$, if $\alpha < 1$, but with help of an example, we prove there exist some $x \in \mathbb{C}_p$ transcendental over \mathbb{Q}_p , with order ≤ 1 (theorem 1). On the other hand, we know that order $\leq \alpha$ is stable by algebraic extension.

At the Oberwolfach meeting, we proved the existence of some $x \in \mathbb{C}_p$ with order $\leq 1 + \epsilon$ whenever $\epsilon > 0$. Here, we give a new kind of number, with order ≤ 1 .

THEOREM. - There exist some numbers $t \in \mathbb{C}_p$, transcendental over \mathbb{Q}_p , of order ≤ 1 .

In the proof of theorem, we will use the following notations.

For any finite set S , let $\text{card}(S)$ be the number of elements of S .

Let H be an algebraic extension of \mathbb{Q}_p , and let $\text{Aut}(H)$ be the group of the \mathbb{Q}_p -automorphisms in H . We know that every algebraic extension of \mathbb{Q}_p is normal [1]

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hence it follows the classical result.

LEMMA. - Let H be an algebraic extension of \mathbb{Q}_p . Then $[H : \mathbb{Q}_p] = \text{card}(\text{Aut}(H))$

Proof of theorem. - Let (u_i) , $i \in \mathbb{N}$, be a sequence in Ω_p such that $v(u_i) = 0$ and $\mathbb{Q}_p[u_i]$ is the unramified extension of degree 2^i over \mathbb{Q}_p . Then we have

$$(1) \quad v(u_i - \sigma(u_i)) = 0 \quad \text{for every } \mathbb{Q}_p\text{-automorphism } \sigma \text{ in } \Omega_p.$$

Let (f_i) , $i \in \mathbb{N}$, be a strictly increasing sequence of positive integers such that the series $\sum_{i=1}^{\infty} f_i/2^i$ converges.

Set $a_i = u_i p^{f_i}$, and let $t = \sum_{i=1}^{\infty} a_i \in \mathbb{C}_p$. We are going to show that t has order ≤ 1 .

Clearly, it is enough to prove the existence of some $C \in \mathbb{R}^+$ such that every irreducible polynomial $F(X) \in \mathbb{Q}_p[X]$, such that $\|F\| = 1$, satisfies

$$(2) \quad v(F(t)) \leq C \deg F.$$

Consider such a polynomial F and set

$$F(X) = \lambda \prod_{i=1}^d (X - b_i), \quad \text{where } b_1, \dots, b_d \text{ are conjugate over } \mathbb{Q}_p.$$

We know that the $v(b_i)$, $1 \leq i \leq d$, are equal [1]. If the $v(b_i)$ are < 0 , we have

$$v(F(t)) = v(\lambda) + d v(b_i) = -\log \|F\| = 0.$$

So we can assume $v(b_i) \geq 0$ to prove the relation (2).

Then, by classical results [1], we have $v(\lambda) = 0$ hence we can assume that F is monic.

Now, by reordering the b_i , we can obviously assume that

$$v(t - b_1) \geq v(t - b_i) \quad \text{for every } i = 2, \dots, d.$$

Set $b = b_1$ and let N be the unique integer such that

$$f_N < v(t - b) \leq f_{N+1}.$$

Write $b = a_1 + \dots + a_N + a'$; then $v(a') \geq v(t - b) > f_N$.

Let $L = \mathbb{Q}_p[a_1, \dots, a_N, a']$. Obviously, $\mathbb{Q}_p[b] \subseteq L$.

We are going to prove $\mathbb{Q}_p[b] = L$. Let $\sigma \in \text{Aut}(L)$ be different of the identical automorphism $\underline{1}$ and let us prove that $\sigma(b) \neq b$. First suppose there exist integers $h \leq N$ such that $\sigma(a_h) \neq a_h$, and let i be the lowest of these integers h . By hypothesis (1), we have $v(\sigma(a_i) - a_i) = v(a_i) = f_i$. It follows that $v(\sigma(b) - b) = f_i$ since the f_N are strictly increasing, and $v(a') > f_N$. Similarly, suppose now i does not exist; then

$$\sigma(b) - b = \sigma(a') - a'.$$

But necessarily $\sigma(a') \neq a'$ since $\sigma \neq 1$ and $\sigma(a_i) = a_i$, $\forall i = 1, \dots, N$.

Thus we see that the $\sigma(b)$ are all different, hence $\text{card}(\text{Aut } \mathbb{Q}_p[b]) \geq \text{card}(\text{Aut } L)$; then by lemma 1, $[\mathbb{Q}_p[b] : \mathbb{Q}_p] \geq [L : \mathbb{Q}_p]$ and finally $\mathbb{Q}_p[b] = L$.

Now, for each $j = 1, \dots, d$, b_j is some $\sigma(b)$ (with $\sigma \in \text{Aut } \Omega_p$).

Then $b_j = \sigma(a_1) + \dots + \sigma(a_N) + \sigma(a')$. Consider

$$b - b_j = \sum_{i=1}^N a_i - \sigma(a_i) + \sum_{i=N+1}^{\infty} a_i - \sigma(a').$$

By (1), we have $v(a_i - \sigma(a_i)) = v(a_i) = f_i$, and

$$v(\sum_{i=N+1}^{\infty} a_i - \sigma(a')) > f_N \text{ since } v(\sum_{i=N+1}^{\infty} a_i) \geq f_{N+1} \text{ and } \sigma(a') > f_N.$$

Thus the $v(b - b_j)$ are necessarily in the form f_ℓ with $\ell \leq N$, when they are not strictly superior to f_N . Then it is the same thing for the $v(t - b_j)$.

For each integer $\ell \leq N$, let

$$E_\ell = \{b_j ; v(t - b_j) = f_\ell\};$$

then it follows $\text{card } E_\ell$ is equal to the number A_ℓ of the $\sigma \in \text{Aut}(L)$ which fix $a_1, \dots, a_{\ell-1}$. Then by definition of the a_n (from u_n), we see clearly that every $\sigma \in \text{Aut}(L)$ which fixes a_i also fixes a_1, \dots, a_{i-1} , hence $A_\ell = d/2^\ell$ and $\text{card}(E_\ell) = d/2^\ell$. So, we have

$$\sum_{\substack{b_j \in E_\ell \\ 1 \leq \ell \leq N}} v(t - b_j) = \sum_{i=1}^N \frac{df_i}{2^i}$$

Similarly the set $\tilde{E}_N = \{b_j ; v(t - b_j) > f_N\}$ has $d/2^N$ elements. Since, for any $j \leq d$, we have $v(t - b_j) \leq v(t - b_1) \leq f_{N+1}$, we see that

$$\sum_{b_j \in \tilde{E}_N} v(t - b_j) \leq \frac{df_{N+1}}{2^N}.$$

Finally

$$v(F(t)) = \sum_{j=1}^d v(t - b_j) \leq \left(\sum_{i=1}^N \frac{df_i}{2^i}\right) + \frac{df_{N+1}}{2^N}.$$

Set $C = \sum_{i=1}^{\infty} f_i/2^{i-1}$; with greater reason, $v(F(t)) \leq C d$.

Thus, every irreducible polynomial F such that $|F| = 1$ satisfies $v(F(t)) \leq C \deg F$, hence by preliminary remarks, theorem 1 is proved.

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