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A NOTE ON THE p-ADIC GAMMA FUNCTION

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Let K be a universal p -adic domain, i. e. K is an algebraically closed field of characteristic zero complete under a valuation extending the p -adic valuation of $\mathbb{Q}$. This valuation is normalized by $|p| = 1/p$, and is denoted additively by $\text{ord } x = -\log |x|/\log p$. We assume $p \neq 2$. Let $U = \mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}$. For r real positive, $D(z, r^-)$ denotes the open disk $\{x; |x - z| < r\}$. We shall use $W_r(\mathbb{Z})$ to denote the union of all disks $\{D(z, r^-)\}$, $z \in \mathbb{Z}$. Clearly this union may be replaced by a finite disjoint union of some of the indicated disks. For

$$r \geq 1, \quad W_r(\mathbb{Z}) = D(0, r^-).$$

We shall avoid the symbol $W_r(\mathbb{Z})$ with $r \geq 1$. For $s \in \mathbb{N}$, let $(x)_s$ denote the polynomial $\prod(x+i)$ the product being over $i \in [0, s-1]$ (and hence $(x)_0 = 1$). For $s \in \mathbb{N}$, we use $\Gamma(s+x)/\Gamma(x)$ to denote $(x)_s$ and $\Gamma(x-s)/\Gamma(x)$ to denote $1/(x-s)_s$. Let $\pi \in K$, $\pi^{p-1} = (-p)$. Let $e = p^{-1} + (p-1)^{-1}$, $\rho = p^{-e}$ (so $1 > \rho > 1/p$). A basis $\{u_i\}_{i \in I}$ of a Banach space will be said to be 0. N. if $\|\sum x_i u_i\| = \sup |x_i|$.

Let θ denote the function $\theta(X) = \exp(\pi(X - X^p))$, which has been used [Dw 1] to give an analytic description of additive characters of finite fields. By comparison with the function $\exp((\pi X)^{p^2}/p^2)$, it is known that the Taylor expansion

$$(1) \quad \theta(X) = \sum_{n=0}^{\infty} c_n X^n$$

satisfies

$$(2) \quad \text{ord } c_n \geq n(p-1)/p^2$$

$$(2') \quad n^{-1} \liminf \text{ord } c_n = (p-1)/p^2$$

$$(3) \quad \text{ord } c_n \geq \frac{n}{p-1} - 2\left[\frac{n}{p^2}\right] - \text{ord } \left[\frac{n}{p^2}\right].$$

We recall the Morita p -adic gamma function, Γ_p , defined on $\mathbb{Z}$ by the initial condition and functional equation

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$$(4) \quad \begin{cases} \Gamma_p(0) = 1 \\ \Gamma_p(1+x)/\Gamma_p(x) = \begin{cases} -1 & \text{if } |x| < 1 \\ -x & \text{if } |x| = 1. \end{cases} \end{cases}$$

The function Γ_p is extended to $W_p(\underline{Z})$ by local analyticity as will be recalled below.

The intimate relation between θ and Γ_p has been examined several times ([Boy], [DW 2], [DW 3], [Ba]). The object of this note is to review this work and to examine more closely the method of BARSKY.

For $y \in D(0, (pp)^-)$, $\mu \in \underline{Z}$, we define

$$(5) \quad h_\mu(y) = \pi^{-\mu} \sum_{ps+\mu \geq 0} c_{ps+\mu} (-\pi)^{-s} \Gamma(y+s)/\Gamma(y).$$

For $x \in W_p(\underline{Z})$, $i \in \underline{Z}$, let

$$(6) \quad g_i(x) = - \sum_{\ell=0}^{\infty} c_\ell \pi^{-\ell} \Gamma(-x+\ell+i)/\Gamma(-x).$$

For $r \in [1/p, 1]$, $x \in W_r(\underline{Z})$, it is known that $|(x)_s| \leq r^{[s/p]}$. This estimate together with (2) shows that aside from a possible finite set of poles at integral values of the argument if μ or i are negative, the function h_μ is analytic on $D(0, (pp)^-)$ and the function g_i is locally analytic of analyticity radius ρ on $W_\rho(\underline{Z})$ (i. e. $g_i|_{D(z, \rho^-)}$ is analytic for each $z \in \underline{Z}$). The sums g_μ are by no means new. In lectures and articles since 1961, they have been associated with the calculation of Gauss sums.

For $x \in W_p(\underline{Z})$, we define $\text{Rep}(-x)$ to be element $\mu \in \{0, 1, \dots, p-1\}$ such that $|x + \text{Rep}(-x)| < 1$. We then define $y \in D(0, (pp)^-)$ by the condition

$$(7) \quad x = -\mu + py.$$

As will again be explained below, with these definitions, we have

$$(8) \quad \Gamma_p(x) = h_\mu(y).$$

This equation with $\mu = 0$ was used by BOYARSKY to show that $\Gamma_p|_{D(0, \rho^-)}$ is an analytic function. The functional equation (4) then shows that Γ_p extends to a locally analytic function of analyticity radius ρ . Local analyticity with radius $|p|$ was known previously [Mo], but the improvement to ρ had not been previously reported.

The analyticity of Γ_p was subsequently studied by BARSKY using noncohomological methods. By his elementary methods one can show (cf. lemma 2 below), for $0 \leq i \leq p-1$,

$$(9)_i \quad g_i(x) = \Gamma_p(1+x) \cdot \chi_{D(i, \rho^-)}$$

where χ_A denotes the characteristic function of the subset A of K .

In particular, BARSKY examined the question of whether Γ_p has analyticity radius greater than ρ . Indeed, one may use either (3) or (9)₀ for this purpose. The point is that, for $r \geq 1$, the Banach space of bounded analytic functions on $D(0, r^-)$ have an O. N. basis deduced by normalization of the functions $\{(x)_s\}_{s \in \mathbb{N}}$ (cf. [Am]). Applying this to equation (8), we see that if Γ_p were to have analyticity radius greater than ρ then

$$\liminf_{s \rightarrow \infty} (ps + \mu)^{-1} \text{ or } c_{ps+\mu} > (p-1)/p^2$$

which according to (2') must be false for at least one $\mu \in \{0, 1, \dots, p-1\}$.

For $r < 1$, the functions $\{(x)_s\}_{s \in \mathbb{N}}$ do not after normalization provide an O. N. basis for bounded analytic functions on $D(0, r^-)$. They do provide a basis [Am 1] for bounded locally analytic functions on $W_p(\underline{Z})$ with local analyticity radius ρ . Applying this, with $1 > r > \rho$, to Barsky's formula (9)₀, one again obtains a contradiction to (2'). (We here fill an omission of BARSKY, who neglected to evaluate g_0 on $D(i, \rho^-)$ for $i \not\equiv 0 \pmod{p}$. In the proof of his theorem 3, he put $x = py$, and incorrectly asserted $\{y \rightarrow (py)_s\}_{s \in \mathbb{N}}$ to be a set of functions which after normalization provide an O. N. basis for the space of bounded analytic functions on $D(0, (pp)^-)$.) In this note, we explain (9)_i by a simplified form of Barsky's method. We then show how it may be deduced cohomologically. We start by giving a rapid evaluation of the magnitude of $\Gamma_p(x)$ since this point has failed to receive a careful explanation (cf. [Ba], theorem 3).

LEMMA 1. - $|\Gamma_p(x)| = 1$, $\forall x \in W_p(\underline{Z})$.

Proof. - We first observe that Γ_p has no zero in $W_p(\underline{Z})$ as if x_0 were a zero then, by (4), $x_0 + p^s$ would be a zero for each $s \in \mathbb{N}$ which, by analyticity on $D(x_0, \rho^-)$, would show that Γ_p is zero on $D(x_0, \rho^-)$, and then, by the functional equation Γ_p , would be zero on $D(0, \rho^-)$ contrary to the initial condition. If now $x_1 \in W_p(\underline{Z})$ then, by (4), there exists $i (= \text{Rep } x_1) \in D(x_1, \rho^-)$ such that $|\Gamma_p(i)| = 1$. If $|\Gamma_p(x_1)| \neq 1$, then, by a well known application of the newton polygon, Γ_p must have a zero in $D(x_1, \rho^-)$. This completes the proof of the lemma.

Note. - Alternate treatments use (2), or (3) together with either (3) or (9), to show $|\Gamma_p(x)| \leq 1$. This is combined with the duality relation

$$(10) \quad \Gamma_p(x) \Gamma_p(1-x) = -(-1)^{\text{Rep}(-x)}$$

to complete the alternate proof.

LEMMA 2. - For $x \in W_p(\underline{Z})$, $0 \leq i \leq p$,

$$g_i(x) = \Gamma_p(1+x) \cdot \chi_{D(i, p^-)}.$$

Proof (Following BARSKY). - We show that, for $N \in \mathbb{N}$,

$$(11) \quad g_i(N+i) = \begin{cases} 0 & \text{if } N \not\equiv 0 \pmod{p}, \\ \Gamma_p(1+N+i) & \text{if } N \equiv 0 \pmod{p}. \end{cases}$$

The lemma then follows from the analyticity properties of the functions g_i (and indeed demonstrates that $\Gamma_p|_{\mathbb{N}}$ may be extended to a locally analytic function on $W_p(\mathbb{Z})$ satisfying (4), the appeal to Mahler's theorem ([La], p. 82) in Lang's account of Barsky's method is quite superfluous).

By equation (1), replacing x by x/π ,

$$(9) \quad \exp \frac{x^p}{p} = \exp(-x) \times \sum c_s x^s / \pi^s$$

and so comparing coefficients

$$\sum_{\ell+k=N} \frac{(-1)^\ell c_k}{\ell! \pi^k} = \begin{cases} 0 & \text{if } N \not\equiv 0(p), \\ 1/(n! p^n) & \text{if } N = pn. \end{cases}$$

Multiplying by $(N+i)!$, we obtain

$$(12) \quad \sum_{\ell+k=N} (-1)^\ell \frac{(N+i)!}{\ell!} \frac{c_k}{\pi^k} = \begin{cases} 0 & \text{if } N \not\equiv (p), \\ (pn+i)!/(n! p^n) & \text{if } N = pn. \end{cases}$$

The right side (12) is zero if $N \not\equiv 0$, and is $(-1)^{1+N+i} \Gamma_p(1+N+i)$ if $N=pn$.

On the other hand with $\ell+k=N$, we compute

$$(N+i)!/\ell! = (-1)^{k+i} (-N-i)_{k+i} = (-1)^{N+i-\ell} \Gamma(-N-i+k+1)/\Gamma(-N-i)$$

from which we recognize that the left side of (12) coincides with $(-1)^{N+i+1} g_1(N+i)$. This completes the proof of (11).

Note. - BARSKY stated ([Ba] equations (16), (25)]

$$\Gamma_p(1+x) = g_0(x) + g_1(x) + \dots + g_{p-1}(x), \quad \forall x \in W_p(\mathbb{Z})$$

$$\Gamma_p(x) = g_0(x), \quad \forall x \in D(0, p^-)$$

Remark. - We have avoided the use of the Laplace transform since it seems to obscure the basic fact that $\exp x$ is the generating function of $1/\Gamma(1+n)$ and that the purpose of equation (9) is to get the relations between $\Gamma(n)$ and $\Gamma(\frac{n}{p})$,

which indeed is approximately the role of $\Gamma_p(n)$.

In this regard, it may be useful to examine the connection between the Boyarsky matrix [Dw 3] for Bessel functions and the relation between the coefficients of the Laurent series

$$(13) \quad \exp \frac{\lambda}{2} \left(t - \frac{1}{t} \right) = \sum_{n=-\infty}^{+\infty} J_n(\lambda) t^n ,$$

as deduced from

$$(14) \quad \exp \frac{\lambda^p}{2^p p} \left(t^p - \frac{1}{t^p} \right) = \exp \frac{\lambda}{2} \left(t - \frac{1}{t} \right) \cdot F ,$$

where $F(\lambda, t) = \theta_0\left(\frac{t\lambda}{2}\right) \theta_0\left(-\frac{\lambda}{2t}\right)$, $\theta_0(x) = \theta(x/\pi)$. Using estimate (2) and differentiating (13), one should be able by means of equation (14) to deduce relations between $(J_n(\lambda), J'_n(\lambda))$ and $(J_{[n/p]}(\lambda^p), J'_{[n/p]}(\lambda^p))$. This is our understanding of how Barsky's method should be interpreted and generalized.

We now give a cohomological explanation of equation (9). The underlying theory has been discussed elsewhere ([Boy], [Dw 2], [Dw 3]) so we shall be brief.

For $a \in U = \mathbb{Q} \cap \mathbb{Z}_p - \mathbb{Z}$, let Ω_a^0 denote the space of all products $\{X^a \xi; \xi \in L_{0,\infty}\}$ where $L_{0,\infty}$ is the space of Laurent series converging in an annulus $\{X; \epsilon_1 > |X| > \epsilon_2\}$, where ϵ_1, ϵ_2 are unspecified real numbers $\epsilon_1 > 1 > \epsilon_2$. We define a differential operator D in Ω_a^0 by the formula

$$D(X^a \xi) = X^a \left(X \frac{d}{dX} + a + \pi X \right) \xi .$$

The factor space $\bar{\Omega}_a = \Omega_a^0 / \Omega_a^0$ has dimension 1 with the image of X^a as a basis. The space $\bar{\Omega}_a$ depends only upon $a \bmod \mathbb{Z}$ but, for $m \in \mathbb{Z}$, the image of X^{m+a} need not coincide with that of X^a , the relation being given by the change in basis formula

$$(15) \quad X^{a+m} \equiv \frac{\Gamma(a+m)}{\Gamma(a)} (-\pi)^{-m} X^a \bmod \Omega_a^0 .$$

For $b \in U$, $pb \equiv a \bmod \mathbb{Z}$, we have the mapping α of Ω_a^0 into Ω_b^0 and a one-sided inverse β given by

$$\alpha : X^a \xi \longrightarrow X^b \psi(\xi X^{a-pb} \theta(X))$$

$$\beta : X^a \frac{1}{\theta(X)} X^{pb-a} \eta \longrightarrow X^b \eta$$

where ψ is the endomorphism $\eta(X) \longrightarrow \eta(X^p)$ of $L_{0,\infty}$ and θ is the one-sided inverse defined by

$$(\psi\theta)(X) = p^{-1} \sum \xi(Y)$$

the sum being over all Y such that $Y^p = X$. From α and β , we deduce a pair of inverse mappings between $\bar{\Omega}_a$ and $\bar{\Omega}_b$. Letting $\gamma_p(a, b)$ denote the "matrix" (it

is one by one) relative to the bases $\{X^a\}$, $\{X^b\}$ of the mapping induced by α , it follows from the definitions and the reduction formula (15) (with a replaced by b) that

$$(16) \quad \gamma_p(a, b) = \pi^{pb-a} h_{pb-a}(b).$$

A similar calculation for the matrix of the inverse mapping induced by β gives

$$(17) \quad (\gamma_p(a, b))^{-1} = \sum_{s=0}^{\infty} (-1)^s c_s (-\pi)^{-s-t} \Gamma(a+s+t)/\Gamma(a),$$

where $t = pb - a$.

Furthermore using (15) as a change in basis formula, we obtain, for $m, n \in \mathbb{Z}$,

$$(18) \quad \gamma_p(a+m, b+n) = \gamma_p(a, b) \frac{\gamma(a+m)}{\gamma(a)} \frac{\Gamma(b)}{\gamma(b+n)} (-\pi)^{n-m}.$$

We now explain the connection with Γ_p . Up to this point, Γ_p is a function of two variables $a, b \in U$, restricted by the condition $pb - a = t \in \mathbb{Z}$. We obtain a function Γ^B of one variable a , by insisting that $t = \text{Rep}(-a) \in \{0, 1, \dots, p-1\}$. We then define $(b = (a + \text{Rep}(-a)) p^{-1})$,

$$(19) \quad \Gamma^B(a) = \gamma_p(a, b) \pi^{-\text{Rep}(-a)}.$$

(The factor $\pi^{-\text{Rep}(-a)}$ serves to make Γ^B defined over $\mathbb{Q}_p$ instead of over $\mathbb{Q}_p(\pi)$.) Using (13) and the definition, we check that Γ^B satisfies the same functional equation as Γ_p

$$(20) \quad \frac{\Gamma^B(a+1)}{\Gamma^B(a)} = \begin{cases} -1 & \text{if } |a| < 1, \\ -a & \text{if } |a| = 1. \end{cases}$$

From equation (16), we deduce

$$(21) \quad \Gamma^B(a) = h_{\text{Rep}(-a)}(b),$$

and so Γ^B may be extended analytically on $\mathbb{W}_p(\mathbb{Z})$ satisfying the initial condition and functional equation of Γ_p as given by equation (4). Thus $\Gamma^B = \Gamma_p$. We now deduce from (17) that, for $a \in U$,

$$(22) \quad \frac{1}{\Gamma_p(a)} = \pi^{\text{rep}(-a)} / \gamma_p(a, b) = (-1)^t \sum_{s=0}^{\infty} c_s(a)_{s+t} \pi^{-s},$$

where $t = \text{Rep}(-a)$. Replacing a by $-a$, t by $\text{Rep}(a)$, and using (10) in the form

$$(23) \quad \Gamma_p(-a) \Gamma_p(1+a) = -(-1)^{\text{Rep } a},$$

we deduce

$$(24) \quad \Gamma_p(1+a) = g_{\text{Rep}(a)}(a).$$

This gives a cohomological explanation of (9)₁ for $x \in D(i, p^-)$. The assertion

that $g_1(a) = 0$ for $a \notin D(1, p^-)$ reduces to the assertion that, for $a \not\equiv 0 \pmod p$, we have

$$(25) \quad X^a / \theta(X) \in DX^a L_{0,\infty}.$$

Since formally $D = (\exp \pi X)^{-1} \circ X \frac{d}{dX} \circ \exp \pi X$, it suffices to show that

$$X^a \exp \pi X^p \in X \frac{d}{dX} (X^a \exp \pi X L_{0,\infty}),$$

or, equivalently, that

$$(26) \quad X^a \exp \pi X^p = X \frac{d}{dX} (X^a \exp \pi X^p \xi)$$

has a solution ξ in $L_{0,\infty}$. The solution is

$$(27) \quad \xi = a^{-1} \sum_{j=0}^{\infty} (-\pi)^j X^{pj} / \left(\frac{a}{p} + 1 \right)_j,$$

which clearly lies in $L_{0,\infty}$.

This completes our cohomological treatment of lemma 2.

The emphasis in our construction of the Boyarsky function, I^B (cf. (19)) has been its characterization by means of the functional equation (20) which is deduced from the change of basis formulae. BARSKY's point of view was to characterize the g_1 by evaluation at a sufficient number of elements of $\underline{\mathbb{Z}}$. We now show how ~~this can be~~ done cohomologically, i. e. by a scientifically acceptable form of manipulation of integral formulae.

We first recognize g_1 as a formal Mellin transform. Let

$$\theta_0(X) = \theta(X/\pi) = \exp\left(X + \frac{X^p}{p}\right).$$

For $a \in U$, we have formally by equation (6)

$$-g_1(-a) = \left(\int_0^\infty e^{-x} x^{i+a} \theta_0(x) \frac{dx}{x} \right) / \int_0^\infty e^{-x} x^a dx/x.$$

More precisely, for $a \in U$, $g_1(-a)$ is specified by the condition

$$(28) \quad -g_1(-a) x^a e^{-x} dx/x \equiv \theta_0(x) e^{-x} x^{i+a} dx/x \pmod{d(e^{-x} x^a \hat{L}_{0,\infty})}$$

where $\hat{L}_{0,\infty}$ is the image of $L_{0,\infty}$ under the substitution $X \rightarrow X/\pi$. This is just a rearrangement of our cohomological treatment of g_1 and is based upon $X^{a+1} e^{-X} dX/X \equiv aX^a e^{-X} dX/X$. Since, $g_1(-a)$ is defined for $a \in \underline{\mathbb{N}}$ we may use equation (28) for this calculation provided we are dealing with a one dimensional space and provided $v \in \underline{\mathbb{N}}$ implies that

$$(29) \quad vX^v e^{-X} dX/X \equiv X^{v+1} e^{-X} dX/X.$$

The formula

$$(30) \quad \Gamma(n) = \int_0^\infty x^n e^{-x} dx/x,$$

in particular, $\int_0^\infty e^{-x} dx = 1$ reminds us that we must not consider $d(e^{-x})$ to be

exact. (Letting $\sigma_v = t^{-v} e^t dt$, the Hankel formula $2\pi i/\Gamma(v) = \int_{-\infty}^{(0+)} \sigma_v$ does not help here as $\sigma_v = v\sigma_{v+1}$.) With this hint, we let $\hat{L}_\infty$ denote the space of power series in X which lie in $L_{C,\infty}$, and we work with the factor space

$$\hat{L}_\infty e^{-X} dX/d(X\hat{L}_\infty e^{-X}).$$

Putting $\omega_n(X) = e^{-X} X^n dX/X$, we have

$$n\omega_n \equiv \omega_{n+1}, \quad \forall n \geq 1,$$

and so

$$(31) \quad \omega_n \equiv \Gamma(n) \omega_1 \pmod{d(X\hat{L}_\infty e^{-X})}.$$

Equation (28) now takes the form ($n > 1$),

$$(32) \quad -g_i(-n) \equiv \omega_{i+n} \exp(X + \frac{X^p}{p}).$$

The left side is $-g_i(-n) \Gamma(n) \omega_1(X)$. The right side is $X^{i+n} \exp(X^p/p) dX/X$ which, for $i+n \not\equiv 0 \pmod{p}$, we show to be of the form $d(\xi \exp \frac{X^p}{p})$ with $\xi \in \hat{L}_\infty$ (cf. equation (26)). We now restrict our attention to the case $n = pm - i$ ($m > 1$, $0 < i < p$). The right side of (32) may be written, letting $-z = X^p/p$, as $(-1)^m p^{m-1} z^m e^{-z} dz/z \equiv (-1)^m p^{m-1} \Gamma(m) \omega_1(z)$. Thus,

$$(33) \quad -g_i(i - pm) \Gamma(pm - i) \omega_1(X) \equiv (-1)^m p^{m-1} \Gamma(m) \omega_1(z).$$

We observe that $\theta_0(X) = e^{X^{-z}}$, and so

$$(34) \quad \omega_1(X) - \omega_1(z) = d((\theta_0(X) - 1) e^{-X}),$$

and the point is that $\theta_0(X) - 1 \in X \hat{L}_\infty$. Thus

$$\frac{1}{g_i(i - pm)} = \frac{\Gamma(pm - i)}{\Gamma(m)} (-p)^{m-1}.$$

On the other hand, by (10)

$$\frac{1}{\Gamma_p(1 + i - pm)} = \Gamma_p(pm - i) (-1)^{i+1} = \frac{(pm - i - 1)!}{(m - 1)! p^{m-1}} (-1)^{pm+1}.$$

This shows that equation (11) may be verified by calculation of Mellin transforms.

We note that h_μ is also a Mellin transform. We leave the details to the reader.

We are reminded by Yvette AMICE [Am 2] that contrary to our impression when writing 21.4.10 in [Dw 2], most of the results concerning radii of convergence may be deduced directly from the original formulae of MORITA [Mo] and DIAMOND [Di]. They showed that, for $x \in p\mathbb{Z}$, we have

$$(35) \quad \log \Gamma_p(x) = \sum b_s x^s,$$

where

$$b_1 = \lim_{k \rightarrow \infty} p^{-k} \sum_{\substack{s=1 \\ (a,p)=1}}^{p^k-1} \log a$$

$$b_s = (-1)^s s^{-1} L_p(s, \omega^{1-s}) \quad (s \geq 2).$$

Here ω denotes the Teichmüller character and L_p the Kubota-Leopoldt L -function. Using elementary properties of L_p and of Bernoulli numbers, one finds, for $s \geq 2$,

$$-L_p(s, \omega^{1-s}) = \lim_{k \rightarrow \infty} (1 - p^{n-1}) B_n/n,$$

where $n = 1 - s + (p-1)p^k$. In fact, one shows that, $a_1 \in \mathbb{Z}_p$,

$$(36) \quad \begin{cases} a_s = 0 & \text{if } s \equiv 0 \pmod{2} \\ sa_s \in \mathbb{Z}_p & \text{if } s \not\equiv 1 \pmod{p-1} \\ |ps(s-1)a_s| = 1 & \text{if } s \equiv 1 \pmod{p-1}. \end{cases}$$

As noted by AMICE, this is sufficient to show that $f(x) \stackrel{\text{def}}{=} \exp \sum b_s x^s$ is analytic for $\text{ord } x > \rho = \frac{1}{p} + \frac{1}{p-1}$. Since $\Gamma_p(x) \equiv 1 \pmod{p}$, for $x \in p\mathbb{Z}$, it follows that f is analytic for $\text{ord } x > \rho$, and coincides with Γ_p on $p\mathbb{Z}$. This shows that Γ_p may be extended to a function analytic on the disk $\text{ord } x > \rho$. This gives the correct lower bound for the radius of analyticity. It is not clear that the upper bound may be verified in this way. Of course, a second proof of lemma 1 may be immediately deduced.

It is well known that, for fixed $a \pmod{p-1}$, the mappings $s \rightarrow L_p(s, \omega^a)$ is analytic (or meromorphic) on the disk $D(0, |p/\pi|^-)$. One may be tempted to use this property to deduce the analytic continuation of the right side of (35) into the region $d(x, \mathbb{Z}_p^*) > |p/\pi|$. It is however better to use the fact that for x close to zero $\log \Gamma_p(x)$ coincides with Diamond's $G_p^*(x)$. Briefly, for $x \in \mathbb{Z}_p$ [Di], with $\lambda(x) = x \log x - x$,

$$(36) \quad G_p(x) = \lim_{k \rightarrow \infty} p^{-k} \sum_{n=0}^{p^k-1} \lambda(x+n)$$

and for $x \notin \mathbb{Z}_p^*$

$$G_p^*(x) = \lim_{k \rightarrow \infty} p^{-k} \sum_{n=0, p \nmid n}^{p^k-1} \lambda(x+n).$$

Diamond's version of the Gauss multiplication formula gives, for $r \geq 1$,

$$(37) \quad G_p(x) = \sum_{a=0}^{p^r-1} G_p\left(\frac{x+a}{p^r}\right),$$

and hence, for $x \notin \mathbb{Z}_p^*$, we have

$$(38) \quad G_p^*(x) = G_p(x) - G_p\left(\frac{x}{p}\right) = \sum_{a=1, p \nmid a}^{p^r} G_p\left(\frac{x+a}{p^r}\right).$$

Thus if $d(x, \mathbb{Z}_p^*) > |p|^r$ by Diamond's Stirling formula for G_p , we have

$$(39) \quad G_p^*(x) - \lambda_r(x) = \sum_{s=1}^{\infty} B_s s^{-1} (s+1)^{-1} p^{rs} \sum_{a=1, p \nmid a}^{p^r-1} (x+a)^{-s},$$

where B_s denotes the s -th Bernoulli number, and

$$(40) \quad \lambda_r = \sum_{a=1}^{p^r-1} \frac{1}{p^r a} \left\{ \left(\frac{x+a}{p^r} - \frac{1}{2} \right) \log \left(\frac{x+a}{p^r} \right) - \frac{x+a}{p^r} \right\}$$

$\log$ being the Iwasawa logarithm. These formulae reduce all questions of analyticity of G_p^* to question concerning λ_r . The analytic continuation of $\frac{d}{dx} G_p^*$ has been discussed by KOBLITZ [Ko], but his results and conjectures do not go beyond these earlier results of DIAMOND. In particular, it follows from equation (39) that, if $\alpha \in \mathbb{Z}_p^*$, then $x \mapsto G_p^*(x) - G_p^*(\alpha x)$ is an analytic function (in the sense of KRASNER, naturally) on the set $K - \mathbb{Z}_p^*$.

We observe that for the analysis of $\Gamma_p(x)$ for $x \in D(0, 1^-)$ along the lines of equation (35), it is better to use equation (39) with $r=1$, recognize that the right side is bounded by $|p|$ for $|x| < 1$, and so reduce the analysis of $\Gamma_p(x)$ to that of $\exp \lambda_1(x)$ for x close to zero. This procedure should again establish p as the precise radius of analyticity of Γ_p .

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