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ON THE REDUCTION OF ABELIAN VARIETIES

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Let A be an abelian variety of dimension n over a **complete non-Archimedean** field k . Viewing A as a rigid analytic group variety over k , we say that an open analytic subgroup $N \subset A$ has semi-abelian reduction if it is smooth over the valuation ring \hat{k} of k (see below) and if the analytic reduction \tilde{N} of N is an extension of an abelian variety \tilde{B} by an affine torus \tilde{T} (everything defined over the residue field \bar{k} of k), i. e., if there is an exact sequence

$$0 \longrightarrow \tilde{T} \longrightarrow \tilde{N} \longrightarrow \tilde{B} \longrightarrow 0.$$

The existence of a subgroup $N \subset A$ having semi-abelian reduction is of indispensable value for the construction of the universal covering \hat{A} of A , see [2] and [3].

If the valuation on k is discrete, one can obtain a subgroup $N \subset A$ of the above type as follows. One considers the formal completion $\hat{\mathcal{N}}$ of the Néron model \mathcal{N} of A . Then $\hat{\mathcal{N}}$ can be viewed as an open analytic subgroup of A , and its identity component $N := \hat{\mathcal{N}}_0$ has potential semi-abelian reduction (meaning that $N \otimes k'$ has semi-abelian reduction for some finite extension field k' of k). It is not yet known how to construct such a group N by analytic means. In this article, we want to discuss this question. In particular, we will characterize the semi-abelian reduction in terms of analytic properties.

Let H be an analytic group variety over k . Then H is called formal if it carries a formal analytic structure (given by some formal affinoid covering) (see [1], § 1), such that the structure is compatible with the group operations. The formal structure of H is unique if it exists. In particular, it gives rise to a well-defined analytic reduction \tilde{H} of H . (The reduction is a scheme of locally finite type over \bar{k} .) A formal analytic group H is called smooth over \hat{k} if the reduction \tilde{H} is geometrically regular and if H is distinguished. The latter means that, for all formal affinoid parts $\text{Sp } C$ of H , the supremum norm on C is a residue norm with respect to some epimorphism $T_m \twoheadrightarrow C$ (where T_m is a free Tate algebra) (see [1], § 2). One knows that \tilde{H} is a group scheme over \bar{k} if H is smooth over \hat{k} .

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In the following we always assume that k is a p -adic field (i. e., a complete field containing \mathbb{Q}_p). The valuation of k can be discrete or dense. We denote by $[p] : A \rightarrow A$ the homomorphism of the abelian variety A obtained by multiplying elements with p .

PROPOSITION 1. - Let N be a connected open analytic subgroup of A , smooth over \bar{k} . Then the following conditions are equivalent :

- (i) N is maximal among all connected open formal analytic subgroups of A .
- (ii) $[p] : N \rightarrow N$ is surjective.
- (iii) N has potential semi-abelian reduction.

The difficult part of the proof is to show that condition (iii) of the proposition implies condition (i). One shows more generally that N contains all connected open formal analytic subgroups of A if it has potential semi-abelian reduction. Consequently, a subgroup $N \subset A$ satisfying the equivalent conditions of the proposition is unique.

Each commutative analytic group of dimension n over a field of characteristic 0 is locally isomorphic to the n -dimensional additive group G_a^n . Therefore one can find an open analytic subgroup $I \subset A$ which is isomorphic to the unit ball in G_a^n . For $v \geq 1$, we denote by I_v the identity component of the affinoid group $[p^v]^{-1}(I)$. Let $A_+ := \bigcup_{v=1}^{\infty} I_v$.

PROPOSITION 2. - Let N be a connected open analytic subgroup of A , smooth over \bar{k} . Let N_+ be the kernel of the reduction map $N \rightarrow \tilde{N}$. Then N has potential semi-abelian reduction if, and only if $N_+ = A_+$.

This result suggests how to proceed with an analytic construction of subgroups $N \subset A$ having semi-abelian reduction. All one has to do is to construct some "quasi-compact closure" of the Stein group $A_+ \subset A$. Two steps are necessary. The first one is established by the following result :

THEOREM. - Modulo extension of the ground field k , the analytic variety A_+ is isomorphic to the "open" unit ball in affine n -space.

The second step is still in the stage of a conjecture.

PROBLEM. - Find a distinguished open affinoid subspace $U \subset A$ containing the unit element $e \in A$ such that $U_+(e) = A_+$ (where $U_+(e) := \pi^{-1}(\pi(e))$ with $\pi : U \rightarrow \tilde{U}$ denoting the canonical reduction map).

It is expected that the problem can be solved, at least if the ground field k is replaced by a finite extension. If U is an open affinoid subspace of A satisfying the stated properties, then it follows from the theorem that U is smooth over

\hat{k} in some formal neighborhood U' of e . We may assume U' is connected. Let N be the subgroup of A generated by U' . Then N is a connected open analytic subgroup, smooth over \hat{k} , such that $N_+ = U'_+(e) = U_+(e) = A_+$. Hence by proposition 2, N has potential semi-abelian reduction. Thereby we see that, modulo finite field extension, an open subgroup $N \subset A$ having semi-abelian reduction can be obtained by an analytic construction, provided the problem stated above can be solved.

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