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DECOMPOSITION OF NON-ARCHIMEDEAN ANALYTIC TORI

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Introduction. - In [3], SHIODA and MITANI prove that a complex abelian surface X is isogenous to the product $C \times C$ of an elliptic curve C with itself if, and only if, X is singular. This means that the rank of the Néron-Severi group of X takes its maximal value. The theorem can be generalised for higher dimensions.

In this paper, we prove an analogous result for abelian varieties which are defined over a complete non-archimedean valued field and which are (analytically) isomorphic to an analytic torus.

1. Definitions and notations.

Let K be a complete non-archimedean valued field. We assume that K is algebraically closed. The multiplicative group $(K^*)^n$ is identified with the affine group $\mathrm{Spec}(K[z_1, z_1^{-1}, \ldots, z_n^{-1}])$ where $z_1, \ldots, z_n$ are variables. This group has a canonical analytic structure (see e. g. [1], [2]).

A lattice Γ in $(K^*)^n$ is a discrete subgroup of $(K^*)^n$ without torsion. This means that

$$\Gamma \cap \{(x_1, \ldots, x_n) \in (K^*)^n ; \; |\pi_1| \leqslant |x_i| \leqslant |\pi_2| \text{ for all } i = 1, \ldots, n\}$$

is a finite set for all $\pi_1, \pi_2 \in K^*$ with $|\pi_1| \leqslant |\pi_2|$.

The factor group $T = (K^*)^n/\Gamma$ is given an analytic structure by requiring that the canonical map

$$\pi : (K^*)^n \longrightarrow (K^*)^n/\Gamma$$

is locally biholomorphic (see e. g. [2]). The analytic space T is called a holomorphic torus of dimension n .

Let A^* be the group of nonvanishing analytic functions on $(K^*)^n$. Each $f \in A^*$, can be written as

$$f = c\, z_1^{r_1} \times \ldots \times z_n^{r_n}$$

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with $c \in K^*$ and $r_1, \dots, r_n \in \mathbb{Z}$.

Let H be the character group of $(K^*)^n$; i. e.

$$H = \{z_1^{r_1} \times \dots \times z_n^{r_n} ; \quad r_1, \dots, r_n \in \mathbb{Z}\} .$$

The action of Γ on Λ^*, defined by $f^\gamma(x) = f(\gamma x)$ for each $f \in \Lambda^*$, $\gamma \in \Gamma$ and $x \in (K^*)^n$, makes Λ^* into a Γ-module.

Providing K^* and H with the trivial Γ-action, we obtain the following exact sequence of Γ-modules :

$$1 \longrightarrow K^* \xrightarrow{\alpha} \Lambda^* \xrightarrow{\beta} H \longrightarrow 1$$

with

$$\alpha(\lambda) = \lambda , \quad \beta(f) = (f(1))^{-1}.f \quad \text{for each } \lambda \in K^* \text{ and } f \in \Lambda^* .$$

Associated is the exact sequence of cohomology groups :

$$\dots \longrightarrow H^1(\Gamma , K^*) \xrightarrow{\alpha^*} H^1(\Gamma , \Lambda^*) \xrightarrow{\beta^*} H^1(\Gamma , H) \xrightarrow{\delta} H^2(\Gamma , K^*) \longrightarrow \dots$$

Since K^* and H are trivial Γ-modules, we have the following isomorphisms :

$$H^1(\Gamma , K^*) \simeq \mathrm{Hom}(\Gamma , K^*) , \quad H^2(\Gamma , K^*) \simeq \mathrm{Hom}(\textstyle\bigwedge^2_{\mathbb{Z}} \Gamma , K^*)$$

and

$$H^1(\Gamma , H) \simeq \mathrm{Hom}(\Gamma , H) .$$

Fixing a basis $\gamma_1, \dots, \gamma_g$ for the free abelian group Γ, we can identify

$$\mathrm{Hom}(\textstyle\bigwedge^2_{\mathbb{Z}} \Gamma , K^*) \quad \text{with} \quad (K^*)^{\binom{n}{2}} .$$

The map δ is then defined by

$$\delta(\sigma) = \left[\frac{q(\gamma_i , \sigma(\gamma_j))}{q(\gamma_j , \sigma(\gamma_i))}\right]_{i<j,\, j=2,\dots,n}$$

where $q : \Gamma \times H \longrightarrow K^*$ is the bilinear form defined by

$$q(\gamma , z) = z(\gamma) .$$

The group $\mathrm{Ker}\,\delta$ is denoted by $N(T)$. It is proved, in [1], that

$$N(T) = \{\sigma \in \mathrm{Hom}(\Gamma , H) ; \quad q(\gamma , \sigma(\gamma')) = q(\gamma' , \sigma(\gamma)) , \text{ for all } \gamma , \gamma' \in \Gamma\} .$$

Furthermore, T is analytically isomorphic to an n-dimensional abelian variety if, and only if, there exists some monomorphism $\sigma \in N(T)$ such that

$$|q(\gamma , \sigma(\gamma)| < 1 \quad \text{for all } \gamma \in \Gamma - \{1\} .$$

In this case, $N(T)$ is isomorphic to the Néron-Severi group of this abelian variety.

2. Decomposition of analytic tori.

We keep the notations of the previous paragraph.

LEMMA 1. - $\underline{\text{Rank}}(\text{Hom}(\Gamma , H)/N(T)) \geqslant \frac{n(n-1)}{2}$.

Proof. - Assume that $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{in})$, for all $i = 1, \ldots, n$, and let $a_{ij} = -\log|\gamma_{ij}|$ for all $i, j = 1, \ldots, n$.

It is easy to see that the matrix A, with entries a_{ij}, has rank n.

(a) Since A is regular, we may assume that, after renumbering the columns of A,

$$a_{11} \neq 0 .$$

For $k = 2, \ldots, n$, we define $\sigma_{1k} \in \text{Hom}(\Gamma , H)$ by

$$\sigma_{1k}(z_\ell) = z_1^{\delta_{\ell k}} \quad \text{where} \quad \delta_{\ell,k} = 0 \quad \text{if} \quad \ell \neq k \quad \text{and} \quad \delta_{k,k} = 1 .$$

For each $\sigma = \sigma_{1,2}^{m_2} \times \cdots \times \sigma_{1,n}^{m_n} \in \text{Hom}(\Gamma , H)$, we have

$$q(\gamma_1 , \sigma(\gamma_k)) = q(\gamma_1 , z_1^{m_k}) = \gamma_{11}^{m_k}$$

and

$$q(\gamma_k , \sigma(\gamma_1)) = q(\gamma_k , 1) = 1$$

for all $k = 2, \ldots, n$. It follows that $\sigma \notin N(T)$ unless $m_2 = \cdots = m_n = 0$, and consequently $\sigma_{12}, \ldots, \sigma_{1k}$ are $\underset{\sim}{Z}$-independent modulo $N(T)$.

(b) Since A is regular, we may assume that, after renumbering all but the first column of A, $\binom{a_{11}}{a_{21}}$ and $\binom{a_{12}}{a_{22}}$ are linear independent over $\underset{\sim}{Z}$.

For $k = 3, \ldots, n$, we define $\sigma_{2k} \in \text{Hom}(\Gamma , H)$ by

$$\sigma_{2k}(z_\ell) = z_2^{\delta_{k\ell}} .$$

One proves in an analogous way as in (a) that $\sigma_{12}, \ldots, \sigma_{1n}, \sigma_{23}, \ldots, \sigma_{2n}$ are linear independent modulo $N(T)$.

Repeating this construction for each $i = 3, \ldots, n$, one defines for each $j>i$ a morphism in $\sigma_{ij} \in \text{Hom}(\Gamma , H)$ such that $\sigma_{1,2}, \ldots, \sigma_{n-1,n}$ are linear independent modulo $N(T)$. It follows that

$$\text{rank}(\text{Hom}(\Gamma , H)/N(T)) \geqslant (1 + 2 + \cdots + (n-1)) = \frac{n(n-1)}{2} ..$$

COROLLARY 2. - $\underline{\text{Rank}}(N(T)) \leqslant \frac{n(n-1)}{2}$.

THEOREM 3. - <u>If</u> $\text{rank}(N(T)) = \frac{n(n-1)}{2}$ <u>then</u> T <u>is analytically isomorphic to an</u> <u>n-dimensional abelian variety. This variety is isogenous to the n-th self product</u>

of an elliptic curve C *of the form* $C \simeq K^*/q\mathbb{Z}$ *with* $q \in K^*$ *and* $|q| < 1$.

Proof. – By means of the basis $\gamma_1, \ldots, \gamma_n$ of Γ, we may identify $H^2(\Gamma, K^*)$ with $(K^*)^{\binom{n}{2}}$ and the map

$$H^1(\Gamma, H) = \mathrm{Hom}(\Gamma, H) \longrightarrow H^2(\Gamma, K^*)$$

is defined by

$$\delta(\sigma) = \left[\frac{q(\gamma_i, \sigma(\gamma_j))}{q(\gamma_j, \sigma(\gamma_i))}\right]_{i<j, j=2,\ldots,n} .$$

For each $i < j$ and $j = 1, \ldots, n-1$, we define $\sigma_{ij} \in \mathrm{Hom}(\Gamma, H)$ such as in the proof of lemma 1. Let S be the group generated by

$$\{\delta(\sigma_{ij}) ; \ i < j, \ i = 1, \ldots, n-1\} .$$

For each $k = 1, \ldots, n$, we define $\sigma_k \in \mathrm{Hom}(\Gamma, H)$ by

$$\sigma_k(z_1) = z_k^{\delta_{k1}} .$$

Since $N(T)$ has rank $\frac{n(n+1)}{2}$, $\mathrm{Im}\,\delta$ has rank $\frac{n(n-1)}{2}$, and hence the group S is of finite index in $\mathrm{Im}\,\delta$.

It follows that $\delta(\sigma_k)^{s_k} \in S$ for some $s_k \in \mathbb{N}$.

An explicit calculation of $\delta(\sigma_{ij})$ and $\delta(\sigma_k)$ shows that

$$\gamma_{ij}^{u_{ij}} \in \gamma_{11}^{\mathbb{Z}} \quad \text{for some } u_{ij} \in \mathbb{N} ; \ i, j = 1, \ldots, n .$$

We may asuume that $|\gamma_{11}| < 1$. (If necessary replace γ_1 by γ_1^{-1}.) Consequently, there exists some $q \in K^*$ with $|q| < 1$, and there exist roots of unity $\xi_{ij} \in K^*$ such that

$$\gamma_{ij} = \xi_{ij}\, q^{m_{ij}} \quad \text{for some } m_{ij} \in \mathbb{Z} ; \ i, j = 1, \ldots, n .$$

Since $\gamma_1, \ldots, \gamma_n$ are linear independent the matrix M with entries m_{ij} is regular. Hence there exists a matrix $B \in GL_n(\mathbb{Z})$ with entries b_{ij} such that

$$M.B = \begin{bmatrix} m & & 0 \\ & m & \\ 0 & & \ddots \ m \end{bmatrix} \quad \text{for some } m \in \mathbb{Z} .$$

Let $N \in \mathbb{N}$ such that $\xi_{ij}^N = 1$ for all $i, j = 1, \ldots, n$, and let $\sigma \in \mathrm{Hom}(\Gamma, H)$ be defined by

$$\sigma(\gamma_i) = z_1^{Nb_{1i}} \times \ldots \times z_n^{Nb_{ni}} \quad \text{for all } i = 1, \ldots, n .$$

For all $i, j = 1, \ldots, n$, we have

$$q(\gamma_i, \sigma(\gamma_j)) = q^{Nm\delta_{ij}} .$$

It follows that $\sigma \in N(T)$ and that $|q(\gamma , \sigma(\gamma))| < 1$ for all $\gamma \in \Gamma - \{1\}$.
Hence T is analytically isomorphic to an abelian variety.

The endomorphism β of $(K^*)^n$, defined by

$$\beta(x_1 , \dots , x_n) = (\prod_{j=1}^{n} x_j^{b_{j1}} , \dots , \prod_{j=1}^{n} x_j^{b_{jn}})$$

maps Γ onto a subgroup Γ' of $(K^*)^n$ which is generated by the elements

$$\gamma'_i = (\tau_{i1} , \dots , \tau_{ii} \ q^m , \ \tau_{ii+1} , \dots , \tau_{in})$$

where $\tau_{ij} \in K^*$ are roots of unity.

Let $M \in \underset{\sim}{N}$ such that $\tau_{ij}^M = 1$ for all $i , j = 1 , \dots , n$.

The endomorphism τ of $(K^*)^n$ defined by

$$\tau(x_1 , \dots , x_n) = (x_1^M , \dots , x_n^M)$$

maps Γ' onto a subgroup Γ'' of $(K^*)^n$, defined by the elements

$$\gamma''_i = (1 , \dots , 1 , \ q^{Mm} , \ 1 , \dots , 1)$$

where q^{Mm} stands on the i-th entry.

Furthermore $\tau \circ \beta$ induces a morphsim

$$\rho : \ (K^*)^n/\Gamma \longrightarrow (K^*)^n/\Gamma'' .$$

It is clear that

$$(K^*)^n/\Gamma'' \simeq (K^*/(q^{Mm})^{\underset{\sim}{Z}}))^n ,$$

and that ρ is finite ; i. e. ρ is an isogeny.

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