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$p$-adic Siegel halfspace


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Results about function theory on the Siegel halfspace $H_n$ over an ultrametric field are given. It is proved that $H_n$ is a Stein domain. Expansions for the analytic functions on $H_n$ are obtained.

(1) Let $K$ be field together with a multiplicative valuation $|\cdot|$. Denote by $H_n(K)$ the set of all symmetric $n \times n$ matrices $x = (x_{ij})$ whose entries $x_{ij} \in K := K - \{0\}$ and for which the associated real symmetric matrix $(-\log |x_{ij}|)$ is positive definite.

Example. — $K = \mathbb{C}$ = field of complex numbers together with the usual absolute value. Let $\mathfrak{c}_n$ be the classical Siegel halfspace of all symmetric $n \times n$ matrices $z = (z_{ij})$ whose entries $z_{ij} \in \mathbb{C}$ and for which the associated matrix $\Im z := (\Im z_{ij})$ is positive definite where $\Im z_{ij}$ is the imaginary part of $z_{ij}$, (see for instance [5], chapter I, § 6, p. 24).

Consider the mapping $e : \mathfrak{c}_n \rightarrow H_n$ given by $e(z_{ij}) := (\exp 2\pi \sqrt{-1} z_{ij})$. As

$$|\exp 2\pi \sqrt{-1} (\Re z_{ij} + \sqrt{-1} \Im z_{ij})| = \exp(-2\pi \Im z_{ij})$$

and

$$-\log |\exp 2\pi \sqrt{-1} z_{ij}| = -\log \exp(-2\pi \Im z_{ij}) = 2\pi \Im z_{ij},$$

we get that a symmetric matrix $z = z_{ij}$ is in $\mathfrak{c}_n$ if, and only if, $e(z) \in H_n(\mathbb{C})$.

Moreover $e(z) = e(z')$ if, and only if, $z - z'$ has entries $\in \mathbb{Z}$.

Thus we see that $H_n(\mathbb{C}) = \mathfrak{c}_n \mod T_n$, where $T_n$ is the group of all integral translations $z \rightarrow t + z$ where $t = (t_{ij})$ is symmetric, and all entries $t_{ij} \in \mathbb{Z}$.

Remark. — Assume that $K$ is complete. Let $x \in H_n(K)$. The multiplicative subgroup of $K^+_n = n$-fold product of the multiplicative group $K_x$ generated by the columns of $x$ is denoted by $\Lambda_x$.

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$A_x$ is a lattice in $\mathbb{K}_x^n$, and the quotient $\mathbb{K}_x^n/A_x$ is an analytic torus and an abelian variety over $K$ (see i.e. [2], (VI 1.3) and (VI 6.1)).

$x$ also determines a polarization given by the zeroes of the principal theta function

$$\theta(z_1, \ldots, z_n) = \theta(z) = (k_1, \ldots, k_n) \in \mathbb{Z}^n \ni \sum x[k] z_1^k \ldots z_n^k$$

where

$$x[k] := \prod_{i,j=1}^n x_{ij}^{k_i k_j}.$$ 

Thus $x$ determines a polarized abelian variety $A_x$ over $K$.

The canonical projection $H_n(K) \times (\mathbb{K}_x^n/A_x) \rightarrow H_n(K)$ gives an analytic family of polarized abelian varieties.

(2) Let $x = (x_{ij})$ be a $m \times n$ matrix with entries $x_{ij} \in \mathbb{K}_x$, and $a = (a_{ij})$ be a $n \times r$ matrix with entries $a_{ij} \in \mathbb{Z}$. We define

$$xa := (y_{ij}) \quad \text{by} \quad y_{ij} := \prod_{k=1}^n a_{jk} x_{ik}.$$ 

$x^a$ is a $m \times r$ matrix with entries $x \in \mathbb{K}_x$.

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All formal rules of matrix manipulations hold also for these products. Especially the set $\mathbb{K}_x^{m \times n}$ of all $n \times n$ matrices with entries in $\mathbb{K}_x$ is a left and a right module over the ring $\mathbb{Z}^{n \times n}$ of all integral $n \times n$ matrices, and these two actions are compatible which means $(a x)^b = a(x^b)$.

Denote by $S_n(K)$ the set of all symmetric $n \times n$ matrices $n = (x_{ij})$ with $x_{ij} \in \mathbb{K}_x$. We consider $S_n(K)$ as a $K$-algebraic torus by identifying as usual $S_n(K)$ with $\mathbb{K}_x^{n(n+1)/2}$. For any $a \in \mathbb{Z}^{n \times n}$ denote by $\xi_a$ the mapping $\xi_a : S_n(K) \rightarrow S_n(K)$ given by $\xi_a(x) := a^x a$, where $a^x$ is the transposed matrix of $a$. We obtain that $\xi_a$ is an algebraic finite covering of degree $|\det a|^{n+1}$ if $\det a \neq 0$ and that $\xi_a(H_n) \subseteq H_n$.

As $\xi_a \circ \xi_b = \xi_{ab}$ and $\xi_a = \xi_b$ if, and only if, $a = \pm b$, we get that

$$\Gamma_n := \{\xi_a; \ a \in \text{GL}_n(\mathbb{Z})\}$$

is a transformation group on $S_n(K)$ isomorphic to $\text{PGL}_n(\mathbb{Z})$.

Remark. - Let $x, x' \in H_n(K)$ and $K$ be ultrametric. Then $A_x$ is isomorphic to
A, as polarized abelian varieties if, and only if, there exists \( \gamma \in \Gamma_n \) such that \( \zeta(x) = x' \).

This result is not true for the complex field \( \mathbb{C} \) (see [5], chapter III, § 6). It can be proved with the help of the lifting theorem in [3].

Thus we see that the orbit space \( H_n(K)/\Gamma_n \) is a subset of the moduli space of all polarized abelian varieties. This motivates the following definitions.

**Definition.** - Let \( K \) be ultrametric and complete. \( H_n(K) \) is called the Siegel halfspace over \( K \), and the transformation group \( \Gamma_n \) on \( H_n(K) \) is called the Siegel modular group.

(3) A \( K \)-valued function \( f(x) \) on \( H_n(K) \) is called \( K \)-analytic if the restriction of \( f \) onto any \( K \)-affinoid polyhedron \( P \) of \( \mathbb{P}^n \) which is contained in \( H_n(K) \) is analytic.

It means for \( K \) algebraically closed that \( f \) can uniformly on \( P \) be approximated by rational functions on \( \mathbb{P}^n \) without poles on \( P \).

In order to determine the analytic functions on \( H_n(K) \), we introduce

\[
M := \{ k = (k_{ij}); \ k \text{ is } n \times n \text{ matrix}; \ k_{ij} = k_{ji} = k_{ii} \in 1/2 \mathbb{Z}; \ k_{ii} \in \mathbb{Z} \} \\
2k_{ij} \text{ is a monomial in the variables } x_{ij}, \ldots, x_{nn}.
\]

**PROPOSITION 1.** - The algebra of \( K \)-analytic functions on \( H_n(K) \) coincides with the algebra of Laurent series

\[
f(x) = \sum_{k \in M} c_k < x, k >, \ c_k \in K,
\]

which converge on all of \( H_n(K) \).

**Proof.** - \( H_n \) is a connected Reinhardt domain (see [4], def. 1.3). For any \( x^0 \in H_n \) one finds \( \rho_{ij} < \rho_{ij}' \) (\( \in |K_n| \)) such that the polyhedron

\[
P := \{ x \in H_n(K); \ \rho_{ij} < |x_{ij}| \leq \rho_{ij}' \}
\]

is contained in \( H_n(K) \) and such that \( x^0 \in P \).

Now \( P \) is the product of ring domains. One knows that any analytic function \( f(x) \) on \( P \) has a Laurent expansion \( \sum_{k \in M} c_k < x, k > \). The coefficients \( c_k \) can not depend on \( P \) which gives the result.

**COROLLARY.** - \( f(x) = \sum_{k \in M} c_k < x, k > \) is \( \Gamma_n \)-invariant if, and only if, \( c_k = c_k' \)
whenever $k' = a^t ka$ with $a \in \text{GL}_n(Z)$.

**Proof.** - $f(a^tx^a) = \sum_{k \in \mathbb{N}} c_k(a^tx^a, k)$. Now

$$\langle x, k \rangle = \text{tr}(x^{k^t}) = \text{tr}(k^tx)$$

where $\text{tr} x := \sum_{i=1}^{n} x_{ii}$.

Thus

$$\langle a^tx^a, k \rangle = \text{tr}(a^tx^{ak^t}) = \langle a^t, ka^t \rangle = \text{tr}(ak^ta^x) = \langle x, ska^t \rangle.$$ 

Thus

$$\sum c_k(a^tx^a, k) = \sum c_k(x, ska^t),$$

which proves the corollary.

For $m \in M$, we denote by $O_m$ the integral orthogonal group with respect to the quadratic form $m$. This means

$$O_m = \{ a \in \Gamma; a^t ma = m \}.$$

Let

$$\theta_m(x) := \sum_{a \in O_m} \langle x, a^t ma \rangle.$$

It is a formal Laurent series in the variables $x_{ij}$. Remark that for any representative $a' \in O_m$ one gets $a^t ma = (a')^t ma'$ because if $a' = b.a$, $b \in O_m$, then

$$(ba)^t mba = a^t b^t ma = a^t ma.$$ 

Also if $a^t ma = (a')^t ma'$, then $a' \in O_a$ because

$$(a'^{-1})^t ma'^{-1} = (a')^{-1}(a')^t ma'^{-1} = a^t ma = m.$$ 

This shows that each coefficient of the Laurent series has either the value 1 or the value 0. In the complex case, one part of the following proposition is known as the theorem of Koecker (see [1], théorème 1).

**Proposition 2.** - $\theta_m(x)$ is an analytic function on $H_n(K)$ if, and only if, $m$ is positive semi-definite.

**Proof.** - Let $s = \{ s \in M; s$ positive semi-definite$\}$.

Let $x \in H_n(K)$ and $v := (- \log |x|) = (v_{ij})$. We will show that, for any given $\rho > 0$, one gets $\langle v, s \rangle \geq \rho$ for almost all $s$.

There is a real orthogonal matrix $b$ such that $b^t vb = \lambda = \begin{pmatrix} \lambda^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda^m \end{pmatrix}$ is a diagonal matrix. As $v$ is positive definite all $\lambda^i > 0$.

Let $\lambda^i \leq \lambda^i$ for all $i$. 

Let \( S' = \{ b^{-1} s b ; s \in S \} \), and all matrices from \( S' \) whose entries have absolute value \( \leq r \).

Then \( S_r' \) is finite, and if \( t = (t_{ij}) \in S'_r \), then \( t \) is not positive semi-definite as

\[
(1, \pm 1, 0, \ldots, 0) \times t \times \begin{pmatrix} \pm 1 \\ \vdots \\ 0 \end{pmatrix} = t_{11} + t_{22} \pm 2t_{12} < 0
\]

for \( + \) or \( - \). This means that

\[
\langle \lambda, t \rangle \geq r \cdot \lambda_1, \text{ for any } t \in S'_r, t \in S'_r.
\]

From this one gets that \( \sum_{s \in S} \langle x, a \rangle \) is convergent on \( H_n(K) \) as well as that any \( \vartheta_s(x) = \vartheta_s'(x) \) if \( s' \) is in the \( \Gamma_n \)-orbit of \( s \) which means that we can write \( \vartheta_s'(x) \) instead of \( \vartheta_s(x) \).

**COROLLARY.** — Let \( f(x) \) be an analytic modular \( ( = \Gamma_n \)-invariant) function on \( H_n(K) \).

Then \( f(x) \) has an expansion

\[
f(x) = \sum_{s \in S} c_s \vartheta_s(x) \quad \text{with} \quad c_s \in K.
\]

**Example.** — Let \( s = (s_{ij}) \) be given by \( s_{ij} = 0 \) for all \( (i, j) \neq (1, 1) \), and \( s_{11} = 1 \). Then

\[
\vartheta_s(x) = \sum_{k \in \mathbb{Z}^n} x[k] \quad \text{where} \quad x[k] = \prod_{i,j=1}^{n} x_{ij}^{k_{ij}}.
\]

**Problem.** — Determine the coefficients of the powers of the modular function

\[
\sum_{s \in S} \vartheta_s(x) = \sum_{s \in S} \langle x, a \rangle.
\]

(4) For any \( \rho > 0 \), define

\[
H_n(\rho) := \{ x \in \mathbb{H}_n \mid \| x[k] \| \leq \rho \| k \| \text{ for all } k \in \mathbb{Z}^n \}
\]

where \( \| k \| = (\sum_{i=1}^{n} k_i^2)^{1/2} \) is the euclidean norm of \( k \).

Then \( H_n = \bigcup_{\rho > 0} H_n(\rho) \).

**Proof.** — Let \( x \in H_n \) and \( v := (- \log |x_{ij}|) \). The function \( f(y) := y^t vy \) for \( y = \left( \begin{array}{c} y_1 \\ \vdots \\ y_n \end{array} \right) \in \mathbb{R}^n \) is positive for \( y \neq 0 \).
As \( S_{n-1} = \{ y \in \mathbb{R}^n : \| y \| = 1 \} \) is compact, there is a constant \( \rho > 0 \) such that \( f(y) \geq \rho \) for all \( y \in S_{n-1} \). But \( f(y) = \| y \| \frac{f(y/\|y\|)}{\|y\|} \) which shows that \( s \in H_n(p) \).

**Lemma.** Given \( 0 < \epsilon < 1, \ 0 < \rho < \rho' < 1 \). There exists an \( r \) which depends on \( \epsilon, \rho, \rho' \), such that

\[
x_r(\rho, \epsilon) := \{ x \in S_n : \epsilon \leq |x_{ij}| \leq \epsilon^{-1} \text{ for all } i, j \}
\]

and

\[
x[k] \leq \rho \| k \|^2 \text{ for all } k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \text{ with } |k_i| \leq r
\]

is contained in \( H_n(\rho') \subset H_n \).

**Proof.** Assume the lemma is not true. Then we find for any \( r \) a matrix \( x_r = x_r(\rho, \epsilon) \) such that \( x_r \notin H_n(\rho') \). Let \( v_r := (- \log |x_{ij}|) \). The entries of \( v_r \) are bounded by \( \log \epsilon^{-1} \). We thus get a point of accumulation \( v^* \) of the sequence \( (v_r) \) which is again a symmetric \( n \times n \) matrix which satisfies

\[
k^t v^* k \geq C \| k \|^2,
\]

where \( C = - \log \rho \), for all \( k \in \mathbb{Z}^n \) because \( k^t v^* k \) is a point of accumulation of the sequence \( (k^t v_r k) \), \( r \geq 1 \), and for large \( r \) we have \( k^t v_r k \geq C \| k \|^2 \).

Let now \( \rho < \rho'' < \rho' \), and let \( D \) be the set of all symmetric real \( n \times n \) matrices \( v = (v_{ij}) \) which satisfy \( k^t v k > C'' \| k \|^2 \) with \( 0 < C'' = - \log \rho'' < C \) for all \( k \in \mathbb{R}^n \).

We claim that \( D \) is open in the space \( \mathbb{R}^{n(n+1)/2} \) of all symmetric real \( n \times n \) matrices. Let \( v \in D \) and \( \epsilon < 0 \) be small such that

\[
n^2 \epsilon < (\inf_{0 \neq k \in \mathbb{R}^n} \frac{k^t v k}{\|k\|^2} - C'')
\]

and, if \( w = (w_{ij}) \) is a symmetric real matrix with \( |w_{ij}| < \epsilon \) for all \( ij \), we obtain

\[
k^t w k = \sum_{i,j=1}^{n} w_{ij} k_i k_j \leq \sum |w_{ij}| |k_i||k_j| \leq \epsilon \left( \sum_{i,j=1}^{n} |k_i||k_j| \right) < n^2 \epsilon \|k\|^2.
\]

Thus

\[
k^t(v + w) k = k^t v k + k^t w k > C'' \| k \|^2
\]

which means that \( v + w \in D \). This proves \( D \) open.

As now \( v^* \in D \), we get that infinitely many \( v_r \) are also in \( D \) as \( D \) is open. If \( v_r \in D \) then \( x(r) \in H_n(\rho') \) which is a contradiction.

**Remark.** One can choose

\[
r = \lfloor n^2 \log \frac{\rho}{\epsilon} \rfloor + 1 \text{ for } \rho' = 1 \text{ where } H_n(1) := H_n.
\]
Theorem. - $H_n(K)$ is a Stein domain on which $\Gamma_n$ acts discontinuously.

Proof. - Let $0 < \delta < 1$, $\rho_m = \frac{m}{\delta^2}$, $\rho'_m = \frac{(m+1)}{\delta^2}$, $\epsilon_m = \delta^m$.

By the lemma, we find $r_m$ such that

$$P_m := \chi_{r_m}(\rho_m, \epsilon_m) \subset H_n(\rho'_m) \subset H_n.$$

$P_m$ is analytic polyhedron in $\mathbb{A}_n(K)$ and $H_n = \bigcup_{m=2}^{\infty} P_m$.

Also $P_m$ is in the interior of $P_{m+1}$. This proves that $H_n$ is a Stein domain (see [6], § 2).

Let $\Gamma_n(m) := \{ \xi \in \Gamma_n ; \xi(P_m) \cap P_m \neq \emptyset \}$. We claim the $\Gamma_n(m)$ is finite. It can be deduced from the fact that for any given $C > 0$, there are only finitely many $\xi \in \Gamma$ such that each column vector of $\xi$ has euclidean norm $\leq C$. This proves that $\Gamma_n$ acts discontinuously.

Let me mention a few open questions:

1° Define the analytic quotient $H_n/\Gamma_n$, and prove that it is a Stein space.

2° Find the algebraic relations between the $\xi(x)$ and its connection with the Satake compactification.

3° Are the Chow coordinates in the sense of Shimura (see [7]), analytic functions on $H_n$?

References:


