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An application of newton iteration procedure to $p$-adic differential equations


<http://www.numdam.org/item?id=GAU_1979-1981__7-8__A6_0>
AN APPLICATION OF NEWTON ITERATION PROCEDURE
TO p-ADIC DIFFERENTIAL EQUATIONS

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This report is based on the author's lectures at Strasbourg, Padova, Grenoble, Groningen and Paris. The motivations of this research were explained in the papers to appear ([3],[5]) and the lecture-notes [4] (joint with S. SPERBER). Therefore, in this paper, we will report only on the technical part.

1. Preliminaries.

Let $K$ be a field of characteristic zero complete with respect to an absolute value $| |$ which is non-trivial and ultrametric. The field of rational number, $\mathbb{Q}$, is a subfield of $K$, and we require that the restriction of $| |$ to $\mathbb{Q}$ is a p-adic absolute value for some prime number $p$. We normalize $| |$ so that $|p| = 1/p$.

For $\rho = \sum_{m=0}^{\infty} a_m x^m \in K[[x]]$, we set

$$|\rho|_0(r) = \sup_{m \geq 0} |a_m| r^m.$$

If $|\rho|_0(r_0) < +\infty$ for some positive constant $r_0$, then $\rho$ is convergent for $|x| < r_0$. The following lemma is fundamental throughout this report.

**Lemma 1.** Assume that $\rho = \sum_{m=0}^{\infty} a_j, m x^m \in K[[x]]$, $j = 1, 2, \ldots$, with the properties:

(i) $\lim_{m \to \infty} a_j, m = a_m$ exists for every $m$;

(ii) $|\rho_j|_0(r) \leq M(r)$ for $0 < r < r_0$, $j = 1, 2, \ldots$, where $r_0$ is a positive number, and $M(r)$ is a non-negative number which depends only on $r$. Then, $\rho = \sum_{m=0}^{\infty} a_m x^m$ is convergent for $|x| < r_0$, and $\lim_{j \to \infty} |\rho_j - \rho|_0(r) = 0$ for $0 < r < r_0$. (Cf. B. DWORK [1].)

2. An example (a rough sketch).

Let us consider a non-linear differential equation

\[ (*) \text{ Texte reçu le 10 juillet 1980.} \]

Partially supported by N. S. F. Grants MCS 79-01998.

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f and g are convergent. We want to find a convergent power series \( \phi \in K[[x]] \) which satisfies the equation (2.1). To do this, we try to construct \( \phi \) in the following form

\[
\phi = \sum_{j=0}^{\infty} \phi_j, \quad \phi_j \in K[[x]].
\]

**Step 1.** First of all, \( \phi_0 \) is determined by the linear differential equation

\[
x \frac{d\phi_0}{dx} + \alpha \phi_0 = f.
\]

**Step 2.** Change \( u \) by \( u = \phi_0 + v \). Then (2.1) becomes

\[
x \frac{dv}{dx} + \alpha v = \phi_0(x)^2 g(x, \phi_0(x)) + \phi_0(x) G(x) v + v^2 g_1(x, v),
\]

where

\[
G(x) = 2g(x, \phi_0(x)) + \phi_0(x) g_u(x, \phi_0(x)) \quad (g_u = \partial g/\partial u),
\]

\[
v^2 g_1(x, v) = \phi_0(x)^2 [g(x, \phi_0(x) + v) - g(x, \phi_0(x)) - g_u(x, \phi_0(x)) v] + 2 \phi_0(x) v [g(x, \phi_0(x) + v) - g(x, \phi_0(x))] + v^2 g(x, \phi_0(x) + v).
\]

We determine \( \phi_1 \) by the linear part of (2.1')

\[
x \frac{d\phi_1}{dx} + \alpha \phi_1 = \phi_0^2 g(x, \phi_0) + \phi_0 G(x) \phi_1.
\]

The other \( \phi_j \) will be determined successively in a similar manner.

This is our Newton iteration procedure.

A closer look at equation (2.3). If \( f = \sum_{m=0}^{\infty} c_m x^m \), then \( \phi_0 \) is given by

\[
\phi_0 = \sum_{m=0}^{\infty} c_m \frac{x^m}{m + \alpha}.
\]

Assuming that \( |f_0(r)| \leq M \) for \( 0 \leq r < r_0 \), where \( r_0 \) and \( M \) are some positive numbers, we want to derive

\[
|\phi_0(r)| \leq M \quad \text{for} \quad 0 \leq r < r_0,
\]

for \( r_0 \) a positive number, as large as possible, such that \( 0 < r_0 \leq r_0' \). To do this, we introduce two assumptions

\[
c_m = 0 \quad \text{for} \quad m < m_0,
\]

\[
|m + \alpha|^{-\delta} \leq c_m^{m-\delta} \quad (m \geq m_0),
\]

where \( m_0 \) is a positive integer, \( C \) is a positive number greater than one, and \( \delta \) is a positive number smaller than one, i.e. \( C > 1, \quad 0 < \delta < 1 \).

The assumption (2.8) may be called "non-Liouville property" of the exponent \( \alpha \). The condition (2.7) may be written

\[
f = 0 \pmod{x^{m_0}}.
\]
Note that, if equation (2.1) admits a formal power series solution, then, we can change (2.1) so that condition (2.7) may be satisfied for any prescribed \( m_0 \). Also note that any algebraic number \( \alpha \) satisfies condition (2.3) for any \( \delta \) if we choose \( C \) and \( m_0 \) suitably.

Under assumption (2.3), set

\[ (2.9) \quad \rho_0 = (1/C)^{m_0^{-\delta}} \]

Then \( 0 < \rho_0 < 1 \), and

\[ \rho_0^n = (\rho_0^{-\delta})^m \leq (1/C)^{m_0^{-\delta}} \leq |m + \alpha| \text{ if } m \geq m_0. \]

Hence, under assumptions (2.7) and (2.8), we have

\[ |\varphi_0|_0(r) = \sup_{m \geq m_0} |m + \alpha|^{-1} |c_m| |r|^m = \sup_{m \geq m_0} |c_m| r^m = |f|_0(r), \]

and

\[ (2.6') \quad |\varphi_0|_0(r) \leq M \text{ for } 0 \leq r < r_0 \rho_0. \]

Equation (2.4) without \( f_0 \) \( G(t) f_1 \). To simplify the explanation, we remove \( f_0 \) \( G(x) f_1 \) from the right-hand member of equation (2.4); i.e., we consider the equation

\[ (2.10) \quad x \alpha \varphi_1 + \varphi_1 = \varphi_0^2 g(x, \varphi_0). \]

We know already that

\[ (2.11) \quad \varphi_0 \equiv 0 \pmod{x^{m_0}}, \]

and that \( \varphi_0 \) satisfies (2.6'). First of all, (2.11) implies that

\[ (2.12) \quad \varphi_0^2 g(x, \varphi_0) \equiv 0 \pmod{x^{2m_0}}. \]

Hence, if we assume that \( g \) satisfies the condition

\[ (2.13) \quad |\varphi_0^2 g(x, \varphi_0)|_0(r) \leq M \text{ for } 0 \leq r < r_0 \rho_0, \]

we have

\[ (2.14) \quad \begin{cases} \varphi_1 \equiv 0 \pmod{x^{2m_0}}, \\ |\varphi_1|_0(r) \leq M \text{ for } 0 \leq r < r_0 \rho_0 \rho_1, \end{cases} \]

where \( \rho_1 = (1/C)^{(2m_0)^{-\delta}} = \rho_0^{-\delta} \).

Suppose that, proceeding inductively as above, we have defined for all \( j > 0 \),

\[ \begin{cases} \varphi_j \equiv 0 \pmod{x^{2^j m_0}}, \\ |\varphi_j|_0(r) \leq M \text{ for } 0 \leq r < r_0 \rho_0 \rho_1 \cdots \rho_j. \end{cases} \]
where $\rho_j = \rho_{j-1} = \rho_0^{-1}$; set

$$\psi_j = \sum_{\beta=0}^{j-1} \phi_\beta, \quad \rho = \sum_{\beta=0}^{\infty} \phi_\beta = \rho_0 (1 - 2^{-\delta})^{-1} > 0.$$  

Then $|\psi_j(r)| \leq M$ for $0 < r < r_0$, and $\psi_j$ converges x-adically to $\rho = \sum_{\beta=0}^{\infty} \phi_\beta$.

Therefore, by virtue of lemma 1, we conclude that $\rho$ is convergent for \(|x| < r_0\).

The argument of this section is not strictly speaking correct, since we removed $\rho_0 C(x) \phi_1$ from the right-hand member of equation (2.4). A correct treatment of equation (2.1) is given in SIBUYA-SPERBER ([2],[4]).

3. Typical results.

In this section, we shall give a rigorous treatment of a problem which is more general than the problem of section 2. We assume that $K$ contains an element $\pi$ such that

$$|\pi| = \left(\frac{1}{p}\right)^{p-1}.$$  

We consider the following situation.

(i) We are given $\alpha_1, \ldots, \alpha_n \in K$ such that

$$|\alpha_j| \leq 1, \quad |m + \alpha_j|^{-1} \leq C^m, \quad |m + \alpha_j - \alpha_j|^{-1} \leq C^m.$$  

for $m \geq 2^k$ and $i, j = 1, \ldots, n$, where $k$ is a non-negative integer, and $C$ and $\delta$ are positive numbers such that $C > 1$, $0 < \delta < 1$.

(ii) We are also given $a_1, \ldots, a_n \in K[[x]]$ such that

$$a_j \equiv 0 \pmod{x}, \quad \left|\int_0^r t^{-1} a_j(t) \, dt\right|_0 < |\pi|,$$  

for $0 < r < r_0$ and $j = 1, \ldots, n$, where $r_0$ is a positive number, and where, for $a = \sum_{m=1}^{\infty} a_m x^m$, we have denoted $\sum_{m=1}^{\infty} (a_m/m) x^m$ by $\int_0^r a(t) \, dt$.

We define two sequences of numbers, $\{\sigma_n\}$ and $\{\tau_n\}$ by

$$\left\{\begin{array}{l}
\sigma_1 = 1/C, \quad \tau_1 = (1/c)^{2(1-2^{-\delta})^{-1}} \\
\sigma_n = \sigma_{n-1}^{2(1-2^{-\delta})^{-1}} \quad \tau_n = (\sigma_n^{h})^{(1-2^{-\delta})^{-1}}.
\end{array}\right.$$  

Note that

$$0 < \tau_n < \sigma_n < \tau_{n-1} < 1.$$  

In this section, we shall prove the following two theorems.
THEOREM 1. - Assume that a differential operator \( H = \sum_{j=0}^{n-1} b_j(x) \partial^j \) (\( \partial = xd/dx \)) satisfies the following conditions:

\[
\begin{cases}
  b_j \in K[[x]] \text{ and } b_j = 0 \pmod{x^{2^k}}, \\
  |b_j|_0(r) < |\pi| \text{ for } 0 \leq r < r_0.
\end{cases}
\]

(3.6)

Then, there exists \( \eta_1, \ldots, \eta_n \in K[[x]] \) such that

\[
\begin{aligned}
  &\eta_j = c \pmod{x^{2^k}}, \\
  &|\int_0^r t^{-1} \eta_j(t) \, dt|_0(r) < |\pi| \text{ for } 0 \leq r < r_0, \quad j = 1, \ldots, n,
\end{aligned}
\]

(3.7)

and that

\[
(\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_n + a_n) - H = (\partial + \alpha_1 + a_1 - \eta_1) \cdots (\partial + \alpha_n + a_n - \eta_n).
\]

THEOREM 2. - Assume that

\[
\begin{cases}
  f \in K[[x]], \quad f = 0 \pmod{x^{2^k}}, \quad |f|_0(r) < 1 \text{ for } 0 \leq r < r_0,
\end{cases}
\]

(3.8)

and that

\[
\begin{cases}
  G = \sum_{\mu_0^+ + \cdots + \mu_{n-1} \geq 2} g_{\mu_0^+ \cdots \mu_{n-1}}(x) v_0^\mu_0 \cdots v_{n-1}^\mu_{n-1} \in K[[x, v_0, \ldots, v_{n-1}]],
\end{cases}
\]

(3.10)

avec \( g_{\mu_0^+ \cdots \mu_{n-1}}(x) \in K[[x]] \),

\[
|g_{\mu_0^+ \cdots \mu_{n-1}}|_0(r) \leq |\pi| \text{ for } 0 \leq r < r_0.
\]

Then, there exists a unique \( \varphi \in K[[x]] \) such that

\[
\varphi = 0 \pmod{x^{2^k}},
\]

(3.11)

and that

\[
(\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_n + a_n)(\varphi) = f + G(x, \varphi, \varphi, \ldots, \varphi^{n-1}).
\]

(3.12)

Furthermore, this power series \( \varphi \) also satisfies the condition

\[
|\varphi|_0(r) < 1 \text{ for } 0 \leq r < r_0, \quad r_0^2 - k^5_n.
\]

(3.13)

Remark 1. - The power series \( \varphi \) is a solution of a non-linear differential equation with purely Fuchsian linear part. This is a prototype of the most difficult situations in the study of \( p \)-adic non-linear problems. The most important part of theorem 2 is the estimate (3.13), i.e. the \( r \)-interval in which \( |\varphi|_0(r) < 1 \) holds.
Remark 2. - Theorem 1 is a Hensel-type lemma. The problem of factorization of a linear differential operator is naturally reduced to a non-linear problem such as that of theorem 2. For example, if the order of the operator is two, the corresponding non-linear problem is a Ricatti equation. In general, if the order of the operator is \( n \), the order of the corresponding non-linear problem is \( n - 1 \). Taking advantage of this situation, we can prove theorem 1 and 2 simultaneously by induction on \( n \). Since the case \( n = 1 \) was treated in SIBUYA-SFERBER [2], we shall prove these theorems for \( n \geq 2 \). (Cf. also SIBUYA-SFERBER [4].)

4. Proof of theorem 1 for \( n \).

In this section, assuming theorem 2 for \( n - 1 \), theorem 1 for \( n = 1 \), and theorem 1 for \( n - 1 \), we shall prove theorem 1 for \( n \). Set

\[
\begin{align*}
L &= (\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_{n-1} + a_{n-1}), \\
\ell &= \partial + \alpha_n + a_n.
\end{align*}
\]

We want to find \( \eta \in K[[x]] \) and \( \tilde{L} = \sum_{j=0}^{n-2} Y_j \partial^j (Y_j \in K[[x]]) \) such that

\[
L\eta - H = (L - \tilde{L})(\ell - \eta).
\]

The relation (4.2) is equivalent to the assertion that

\[
L\eta(u) - H(u) = 0
\]

for all \( u \) belonging to a sufficiently large extension of \( K[[x]] \) such that

\[
(\ell - \eta)(u) = 0.
\]

Therefore, (4.2) is equivalent to the assertion that

\[
L(u\bar{\eta}) = H(u) \quad \text{for all such } u \text{ satisfying } \ell(u) = u\bar{\eta}.
\]

Observe that

\[
(\partial + \alpha_j + a_j)(uv) = u(\partial + (\alpha_j - \alpha_0) + (a_j - a_0) + \eta)(v),
\]

if \( \ell(u) = u\bar{\eta} \). Hence

\[
L(u\bar{\eta}) = u(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) + \eta)
\]

\[
\cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) + \eta)(\eta),
\]

if \( \ell(u) = u\bar{\eta} \). We can write

\[
(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) + \eta) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) + \eta)(\eta)
\]

\[
= (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(\eta)
\]

\[
- \tilde{F}(x, \bar{\eta}, \cdots, \partial^{n-2})(\tau),
\]
where
\[
\tilde{F} = \sum_{\mu_0 \cdots \mu_{n-2} \geq 2, \mu_j \geq 0} \tilde{F}^{\mu_0 \cdots \mu_{n-2}}(x) v_0 \cdots v_{n-2} \in K[[x]][v_0, \ldots, v_{n-2}],
\]

On the other hand, if \( u = u_{n-2} \), we have
\[
F u = u(-\alpha_n - a_n + \eta), \quad \delta^2 u = u \{(-\alpha_n - a_n + \eta)^2 + \delta(-\alpha_n - a_n + \eta)\}, \quad \text{etc.}
\]

Hence, \( H(u) \) has the following form
\[(4.5) \quad H(u) = u F(x, \eta, \ldots, \eta^{n-2}), \]

where
\[
F = \sum_{\mu_0 \cdots \mu_{n-2} \geq 0, \mu_j \geq 0} F^{\mu_0 \cdots \mu_{n-2}}(x) v_0 \cdots v_{n-2} \in K[[x]][v_0, \ldots, v_{n-2}],
\]

\[
F^{\mu_0 \cdots \mu_{n-2}} \in K[[x]], \quad F^{\mu_0 \cdots \mu_{n-2}} = 0 \quad \text{(mod } x^k),
\]

and
\[
|F^{\mu_0 \cdots \mu_{n-2}}|_0(r) < |\eta| \quad \text{for } 0 < r < r_0.
\]

Thus, we derive from (4.3) the equation for \( \eta \):
\[
(\delta + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\delta + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(\eta) = F + \tilde{F}.
\]

Set \( \eta = \pi w \), and \( \tilde{f}(x) = F_{\omega_0}(x) \), \( \tilde{H} = \sum_{j=0}^{n-2} \tilde{b}_j(x) \delta^j \), where
\[
\sum_{j=0}^{n-2} \tilde{b}_j(x) v_j = \sum_{\mu_0 \cdots \mu_{n-2} \geq 1, \mu_j \geq 0} \tilde{F}^{\mu_0 \cdots \mu_{n-2}}(x) v_0 \cdots v_{n-2},
\]

and
\[
\tilde{G}(x, v_0, \ldots, v_{n-2}) = \sum_{\mu_0 \cdots \mu_{n-2} \geq 2, \mu_j \geq 0} \{\tilde{F}^{\mu_0 \cdots \mu_{n-2}}(x) + \tilde{F}^{\mu_0 \cdots \mu_{n-2}}(x)\} v_0 \cdots v_{n-2}.
\]

Then the equation for \( w \) is given by
\[(4.5) \quad (\delta + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\delta + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(w)
\]
\[
= (1/\pi) \tilde{f} + \tilde{H}(w) + (1/\pi) \tilde{G}(x, \pi w, \pi^2 w, \ldots, \pi^{n-2} w)
\]

Utilizing theorem 1 for \( n - 1 \), we find \( \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1} \in K[[x]] \) such that
that applying (4.6) theorem 2 for $n - 1$, we find a unique solution $w(x)$ such that

$$w \equiv 0 \pmod{x^2},$$

$$|w|_0(r) < 1 \text{ for } 0 < r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k_5}}.$$

Thus, we constructed $\tilde{\eta}$ so that (4.3) is satisfied and

$$\tilde{\eta} \equiv 0 \pmod{x^2},$$

$$|\tilde{\eta}|_0(r) < |\eta| \text{ for } 0 < r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k_5}}.$$

To compute $L$, we derive $L(\xi - \eta) = H - L\eta$. Putting

$$H - L\eta = \sum_{j=0}^{n-1} \hat{b}_j(x) \xi^j, \quad b_j \in \mathbb{K}[[x]],$$

we get

$$\hat{b}_j = 0 \pmod{x^2},$$

$$|\hat{b}_j|_0(r) < |\eta| \text{ for } 0 < r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k_5}};$$

furthermore,

$$Y_{n-2} = \hat{b}_{n-1}, \quad \mu = \hat{b}_{\mu + 1} - \sum_{j=\mu + 1}^{n-2} f_{j,\mu + 1} Y_j, \quad \mu = 0, \ldots, n - 3,$$

where $f_{j,\mu} \in \mathbb{K}[[x]]$, and $|f_{j,\mu}|_0(r) < 1$ for $0 < r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k_5}}$.

Finally, applying to $L - \tilde{L}$ theorem 1 for $n - 1$, and to $\xi - \eta$ theorem 1 for $n = 1$, and utilizing the inequality $\sigma_{n-1} < \sigma_1$, we complete the proof.

5. Proof of theorem 2 for $n$.

In this section, assuming theorem 1 for $n$, and theorem 2 for $n = 1$, we shall prove theorem 2 for $n$. Setting

$$\psi = \sum_{j=0}^{\infty} \frac{x^j}{\xi^j} \psi_j = \psi_{j-1} + \xi^j \psi_j,$$

we determine $\psi_j \in \mathbb{K}[[x]]$ by
\[ (5.2) \quad (\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_n + a_n)(\psi_j) = f + G(x, \psi_{j-1}, \partial^1 \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) + \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) \partial^i \phi_j, \]

where \( G_{v_i} = \partial G/\partial v_i \). This means that the \( \phi_j \) are determined by linear differential equations:

\[ (5.3) \quad L_j(\phi_j) = f_j \quad (j = 0, 1, \ldots), \]

where

\[
\begin{align*}
L_{j} &= L_{0} - \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) \partial^i \quad (j \geq 1); \\
L_{0} &= (\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_n + a_n),
\end{align*}
\]

\[ (5.4) \quad \begin{cases}
  f_0 = f \\
  f_j = G(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) - G(x, \psi_{j-2}, \ldots, \partial^{n-1} \psi_{j-2}) \\
  - \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-2}, \ldots, \partial^{n-1} \psi_{j-2}) \partial^i \phi_{j-1}\end{cases} \quad (j \geq 1)
\]

where \( \phi_0 = 0 \) if \( \lambda < 0 \).

We want to construct the \( \phi_j \) so that

\[ (5.6) \quad \begin{cases}
  \phi_j \equiv 0 \pmod{x^2^{k+j}} \\
  |\phi_j|^2(r) < 1 \quad \text{for} \quad 0 \leq r < r_0 \sigma_1 \sigma_n 2^{-(k+\lambda)\delta} \prod_{\lambda=0}^{j-1} (\sigma_1 \sigma_n 2^{-(k+\lambda)\delta}).
\end{cases} \]

To do this, set

\[ (5.7) \quad L_j = L_{j-1} - H_j \quad (j \geq 1), \]

where by (5.4)

\[ (5.3) \quad H_j = \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) \\
- G_{v_i}(x, \psi_{j-2}, \ldots, \partial^{n-1} \psi_{j-2}) \partial^i. \]

Using an induction on \( j \), we can achieve a factorization of \( L_j \) into linear factors, by virtue of theorem 1 for \( n \), for

\[ |x| < r_0 \prod_{\lambda=0}^{j-1} (\sigma_1 \sigma_n 2^{-(k+\lambda)\delta}). \]

Then, by using theorem 2 for \( n = 1 \) (\( n \)-times), we can achieve (5.6).
Thus, we get

\[ |\psi_j|_0(r) < 1 \text{ for } 0 < r < r_0 \tau_n^{-k^5}, \quad j = 0, 1, \ldots, \]

and \( \psi_j \) converges \( x \)-adically to \( \phi = \sum_{z=0}^{\infty} \alpha_z. \) Hence, by lemma 1 of section 1,

\[ |\phi|_0(r) < 1 \text{ for } 0 < r < r_0 \tau_n^{-k^5}. \]

Finally, letting \( j \) tend to infinity on the both sides of (5.2), we complete the proof.

Results for more general cases, applications, and treatments of systems of differential equations were given in SIBUYA–SPERBER ([3],[4]).

REFERENCES


