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ON APERY'S DIFFERENTIAL OPERATOR

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Let L be an ordinary linear differential operator of order n with coefficients in $\underset{\sim}{Q}(x)$,

(1) $$L = D^n + a_1 D^{n-1} + \dots + a_n$$

where $D = d/dx$, $a_j \in \underset{\sim}{Q}(x)$. For each prime p , we may consider the generic point t , and ask for the maximal common radius of convergence , r_p , of all the solutions of L at t . We restrict our attention to those primes for which the Gauss norm of a_j is bounded by unity for $1 \leq j \leq n$. With this qualification, the solutions at t certainly converge in the disk

(2) $$\operatorname{ord}(x - t) > \frac{1}{p - 1} .$$

The mod p reduction of L is said to be of nilpotent p-curvature if there exists $\epsilon > 0$ such that all the solutions of L at t converge in the disk,

(3) $$\operatorname{ord}(x - t) > \frac{1}{p - 1} - \epsilon .$$

It is known in this case that, in fact, the solutions converge in the disk

(4) $$\operatorname{ord}(x - t) > \frac{1}{p - 1} - \frac{1}{pn} .$$

This shows that the maximal radius of convergence cannot assume arbitrary values.

It is known (KATZ) that nilpotence for an infinite set of primes implies that the singularities of L are all regular. Furthermore nilpotence for almost all primes implies the exponents of L are rational.

The converse is false. Let $f(x) = 4(x - e_1)(x - e_2)(x - e_3)$

(5) $$L = f(x) D^2 + \frac{1}{2} f'(x) D - B .$$

For suitable constant B , the estimate (4) fails (for an infinite set of primer) to be valid. The equation clearly has regular singular points and rational exponents.

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At this point, we mention two conjectures.

I (BOMBIERI) : Nilpotence for almost all p implies $r_p = 1$ for almost all p.

II : If $r_p = 1$ for almost all p, then L "comes from geometry".

We say that L comes from geometry if at one regular point it has n independent analytic solutions which lie in the class of functions at that point which come from geometry.

The class of functions at zero which come from geometry is defined to the minimal set satisfying the following conditions :

(a) It is a vector space over $\underset{\sim}{Q}^{alg}$, the algebraic closure of $\underset{\sim}{Q}$.

(b) It is a ring.

(c) It contains all functions algebraic over $\underset{\sim}{Q}(x)$.

(d) It is closed under formal integration.

(e) It is closed under conposition with elements of $\underset{\sim}{Q}(x)$.

(f) For each $F \in Q[\lambda , v_1 , \ldots , v_m]$ which is a form in the variables $v_1 , \ldots , v_m$, let

$$D_i = v_i \frac{\partial}{\partial v_i} + v_i \frac{\partial}{\partial v_i} F \quad i = 1 , \ldots , m$$

$$\mathfrak{W}_F = \underset{\sim}{Q}(\lambda)[v]/\Sigma_{i=1}^{m} D_i \underset{\sim}{Q}(\lambda)[v] ,$$

Then $\mathfrak{W}_F$ is a finite dimensional $\underset{\sim}{Q}(\lambda)$ space and the derivation $\frac{d}{d\lambda}$ of $\underset{\sim}{Q}(\lambda)$ is extended to $\mathfrak{W}_F$ by means of

$$\sigma_F = \frac{\partial}{\partial \lambda} + \frac{\partial F}{\partial \lambda} .$$

Under σ_F, $\mathfrak{W}_F$ becomes a differential module.

We insist that the solutions at the origin of the associated differential equation lie in the class of functions coming from geometry.

The object of this talk is to discuss an example in which this conjecture is correct.

APERY has discussed the differential operator

(6) $$L = (x - 11x^2 - x^3) D^2 + (1 - 22x - 3x^2) D - (3 + x) .$$

The unique solution regular at the origin is given by $y = \Sigma b_n x^n$, where

$$b_n = \Sigma_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j} \in \underset{\sim}{Z} .$$

It is not at all obvious that this series gives a solution of (6). I am indebted to R. ASKEY for the explanation that

$$b_n = {}_3F_2\left(\begin{matrix} -n , -n , n+1 ; \\ 1 , 1 \end{matrix} \; x = 1\right)$$

and that the contiguity relations for such a generalized hypergeometric function ([1]) gives a linear recursion relation for the sequence $\{b_n\}$ which implies that the series is indeed a solution of (6).

Since (6) has a solution in $\mathbb{Z}[[x]]$, we may conclude that for each prime p there is at least one bounded solution, u , converging on the generic disk $D(t , 1^-)$. The wronskian, w , of (6) is obviously given by $1/(x - 11x^2 - x^3)$ and so a second solution at t is given by the formal integral

$$v = u \int_t^x \frac{w}{u^2}\, ds$$

which again represents a function analytic (but perhaps unbounded) on $D(t , 1^-)$. The point here is that u is bounded and hence has at most a finite set of zeros. Thus w/u^2 may be represented as the sum of a function analytic on $D(t , 1^-)$ together with a rational function having poles of order two on $D(t , 1^-)$. There can be no residues at there poles as otherwise v would have a singularity in $D(t , 1^-)$ which contradicts the fact that the singularities of L lie among the zeros of $x - 11x^2 - x^3$.

This then shows that for (6), $r_p = 1$ for all p . This is troubling since (6) is an example of a Halphen transform of the Lamé equation. With $f(x)$ as in (5), the Lamé equation is

(7) $$f(x)\, D^2 + \frac{1}{2} f'(x)\, D - [m(m+1)\, x + B]$$

with Riemann data

(8) $$\begin{pmatrix} e_1 & e_2 & e_3 & \infty \\ 0 & 0 & 0 & \frac{m+1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -m/2 \end{pmatrix}$$

The halphen transformation is abtained by putting

(9) $$x = \wp(u) \quad \left(u = \int_x^\infty \frac{dt}{f(t)}\right)$$

letting $v = u/2$ and changing the dependent variable y by setting

([1]) Cf. RAINVILLE (E.). - Special functions. - New York, Macmillan Company, 1960.

(10) $$Y = y\cdot\wp(u)^{m}$$

and the independent variable by setting

(11) $$X = \wp(v)$$

so that x is a rational function of X.

The transformed equation is

(12) $$0 = \left[f(X)\frac{d^2}{dX^2} + \left(\frac{1}{2} - m\right) f'(X)\frac{d}{dX} + 4(m(2m-1)X - B)\right] Y .$$

Equation (6) is a specialization of (12) with $m = -\frac{1}{2}$. We have remarked that there are examples of type (5) which do not come from geometry, and in general there are no integral formulae representing the solutions of the Lamé equation. Thus there is no general way by which we may show that (6) commes from geometry.

We write $\binom{n+j}{j} = \binom{-n-1}{j}(-1)^j$ and so

(13) $$b_n = \sum_j \binom{n}{j}^2 \binom{-n-1}{j}(-1)^j$$

$$= \mathrm{Res}_\infty \frac{du}{u}\cdot\frac{dv}{v}\left(1 - \frac{1}{uv}\right)^{-(n+1)} (1+u)^n (1+v)^n .$$

By this we mean if we let γ_1, γ_2 be circular paths in the u and v planes respectively with centers at infinity such that $|uv| > 1$ for all $(u, v) \in \gamma_1 \times \gamma_2$, then

(14) $$b_n = \frac{1}{2\pi i}\int_{\gamma_1}\int_{\gamma_2} \frac{du}{u}\cdot\frac{dv}{v}\left(1 - \frac{1}{uv}\right)^{-1} H^n$$

where

$$H = (1+u)(1+v)/\left(1 - \frac{1}{uv}\right) .$$

Thus for x small, i. e. for $|xH| < 1$ for all $(u, v) \in \gamma_1 \times \gamma_2$, we have

(15) $$y = \sum x^n b_n = \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2} \frac{du}{u}\cdot\frac{dv}{v}\left(1 - \frac{1}{uv}\right)^{-1} \frac{1}{1 - xH}$$

$$= \frac{1}{(2\pi i)^2}\int_{\gamma_1} du \int_{\gamma_2} \frac{dv}{uv - 1 - x(1+u)(1+v)\,uv}$$

For fixed u, we write the denominator as

$$-xu(1+u)(v - \theta_1)(v - \theta_2)$$

where θ_1, θ_2 are algebraic functions of u and x.

When $x = 0$, one solution is $v = 1/u$, and the other is infinite. From this, we may convince ourselves that for x small, γ_1 and γ_2 being of radii strictly greater than one, then for each $u \in \gamma_1$ θ_2 lies outside γ_2 (i. e. closer to ∞) and θ_1 lies inside γ_2. This then shows that

(16) $$y = \frac{1}{2\pi i}\int_{\gamma_1} \frac{du}{-xu(1+u)(\theta_1 - \theta_2)} .$$

Hence by a calculation of the discriminant of the denominator in (15), we obtain

(17) $$y = \frac{1}{2\pi i}\int_{\gamma_1} \frac{du}{z} ,$$

where

(18) $$z^2 = u[u^3 x^2 + 2u^2(x^2 - x) + u(x^2 - 6x + 1) - 4x]$$

which identifies (6) with the equation satisfied by periods of the differential of the first kind of the elliptic curve (18). This implies that (6) is the pull back of a hypergeometric differential equation by means of a rational map.

This calculation supports our conjecture II, but of course does not confirm it.

We have not carried out a similar calculation for

$$L_2 = x(8x - 1)(x + 1)\, D^2 + (24x^2 + 14x - 1)\, D + (8x + 2)$$

with unique solution at the origin given by $\sum a_n x^n$ with

$$a_n = \sum_{j=0}^{n} \binom{n}{j}^3 .$$

We have carried out a similar calculation for

$$L_3 = (1 - 34x + x^2)\, x^2 D^3 + (3 - 153x + 6x^2)\, xD^2 + (1 - 112x + 7x^2)\, D + (x - 5)$$

with unique solution analytic at the origin, $\sum c_n x^n$,

$$c_n = \sum \binom{n}{j}^2 \binom{n+j}{j}^2 .$$

(All of these examples are due to APERY). We are indebted to APERY for the observation that L_3 is the symmetric square of

$$L_4 = x((1 - 34)\ x + x^2)\ D^2 + ((1 - 5)\ x + 2x^2)\ D + \frac{x - 10}{4}$$

while we have obtain an integral formula for the solution of L_3, we do not have an integral formula for the solution of L_4.