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NILPOTENT SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS
WITH FUCHSIAN SINGULARITIES

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Let K be a field of characteristic $p \neq 2$ say algebraically closed. Let L be a linear differential operator

$$(0.0) \quad L = D^2 - aD - b \in K(x)[D]$$

with $D = d/dx$. Let $\{\gamma_1, \dots, \gamma_m, \gamma_\infty = \infty\} = T$ be the set of singularities of L and let

$$(0.1) \quad f(x) = \prod_{i=1}^m (x - \gamma_i) .$$

We assume

(0.2) All the singularities of L are fuchsian.

(0.3) The exponents of L at each singularity lie in $\frac{\mathbb{F}}{\sim p}$.

(0.4) L is nilpotent but does not have two solutions in $K(x)$ linearly independent over $K(x^p)$.

By "nilpotent", we mean that L has a non-trivial solution in $K(x)$, and that the equation for the wronskian,

$$(0.4.1) \quad Dw = wa ,$$

has a non-trivial solution in $K(x)$. We may assume that the zeros and poles of w lie in T .

We use the word "exponent" to refer to a root of the indicial polynomial.

For $i = 1, \dots, m, \infty$, let e_i, e'_i be the exponents at γ_i .

We choose a solution u of L in $K[x]$, unique up to factor in K , by the condition that no zero of u is of order greater than $p - 1$.

We write

$$(0.5) \quad u = g(x) \prod_{i=1}^m (x - \gamma_i)^{\tilde{e}_i} ,$$

(*) Texte reçu le 2 juillet 1981.

where

$$g \in K[x], \quad (g, f) = 1, \quad \tilde{e}_i \in [0, p-1].$$

We define \tilde{e}_∞ by the condition that $\tilde{e}_\infty \in [0, p-1]$,

$$(0.6) \quad \tilde{e}_\infty \equiv -\deg u \pmod{p}.$$

Clearly the \tilde{e}_i represent exponents of L . For all $s \in \mathbb{N}$, we write

$$(0.7) \quad D^s = a_s D + b_s \pmod{K(x)[D] L},$$

with $a_s, b_s \in K(x)$. It is known that

$$a_s = 0 = b_s, \quad \forall s \geq 2p.$$

An ad hoc proof is given in § 4.5 below :

Having defined $\tilde{e}_i (i = 1, \dots, m, \infty)$, we define $e_i (\in \mathbb{F}_p)$ to be the class of \tilde{e}_i , and we define e'_i to be the other exponent at γ_i (of course we may have $e_i = e'_i$). Thus we have uniquely defined the difference, $e_i - e'_i$, of exponents at γ_i . We define $t_i \in [0, p-1]$

$$(0.8) \quad t_i \pmod{p} = e_i - e'_i \quad (i = 1, \dots, m, \infty).$$

The object of this section is to prove the following lemma.

1. LEMMA.

$$(1.1) \quad (p-1)(m-1) = 2 \deg g + (t_1 + \dots + t_m + t_\infty) + pt$$

where $t \in \mathbb{N}$, $t \geq 0$.

2. LEMMA.

$$(2.1) \quad f(x)^{p-1} a_p = g(x)^2 \prod_{i=1}^m (x - \gamma_i)^{t_i} \theta(x^p),$$

where

$$\theta \in K[x]$$

$$g \in K[x]$$

g is prime to f

g has only simple zeros.

We commence our treatment with an elementary proposition.

3. PROPOSITION. - For each $s \in \underline{N}$, $a_s f(x)^{s-1} \in K[x]$,

$$(3.1) \quad \deg a_s f(x)^{s-1} \leq (s-1)(m-1).$$

Proof. - By differentiating (0.7) and using L to reduce the D^2 on the right hand side, we obtain the recursion formula

$$(3.2) \quad \begin{pmatrix} a_{s+1} \\ b_{s+1} \end{pmatrix} = \begin{pmatrix} a'_s \\ b'_s \end{pmatrix} + \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} a_s \\ b_s \end{pmatrix}$$

On the other hand,

$$(3.3) \quad \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

By hypothesis for $1 \leq i \leq m$, $a_2 = a$ (resp. $b_2 = b$) has a pole at γ_i of order not greater than one (resp. two). By induction on s and the recursion formula, we show that, for $s \geq 1$,

$$(3.4) \quad a_s \text{ (resp. } b_s) \text{ has a pole at } \gamma_i \text{ of order not greater than } s-1 \text{ (resp. } s).$$

This shows that $a_s f(x)^{s-1}$ is a polynomial.

The condition that L is fuchsian everywhere implies that we may write L in the form

$$(3.5) \quad L = D^2 + \sum_{i=1}^m \frac{A_i}{x - \gamma_i} D + \sum_{i=1}^m \left(\frac{B_i}{x - \gamma_i} + \frac{C_i}{(x - \gamma_i)^2} \right),$$

where $A_i, B_i, C_i \in K$ for $i = 1, 2, \dots, m$.

The condition at infinity implies that

$$(3.6) \quad \sum_{i=1}^m B_i = 0.$$

Thus $\deg a_2$ (resp. $\deg b_2$) ≤ -1 (resp. -2) and by (3.2) and induction we show that

$$(3.7) \quad \deg a_s \text{ (resp. } \deg b_s) \leq -(s-1) \text{ (resp. } -s).$$

This completes the proof of the proposition.

4. - We now commence the proofs of the lemmas. By hypothesis, the $K(x^p)$ -space of solutions of L in $K(x)$ has dimension one but in a suitable differential extension field F , the F^p space of solutions of L has dimension two.

More explicitly, we choose F so as to contain, τ , a solution of

$$(4.1) \quad \tau' = w/u^2$$

and then, by a well known calculation using (0.4.1),

$$L(u\tau) = \tau L(u) + \frac{w}{u} \left(\tau' \frac{u^2}{w} \right) = 0,$$

while the wronskian

$$(4.2) \quad \begin{vmatrix} u\tau & (u\tau)' \\ u & u' \end{vmatrix} = -w$$

which shows that u , $u\tau$ are linearly independent over the kernel of D in F .

We now apply (0.7) and conclude that

$$(4.3) \quad \begin{aligned} (\tau u)^{(s)} &= a_s (\tau u)' + b_s (\tau u) \\ u^{(s)} &= a_s u' + b_s u. \end{aligned}$$

Eliminating b_s , we obtain

$$(4.4) \quad a_s = \sum_{\substack{i+j=s \\ i \geq 1}} \frac{\tau^{(i-1)}}{\tau'} \frac{u^{(j)}}{u} \binom{s}{i},$$

a formula involving u and τ' but not τ . We observe that this formula is independent of the characteristic.

(4.5) Remark. - Since u and τ' lie in $K(x)$ they are annihilated by D^p . For $s \geq 2p$ either $i - 1$ or j on the right side of (4.4) exceeds $p - 1$ which shows again that $a_s = 0$ for $s \geq 2p$.

In particular, for $s = p$, the above formula gives

$$(4.6) \quad a_p = (u^2/w) D^{p-1} (w/u^2).$$

Now $a_p \neq 0$ as otherwise $\frac{w}{u^2}$ would lie in the kernel of D^{p-1} in $K(x)$, i. e.,

$$\frac{w}{u^2} \in K(x^p) + K(x^p)x + \dots + K(x^p)x^{p-2}$$

which would show that (4.1) has a solution τ in $K(x)$ contrary to hypothesis concerning the dimensionality of the kernel of L in $K(x)$ (as $K(x^p)$ space).

By the same argument since $1, x, \dots, x^{p-1}$ is basis of $K(x)$ as $K(x^p)$ space, we conclude that D^{p-1} maps $K(x)$ into $K(x^p)$. Hence

$$(4.7) \quad a_p \in \frac{u^2}{w} K(x^p).$$

Putting

$$Q_p = a_p f(x)^{p-1},$$

we have

$$(4.8) \quad Q_p \in \frac{u^2}{w} \frac{1}{f(x)} K(x^p).$$

We have defined g as the factor of u prime to $f(x)$. If x_0 is a zero of g then the indicial polynomial of L at x_0 has $0, 1 \pmod{p}$ as zeros and by definition u has no zero of order greater than $p-1$. This shows that the zeros of g are simple.

5. - We continue our proof of the lemmas. We will show

$$(5.1) \quad \frac{u^2}{w} \frac{1}{f(x)} \in g(x)^2 \prod_{i=1}^m (x - \gamma_i)^{t_i} K(x^p).$$

With this in mind, we use (3.5) to deduce

$$(5.2) \quad e_i + e'_i = 1 - A_i, \quad i = 1, \dots, m$$

$$e_\infty + e'_\infty = \sum_{i=1}^m A_i - 1$$

while

$$(5.3) \quad w \in \prod_{i=1}^m (x - \gamma_i)^{-\bar{A}_i} K(x^p),$$

where \bar{A}_i is a representative in \mathbb{N} of A_i ($1 \leq i \leq m$). Thus the order of γ_i as zero of the left side of (5.1) is congruent mod p to $2e_i + A_i - 1 = e_i - e'_i \equiv t_i$.

This together with our discussion of g , the factor of u prime to f , concludes the demonstration of (5.1).

We now estimate the degree of the left side of (5.1). By hypothesis $\deg u = -\tilde{e}_\infty \equiv -e_\infty$ and so

$$\deg \frac{u^2}{wf(x)} \equiv -2e_\infty - m + \sum_{i=1}^m A_i.$$

By (5.2) this is the same as $-m + 1 - t_\infty$. Thus from (5.1),

$$(5.4) \quad m + 2 \deg g + t_\infty + \sum_{i=1}^m t_i \equiv 1 \pmod{p}.$$

By (4.8), (5.1), we obtain (2.1) with $\theta \in K(x)$. We assert that θ is a polynomial. Indeed Q_p is a polynomial and $Q_p/\theta(x^p)$ is, by (2.1), a polynomial with zeros of order bounded by $p-1$. Thus θ must be a polynomial. This completes the proof of Lemma 2. We continue with the proof of Lemma 1. By proposition 3,

$$(5.5) \quad (p-1)(m-1) \geq \text{degree } Q_p = 2 \text{ degree } g + \sum_{i=1}^m t_i + p \text{ degree } \theta.$$

Let

$$(5.6) \quad \rho = (p-1)(m-1) - 2 \text{ degree } g - \sum_{i=1}^m t_i.$$

Then

$$(5.7) \quad \rho \geq p \text{ degree } \theta \geq 0.$$

On the other hand, by (5.4) and (5.6),

$$\rho \equiv t_\infty \pmod{p}.$$

And hence, by (5.7),

$$\rho = t_\infty + pt$$

for some $t \geq 0$. Substitution in (5.6) completes the proof of Lemma 1.

6. Remark. - We view the sum of the t_i as the analogue of the sum of the angles of the image of the upper half plane under a ratio of solutions of L if K were say the reals and the γ_i were all real.

7. - In general we are given L but not u and so there are two choices of t_i for each i . Thus in applying Lemma 1 there are 2^{m+1} choices for $(t_1, \dots, t_m, t_\infty)$ and t is not known.

COROLLARY. - If $m = 2$ then under hypotheses (0.2)-(0.4), we have $t = 0$, and

there is just one possible choice for t_0, t_1, t_∞ . Equation (1.1) takes the form

$$p - 1 = 2 \deg g + t_0 + t_1 + t_\infty .$$

Proof. - It is clear from (1.1) that $t = 0$. Since $p \neq 2$, it follows that

$$(7.1) \quad t_0 + t_1 + t_\infty \equiv 0 \pmod{2}$$

$$p - 1 \geq t_0 + t_1 + t_\infty .$$

Now each t_i is fixed by L up to the transformation

$$t_i \longrightarrow p - t_i .$$

The condition of parity shows that such a transformation, if applied at all, must be applied to two of the t_i , say to t_0, t_1 and we would then have

$$(7.2) \quad p - 1 \geq p - t_0 + p - t_1 + t_\infty .$$

This is inconsistent with (7.1) as the sum would give

$$p - 1 \geq 2p + 2t_\infty \geq 2p .$$

Remark. - The degree of g is at most $\frac{p-1}{2}$ and this occurs precisely, when $t_0 = t_1 = t_\infty = 0$, for example in the case of the differential operator associated to the hypergeometric function $F(1/2, 1/2, 1; x)$.
