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On Morita’s $p$-adic $\Gamma$-function


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Y. MORITA proved that, for each prime number $p$, one can define a $p$-adic continuous function $\Gamma_p(x)$ from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, interpolating the sequence

$$n \rightarrow (-1)^n \prod_{m=1}^{n} m,$$

where $m$ runs through the integers $m$ prime to $p$ with $1 \leq m < n$. Our aim is to show how this result is related to DWORK's result on the radius of convergence of $\exp(X + (X^p/p))$.

1. Introduction and notations.

Let $p$ be a prime, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ as usual [2]. The absolute value (resp. the valuation) on $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ is denoted by $| \cdot |$ (resp. $v(\cdot)$), and normalized by $|p| = p^{-1}$ (resp. $v(p) = 1$). The sequence $n \rightarrow n!$ cannot be the restriction to $\mathbb{N}$ of a $p$-adic continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_p$. However, MORITA [10] proved that the sequence $n \rightarrow (-1)^n \prod_{m=1}^{n} m = \Gamma_p(m)$, $m$ prime to $p$, $1 \leq m < n$, can be interpolated by a continuous $p$-adic function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ denoted by $\Gamma_p(x)$. We shall prove, by means of the formal Laplace transform ([5] or [3]), that this result can be deduced from DWORK's estimate of the $p$-adic absolute value of the coefficients $b_{n}/n!$ of the series

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = \exp(x + \frac{x^p}{p}).$$

We actually prove that the result of DWORK is equivalent to the existence of a locally analytic function $x \rightarrow \Gamma_p(px)$ on $\mathbb{Z}_p$, with local analyticity radius $\rho_p = p^{-(1/p)-(1/(p-1))}$, bounded by $1$ on

$$\mathbb{L}_p(\mathbb{Z}_p) = \{x \in \mathbb{C}_p ; |x - z| < \rho_p \text{ for all } z \in \mathbb{Z}_p\},$$

interpolating the sequence $n \rightarrow (-1)^n \prod_{m=1}^{n} (m/p)(n!)$ = $\Gamma_p(pn)$. Let us denote

$$B(a, \rho^+) = \{x \in \mathbb{C}_p ; |x - a| < \rho \}, \quad a \in \mathbb{C}_p, \quad \rho \in \mathbb{R}_+ = \{x \in \mathbb{R} ; x > 0\}.$$

Recall [3] that the formal Laplace transform $\mathcal{L}(f(x))$ of $\bar{f}(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n \in \mathbb{C}_p[[x]]$ is

$$\mathcal{L}(f(x)) = \bar{f}(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}_p[[x]].$$

(*) Visiting Research Associate of Princeton University, 1977/78. L'exposé, au groupe d'étude, a été prononcé par Philippe ROBBA.
The following properties of \( \mathcal{E} \) are obvious from the definition:

- \( \mathcal{E} \) is a \( \mathbb{C}_p \)-linear map from \( \mathbb{C}_p[[x]] \) onto \( \mathbb{C}_p[[x]] \).

- \( \mathcal{E} \) is continuous for the \((p, x)\)-adic topology on \( \mathbb{C}_p[[x]] \).

Let

\[
\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}, \quad \binom{x}{0} = 1.
\]

2. Morita's \( p \)-adic \( \Gamma \)-function.

**Definition 1** (Morita [10]). - Let \( p(n) \) be defined, for \( n \in \mathbb{N} \), by

\[
\Gamma_p(n) = \prod_{\substack{0 \leq m < n \leq m, \ m \ prime \ to \ p}} (-1)^n \prod_{m < n} m.
\]

**Lemma 1.** - For \( 1 \leq i \leq p \), we have

\[
\Gamma_p(pn + i) = (-1)^{pn+i} \frac{(pn + i - 1)!}{p^n (ni)!}.
\]

**Proof.** - Obvious from definition 1. Q. E. D.

**Lemma 2.** - The generating function of the sequence \( n \to (-1)^n \Gamma_p(n) \),

\[
F(x) = \sum_{n \geq 0} (-1)^n \Gamma_p(n+1) x^n,
\]

is the formal Laplace transform of

\[
F(x) = \sum_{i=1}^{p} x^{i-1} \exp(x/p).
\]

**Proof.**

\[
\sum_{i=1}^{p} x^{i-1} \exp(x/p) = \sum_{n \geq 0} \sum_{i=1}^{p} \frac{x^{pni-1}}{n!}.
\]

So

\[
F(\sum_{i=1}^{p} x^{i-1} \exp(x/p)) = \sum_{i=1}^{p} \sum_{n \geq 0} \frac{(pn + i - 1)!}{p^n n!} x^{pni-1},
\]

the conclusion follows from lemma 1. Q. E. D.

Let

\[
\sum_{n \geq 0} \frac{b_n}{n!} x^n
\]

be the formal Taylor series at the origin of \( \exp(x + \frac{x^p}{p}) \).

It is clear the \( \sum_{n \geq 0} \frac{b_n}{n!} x^n \in \mathbb{C}_p[[x]] \).

**Lemma 3** (Dwork [6]). - The coefficients \( b_n \) of the Taylor series of \( \exp(x + (x^p/p)) \) satisfy the following \( p \)-adic estimate

\[
\nu\left(\frac{b_n}{n!}\right) \geq -n \frac{2p - 1}{p^2(p - 1)}.
\]

For the proof see [9] or [6].
LEMMA 4. - If $E(f(x)) = f(x)$, then

\begin{equation}
E((\exp \alpha x) \tilde{f}(x)) = \frac{1}{1 - \alpha} f\left(\frac{x}{1 - \alpha}\right), \quad \alpha \in \mathbb{C}_p.
\end{equation}

Proof. - Let $\tilde{f}(x) = \sum_{n \geq 0} a_n (x^n/n!)$, then

\begin{equation}
(\exp \alpha x) \tilde{f}(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \alpha^k a_{n-k} \frac{n!}{k!}\right) \frac{x^n}{n!}.
\end{equation}

This implies

\begin{equation}
E((\exp \alpha x) \tilde{f}(x)) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \alpha^k a_{n-k} \frac{n!}{k!}\right) \frac{x^n}{n!} = \sum_{n \geq 0} a_n \frac{x^n}{(1 - \alpha)^{n+1}}.
\end{equation}

THEOREM 1. - The following identity holds

\begin{equation}
F(x) = \sum_{n \geq 0} (-1)^{n+1} \left(\sum_{i=1}^{p} b_n \frac{x^n}{n!}ight) + \sum_{n \geq 0} b_n = b_{n-i+1} \frac{n!}{(n+i-1)!}.
\end{equation}

Proof. - $F(x) = E(\exp(-x) \left(\sum_{i=1}^{p} x^{i-1} \exp(x + (x^p/p))\right))$. Applying lemma 4 and 2, we obtain

\begin{equation}
F(x) = \sum_{i=1}^{p} \sum_{n \geq 0} b_n \frac{x^{n+i-1}}{n! + x^{n+i-1}}
\end{equation}

COROLLARY 1. - For $n \in \mathbb{N}$, we have

\begin{equation}
\Gamma_p(n+1) = - \sum_{k=1}^{P} (-1)^k b_k \frac{n!}{k!}.
\end{equation}

Proof. - $F(x) = \sum_{i=1}^{p} \sum_{n \geq 0} b_n \frac{x^n}{(1 + x)^{n+i+1}} = \sum_{i=1}^{p} \sum_{n \geq 0} b_n \frac{x^n}{(1 + x)^{n+i+1}} = \sum_{i=1}^{p} \sum_{n \geq 0} b_n \frac{x^n}{(1 + x)^{n+i+1}}$

THEOREM 2. - The sequence $n \rightarrow \Gamma_p(n)$ is the restriction to $\mathbb{N}$ of a unique locally analytic function on $Z_p$, $\Gamma_p(x)$, with local analyticity radius

\begin{equation}
\rho_p = p^{-(1/p) - (1/p-1)}.
\end{equation}

Proof. - Recall [1], that a function from $Z_p$ to $\mathbb{C}_p$ is locally analytic on $Z_p$, if, for each point $\alpha \in Z_p$, there exist $\rho_\alpha \in R$ such that, on $B(\alpha, \rho_\alpha) \cap Z_p$, $f$ is the restriction of a function $f_\alpha(x) = \sum_{n \geq 0} a_n (x - \alpha)^n$, analytic on $B(\alpha, \rho_\alpha)$. The local analyticity radius of $f$ is

\begin{equation}
\rho = \inf_{\alpha \in Z_p} \rho_\alpha > 0 \quad (\text{because } Z_p \text{ is compact}).
\end{equation}

Take the series

\begin{equation}
\Gamma_p(x + 1) = - \sum_{i=1}^{p} \sum_{n \geq 0} (-1)^n b_n \frac{x^n}{n!}.
\end{equation}

By lemma 3, we get
where \([a]\) is the integral part of \(A \in \mathbb{R}_+\), that is \([a] \in \mathbb{N}\) and \(a - 1 < [a] \leq a\). This (17) implies

\[
\lim_{n \to \infty} |\sum_{i=1}^{N} b_n(i)| = 0.
\]

By Kähler's theorem [1], this implies that (16) defines the unique continuous functions from \(\mathbb{Z}_p\) to \(\mathbb{C}_p\) (actually to \(\mathbb{Q}_p\)) interpolating the sequence \(n \to \Gamma_p(n)\).

Actually (17) gives us a stronger result. Let \(1/p < \rho = p^{-\alpha} \leq 1\), \(\alpha \in \mathbb{R}_+\), \(1 > \alpha > 0\). Let \(W_p(Z_p) = \{x \in \mathbb{C}_p; |x - z| < \rho\} \) for all \(z \in \mathbb{Z}_p\).

\[
\inf_{x \in W_p(Z_p)} v((x)_n) > \left[\frac{n}{\rho}(1 - \alpha) + \sum_{i \geq n} \frac{n}{i^\alpha}\right].
\]

\[
\inf_{x \in W_p(Z_p)} v((x)_n) > -\frac{n}{\rho}(1 - \alpha + \frac{1}{p^{-\alpha} - 1}).
\]

From (20) and (17), we obtain
\[
\lim_{n \to \infty} \sup_{x \in W_p(Z_p)} \left|\sum_{i=1}^{N} b_n(i)(x)_n\right| = 0
\]
if, and only if,

\[
\alpha > \frac{1}{p^{-\alpha} - 1} + \frac{1}{p}.
\]

Q. E. D.

But, we can get more the following theorem.

**THEOREM 3.** - The following two propositions are equivalent.

(i) The sequence \(n \to \Gamma_p(pn)\) is the restriction of a unique locally analytic function on \(\mathbb{Z}_p\), \(\Gamma_p(px)\), with local analyticity radius \(r_p = p^{1/(1/p)-(1/p-1)}\), and bounded by one on \(\mathbb{N}_r(Z_p)\).

(ii) \(\exp(x + (x^p/p)) = \sum_{n \geq 0} \frac{b_n}{n!} x^n\), with \(v((n)_n) > -n \frac{2p-1}{p^2(p-1)}\).

**Proof.**

\[
\Gamma_p(pn) = (-1)^{pn} \frac{(p(n-1) + p - 1)!}{p^{n-1}(n-1)!} = (-1)^{pn} \frac{(pn)!}{p^n n!}.
\]

From (22), it is clear that

\[
\sum_{n \geq 0} \frac{(pn)!}{p^n n!} x^n = \mathcal{F}(\exp(x^p/p)) = \mathcal{F}(\exp(-x) \exp(x + x^p/p)).
\]

We obtain, as in corollary 1,

\[
\Gamma_p(pn) = \sum_{k=0}^{n} (-1)^k \frac{(pn)!}{p^k k!} b_k(pn).
\]

Consider the series
We obtain, as in (19),

\begin{equation}
(25) \quad \Gamma_p(px) = \sum_{n=0}^{\infty} \frac{(-1)^n b_n(px)}{n!}.
\end{equation}

We obtain, as in (19),

\begin{equation}
(26) \quad \sup_{x \in \mathbb{Z}_p} \left| \frac{b_n(px)}{n!} \right| = \frac{b_n}{n!} \left| \frac{[n]}{p} \left( 1 + \frac{1}{p-1} \right) \right|.
\end{equation}

Assuming (ii), we obtain

\begin{equation}
(27) \quad \sup_{x \in \mathbb{Z}_p} \left| \frac{b_n(x)}{n!} \right| \leq \frac{1}{p^{n-1}} \left( 1 + \frac{1}{p-1} \right) \left( 1 - \frac{n}{p} \right) \leq 1,
\end{equation}

and, for all \( 0 < r < r_p \),

\[ \lim_{n \to \infty} \sup_{x \in \mathbb{Z}_p} \left| \frac{b_n(x)}{n!} \right| = 0, \]

so (i) is true.

Assuming (i), we obtain

\[ \left| \frac{b_n}{n!} \right| \leq \frac{[n]}{p} \left( 1 + \frac{1}{p-1} \right) \leq p^{n-2} \left( 1 - \frac{1}{p-1} \right), \]

so (ii) is true.

This theorem shows the close relations that exist between \( \exp(x + \frac{x^p}{p}) \) and \( \Gamma_p(x) \). Actually, what formulas (24) and (25) say, is that the coefficients of the Mahler expansion [1] of the function \( x \to \Gamma_p(px) \) are given by the coefficients \( b_n \) of the Taylor expansion of \( \zeta(\exp(x + \frac{x^p}{p})) \). This suggests the possibility of a direct proof of Koblitz's formula ([4], [7], [8]) between Gauss sums and the p-adic \( \Gamma \)-function of Norita.

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