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CONSTRUCTION OF AN HOMOLOGY AND A COHOMOLOGY THEORY

ASSOCIATED TO A FIRST ORDER FORMULA

by René GUITART

RESUME - On montre comment chaque formule ϕ d'un langage $\mathcal{L}$ détermine une théorie d'homologie (et une théorie de cohomologie) sur la catégorie des interprétations de $\mathcal{L}$, dont la valeur sur chaque interprétation I de $\mathcal{L}$ est une obstruction à $I \models \phi$ "à des co-équations près" (et "à des équations près").

This paper is a sequel of [7].

O. Let $\mathcal{L}$ be a first order language, let I be an interpretation of $\mathcal{L}$, and let ϕ be a formula of $\mathcal{L}$. The aim of this note is to indicate a way in which it is possible to measure partially and to compute how I is far from the models of ϕ .

In the papers [7] and [8] it is shown how this question is connected with the possibility of a geometrical study of algorithms and ambiguities.

1. PROPOSITION. Let $\mu(x_1, \dots, x_n)$ be a first order formula of $\mathcal{L}$, and let $\text{Mod}_\mu \phi$ be the category with objects the models of ϕ , and with morphisms from M to M' the morphisms (of models of ϕ) $m : M \longrightarrow M'$ such that

$$\forall x_1, \dots, x_n [\mu(m(x_1), \dots, m(x_n)) \longrightarrow \mu(x_1, \dots, x_n)]$$

Then there is a small mixed sketch σ such that $\text{Mod}_\mu \phi \cong \text{Mod} \sigma$.

The existence of σ is proved by the juxtaposition of proposition 3 p.8 of [6], théorème 2.1 p.26 of [5], and proposition 3 p.301 of [7],II. In fact, this juxtaposition shows more than our proposition here.

2. For C a category, let $BC = |NC|$ be the geometric realization of the nerve of C . BC is a cw-complexe, and $\pi_1 BC \cong C[C^{-1}]$ (the category of fractions of C). Of course if C is a class, BC is a class too. But, if $C = \text{Mod}\sigma$ for a small sketch σ , then in BC we can construct a set $g\sigma$ such that the inclusion $g\sigma \longrightarrow B\text{Mod}\sigma$ is an equivalence of homotopy. In particular we get

PROPOSITION. $\text{Mod}\sigma[(\text{Mod}\sigma)^{-1}]$ is a small groupoid, up to equivalence. We call it the fundamental groupoid of σ , and we denote it by $\pi_1 g\sigma$.

The existence of the set $g\sigma$ comes from [7].

3. Let $\text{Mod}_\mu \phi / I$ be the category with objects the morphisms (of interpretations of $\mathcal{L}$) $f : M \longrightarrow I$ where M is a model of ϕ , and with morphisms, from $f : M \longrightarrow I$ to $f' : M' \longrightarrow I$, the morphisms of models (morphisms of $\text{Mod}_\mu \phi$) $g : M \longrightarrow M'$ such that $f'.g = f$.

Then

PROPOSITION. There is a small sketch $\sigma = \sigma(\mathcal{L}, I, \phi, \mu)$ such that

$$\text{Mod}_\mu \phi / I \cong \text{Mod}\sigma.$$

So we get a small cw-complexe $g\sigma(\mathcal{L}, I, \phi, \mu)$, which is a geometric description of the position of I with respect to $\text{Mod}_\mu \phi$.

4. Let $\mathbf{Ab}$ be the category of small abelian groups, and let $F : \text{Mod}_\mu \phi \longrightarrow \mathbf{Ab}$ be a functor. (In particular F could be the constant functor on a fixed abelian group A , or it could be a "canonical" functor if $\mathcal{L}$ is a language over the language of abelian groups, etc).

The André's homology measures "how I is far from $\text{Mod}_\mu \phi$, from the point of view of F ". In order to do that we consider the chain complexe

$$\longrightarrow C_2(I, F) \xrightarrow{d_2} C_1(I, F) \xrightarrow{d_1} C_0(I, F) \xrightarrow{d_0} 0$$

which is

$$\begin{array}{ccccccc} \dots & \longrightarrow & \sum FM_2 & \xrightarrow{d_2} & \sum FM_1 & \xrightarrow{d_1} & \sum FM_0 \xrightarrow{d_0} 0 \\ & & M_2 \rightarrow M_1 \rightarrow M_0 \rightarrow I & & M_1 \rightarrow M_0 \rightarrow I & & M_0 \rightarrow I \end{array}$$

with $d_1 = s_1^0 - s_1^1$, where

$$\begin{array}{l} s_1^0 : (FM_1)_{(M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} I)} \xrightarrow{\text{Id}} (FM_1)_{(M_1 \xrightarrow{\beta \cdot \alpha} I)} \xrightarrow{\text{Inc}} \sum_{M \rightarrow I} FM \\ s_1^1 : (FM_1)_{(M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} I)} \xrightarrow{F(\alpha)} (FM_0)_{(M_0 \xrightarrow{\beta} I)} \xrightarrow{\text{Inc}} \sum_{M \rightarrow I} FM \end{array}$$

and so on, and we define

$H_0(I, F) = \ker d_0 / \text{Im } d_1 = \text{coker } d_1$, $H_1(I, F) = \ker d_1 / \text{Im } d_2$, and, for every $n \geq 0$, $H_n(I, F) = \ker d_n / \text{Im } d_{n+1}$.

PROPOSITION. $H_n(I, F)$ is a function of F , I , μ , ϕ , which in fact depends only of the homotopy type of $\text{Mod } \phi/I$ and of F and could be denoted by $H_n(\text{Mod } \mu \phi/I, F)$.

see [1], [2], [3] and [4].

Let $\text{Int}\mathcal{L}$ be the category of interpretations of $\mathcal{L}$, let $J : \text{Mod}\phi \longrightarrow \text{Int}\mathcal{L}$ be the canonical inclusion. Then the inductive Kan extension of F along J is given by

$$[\underline{\text{Ext}}_J F](I) = \lim_{M_0 \longrightarrow I} F(M_0)$$

and we have

$$H_0(I, F) = [\underline{\text{Ext}}_J F](I).$$

$$\text{If } I \models \phi, \text{ then } H_n(\text{Mod } \mu \phi/I, F) = \begin{cases} F(I) & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

5. Now, the point is that, because of the results hereover (§§ 1 to 4), we get

PROPOSITION. *The tools of [1] and of [3], available in the situation where a full and small category $\mathbf{M}$ (called a category of "models") lives inside a big category of "spaces", are also available in the situation where a (possibly big and not necessarily full) category $\text{Mod}_\mu \phi$ of models of a theory lives inside a big category of interpretations of a language $\mathcal{L}$ (compare with the idea of "paires adéquates" p. 43 of [1]). Precisely here we get the fact that the $H_n(\text{Mod}_\mu \phi/I, F)$ are small.*

6. After the existence of $g\sigma$ proved in [7], the theorem hereunder §9 is just a second stone for a work to be pursued. Theoretically the computation of our H_n is based on the effective construction of a "locally cofree diagram", and more precisely on the construction of a "relatively cofiltered locally cofree diagram" (r.cf.l.cf.d.)(see [5] and [6]) (in the category $\text{Mod}_\mu \phi$) generated by I . This r.cf.l.cf.d. contains all the information we need, and it will be the starting point of an absolute calculus. But for concrete situations we need a relative calculus, by the way of comparaisons between various H_n . For that it will be essential to go toward effective relative calculation of these small H_n , and especially we need a description of the link between these calculations and the theory of demonstrations. For example we need relations among $H_n(\text{Mod}_\mu \phi/I, F)$, $H_n(\text{Mod}_\mu \gamma/I, G)$, $H_n(\text{Mod}_\mu [\phi \wedge \gamma]/I, K)$, $H_n(\text{Mod}_\mu [\phi \Rightarrow \gamma]/I, L)$ (for convenient K and L). For that it will be necessary to describe the category $\text{For}(\mathcal{L})$ of formulas of the language $\mathcal{L}$. At first this will be usefull to precise the functoriality of the $H_n(\text{Mod}_\mu \phi/I, F)$ with respect to ϕ and μ .

7. The first purpose of this paper was to show precisely how each classical first order formula ϕ of a language $\mathcal{L}$ determines a "small" homology theory on the category of interpretations of $\mathcal{L}$.

Now, the continuation of this research pass trough the description of $\text{For}(\mathcal{L})$. With respect to that, I would like to make the following remark : what have to be morphisms between formulas ? it is not so clear a priori ; they have to be "demonstrations" or "proofs", but there is no

canonical idea of what is a demonstration.

But if we decide to stay in (or to come back to) the style of sketches, a first picture is easy to give. In fact $\mathcal{L}$ "is" a sketch σ_0 (i.e. the category of interpretations of $\mathcal{L}$ is isomorphic to $\text{Mod}\sigma_0$), the formula ϕ (or ${}_{\mu}\phi$) is a sketch σ , and the inclusion of the category of models of ϕ (of ${}_{\mu}\phi$) in the category of interpretations of $\mathcal{L}$ is induced by a morphism of sketches $P : \sigma_0 \longrightarrow \sigma$. This P is the "proof" that a model of ϕ (of ${}_{\mu}\phi$) is an interpretation of $\mathcal{L}$. In fact P is not a general morphism of sketches, but determines σ as a σ_0 -sketch (see [6] p.10 for the precise definition). So we choose to say now that a formula for σ_0 (in the place of a $\mathcal{L}$ -formula) is nothing but such a P , a σ_0 -sketch. In [6] the boolean calculus of σ_0 -sketches (conjunctions, disjunctions, complements) is exposed as construction in the category of sketches. Then we can defined the category $\text{For}(\sigma_0)$ as being the category of σ_0 -sketches, as objects, with morphisms from P to P' the morphisms of sketches $f : \sigma \longrightarrow \sigma'$ which determine σ' as a σ -sketch, such that $f.P = P'$.

At this level of language, we can change our notations, replacing $\text{Mod}_{\mu}\phi$ by $\text{Mod}\sigma$, or even, more precisely, by P , and the $H_n(\text{Mod}_{\mu}\phi/I, F)$ will be denoted by $H_n(P/I, F)$. Of course for general mixed sketches (and not only for those associated to first order formulas) the result in §5 works, and the abelian groups $H_n(P/I, F)$ are smalls. Now

PROPOSITION. *The functoriality of these H_n , with respect to P , I and F are trivial facts.*

8. In a dual way, given a functor $F : \text{Mod}_{\mu}\phi \longrightarrow \mathbf{Ab}$ and an interpretation I of $\mathcal{L}$, the cohomology of I with coefficient in F is defined by considering the cochain complexe

$$\longleftarrow C^2(I, F) \xleftarrow{d^1} C^1(I, F) \xleftarrow{d^0} C^0(I, F)$$

which is

$$\begin{array}{ccccc}
& & d^1 & & d^0 \\
\longleftarrow & \prod FM_2 & \longleftarrow & \prod FM_1 & \longleftarrow & \prod FM_0 \\
M_2 \leftarrow M_1 \leftarrow M_0 \leftarrow I & & M_1 \leftarrow M_0 \leftarrow I & & M_0 \leftarrow I
\end{array}$$

with

$$d^1(x)_{(I \xrightarrow{\gamma} M_0 \xrightarrow{\lambda} M_1)} = F(\lambda)(x_{(I \xrightarrow{\gamma} M_0)}) - x_{(I \xrightarrow{\lambda \cdot \gamma} M_1)}$$

and so on, and we define $H^n(I, F) = \ker d^n / \text{Im } d^{n-1}$.

For these cohomology groups, the same result is true, that is to say that they are small. But now, the computation is based on the effective construction of a "relatively filtered locally free diagram" (r.f.l.f.d.) (in the category $\text{Mod}_\mu \phi$) generated by I . These cohomology groups will be denoted by $H^n((I/\text{Mod}_\mu \phi)^{\text{op}}, F)$.

9. Collecting the results of §5, §7 and §8, we get :

THEOREM : The abelian groups $H_n(\text{Mod}_\mu \phi / I, F)$ and $H^n((I/\text{Mod}_\mu \phi)^{\text{op}}, F)$ are small, i.e. they are elements of the category **Ab**, they are functorial with respect to I , F , μ and ϕ , and if $I \models \phi$, then $H_n(\text{Mod}_\mu \phi / I, F) = 0$, for every $n > 0$, and $H^n((I/\text{Mod}_\mu \phi)^{\text{op}}, F) = 0$, for $n > 0$. In fact, more precisely, we have $H_n(\text{Mod}_\mu \phi / I, F) = 0$, for every $n > 0$, if there is a cofree model generated by I , and we have $H^n((I/\text{Mod}_\mu \phi)^{\text{op}}, F) = 0$, for $n > 0$, if there is a free model generated by I . So they are small obstructions to the satisfaction of ϕ in I "up to co-equations" and "up to equations".

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