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# CHARACTERIZING POMONOIDS S BY COMPLETE S-POSETS

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**RESUME.**

Un ensemble partiellement ordonné (ou 'poset') muni d'une action d'un monoïde partiellement ordonné S est appelé S-poset. Pour un S-poset, il y a deux notions de complémentarité, la première en le cosidérant seulement en tant que poset, la seconde en tenant compte aussi des actions qui sont distributives sur les suprema.

Dans cet article, en cherchant à comparer ces deux notions, nous obtenons des caractérisations de certains monoïdes partiellement ordonnés.

**ABSTRACT.**

A poset with an action of a pomonoid S on it is called an S-poset. There are two different notions of completeness for an S-poset : one just as a poset and the other as a poset as well as the actions being distributive over the joins.

In this paper, comparing these two notions with each other, we find characterizations for some pomonoids.

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Key words: S-poset, completeness, continuous completeness.

## 1. Introduction and Preliminaries

General ordered algebraic structures play a role in a wide range of areas, including analysis, logic, theoretical computer science, and physics. One of the most important notion in any ordered algebraic structure is *completeness*. The purpose of the present article is to give some homological classification of pomonoids by *continuous completeness* of $S$-posets; complete posets with an action of a pomonoid $S$ on them which is compatible with the joins (supremums).

A number of papers [2, 3, 4, 5, 8, 9, 11, 12] have studied some properties of $S$-posets, and the papers [8], [9], [11] also deal with completeness.

Our aim is to find some necessary or sufficient conditions on the pomonoid $S$ under which completeness (as posets) and continuous completeness of $S$-posets coincide.

Introducing the notion of a *strongly indecomposable* pomonoid (having no prime ideal $P$ satisfying $\forall p \in P, \forall s \in S\backslash P, p \not\leq s$ or $\forall p \in P, \forall s \in S\backslash P, s \not\leq p$) we get a necessary condition on $S$ which plays an important role in this study.

Also, introducing the notion of a *strongly left residuated* pomonoid ($\forall s \in S, \exists t \in S, ts \leq 1 \leq st$), we prove that for such class of pomonoids, as well as for pogroups, complete and continuously complete $S$-posets are the same.

Finally, some classification of pomonoids by considering the additional condition "for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$" on $S$, and using completeness and continuous completeness of $S$-posets, are presented.

In the following we give a brief review of $S$-acts, posets, and $S$-posets needed in the sequel.

Let $S$ be a monoid with identity 1. Recall that a (*right*) *$S$-act* $A$ is a set equipped with a map $\lambda : A \times S \to A$, called its *action*, such that, denoting $\lambda(a, s)$ by $as$, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. The category of all $S$-acts, with action preserving ($S$-act) maps ($f : A \to B$ with $f(as) = f(a)s$, for $s \in S$, $a \in A$) between them, is denoted by **Act-S**.

An element $\theta$ of an $S$-act is called a *zero* or a *fixed* element if $\theta s = \theta$ for all $s \in S$. For more information about $S$-acts see [10] and [7].

Let **Pos** denote the category of all partially ordered sets (posets) with order preserving (monotone) maps between them. Recall that a poset is said to be *complete* if each of its subsets has an infimum and a supremum, in particular, a complete poset is bounded by the least (bottom) element $\perp$ and the greatest (top) element $\top$.

A monoid (semigroup, group) $S$ is said to be a *pomonoid* (*posemigroup, pogroup*) if it is also a poset whose partial order $\leq$ is compatible with the binary operation($s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$). In this paper $S$ denotes a pomonoid, unless otherwise stated.

By a *monoid ideal* $I$ of a pomonoid $S$, we mean a left ideal ($SI \subseteq I$) and a right ideal ($IS \subseteq I$) of $S$. Also, a proper nonempty monoid ideal $P$ of $S$ satisfying the property that $st \in P$ implies $s \in P$ or $t \in P$, for $s, t \in S$, is called a *prime ideal.*

A *right poideal* of a pomonoid $S$ is a (possibly empty) subset $I$ of $S$ if it is both a monoid right ideal ($IS \subseteq I$) and a down set ($a \leq b, b \in I$ imply $a \in I$).

A (*right*) *S-poset* is a poset $A$ which is also an $S$-act whose action $\lambda : A \times S \to A$ is order-preserving. Here, $A \times S$ is considered as a poset with componentwise order.

An *S-poset map* (or *morphism*) is an action preserving monotone map between $S$-posets. We denote the category of all right $S$-posets, with $S$-poset maps between them, by **Pos-S**.

## 2. Continuous completeness and Completeness

Being an ordered set as well as an algebraic structure, an $S$-poset may be defined to be complete in two different ways: one just as a poset and the other involving the action, too.

In this section first we define what we mean by these two notions in **Pos-S**, then we see that they are actually different and give some necessary conditions on $S$ for them to coincide.

**Definition 2.1.** Let $S$ be a pomonoid (posemigroup). An $S$-poset $A$ is called

(i) *complete* if it is complete as a poset,

(ii) *continuously complete* if it is a complete poset and the actions are compatible with the joins (supremums); that is, for every $X \subseteq A$ and $s \in S$, $(\bigvee X)s = \bigvee(Xs)$.

In the following proposition and remark, we see that completeness does not necessarily implies continuous completeness.

**Proposition 2.2.** *If all complete S-posets are continuously complete then S must have an identity.*

*Proof.* On the contrary, consider a posemigroup $S$ without 1. We construct a complete $S$-poset which is not continuously complete. Take the poset $A = \{\bot, a, b, c, \top\}$ with the order given by $a, b \leq c$, and $a, b$ are incomparable. Define the action on $A$ as $cs = \top$, for every $s \in S$, and $\bot, a, b, \top$ are fixed (zero) elements. To see that $A$ is an $S$-poset, it suffices to note that $(cs)t = \top t = \top = c(st)$, for every $s, t \in S$. Now, although $A$ is complete, it is not continuously complete. This is because $(a \vee b)s = cs = \top$ while $as \vee bs = a \vee b = c$, for every $s \in S$. $\square$

**Remark 2.3.** Notice that, even if $S$ is a pomonoid, not all complete $S$-posets are continuously complete. For example, consider the poset $S = (\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \preceq)$ of non-negative integers ordered by division: $m \preceq n \Leftrightarrow m|n$. Then $S$ is a lattice in which $m \vee n = \text{lcm}\{m, n\}$ (the least common multiple) and $m \wedge n = \gcd\{m, n\}$ (the greatest common divisor). Also $S$ is complete, since for $X \subseteq S$, we have $\bigvee X = \text{lcm}X$ if $X$ is finite, and $\bigvee X = 0$ if $X$ is infinite. Note that $S$ is a pomonoid with the multiplication $st = s \wedge t$. Therefore, $S$ is a complete $S$-poset. But $S$ is not continuously complete, because taking $X = \{3, 5, 7, ...\}$ we get $2 \wedge \bigvee X = 2 \wedge 0 = 2$, but $\bigvee\{2 \wedge x \mid x \in X\} = \bigvee\{1\} = 1$.

Since we intend to seek necessary as well as sufficient conditions on $S$ under which all complete $S$-posets are continuously complete, the above proposition justifies why we have taken $S$ to be a pomonoid.

We now give some necessary conditions on the pomonoid $S$ under which continuous completeness coincides with completeness.

**Definition 2.4.** A pomonoid $S$ is called *strongly decomposable* if $S = T\dot{\cup}I$, for some subpomonoid $T$ and some nonempty proper monoid ideal $I$ of $S$ such that $\forall t \in T, \forall i \in I, t \not\leq i$ or $\forall t \in T, \forall i \in I, i \not\leq t$. In this case, the pair $(T, I)$ is called a *strong decomposition* of $S$. Otherwise, $S$ is said to be *strongly indecomposable.*

**Remark 2.5.** Notice that in the above definition, $I$ is necessarily a prime ideal, because $T = S\backslash I$ is a monoid. Thus, the existence of a strong decomposition for $S$ is equivalent to the existence of a prime ideal $P$ of $S$ with the property that $\forall p \in P, \forall s \in S\backslash P, p \not\leq s$ or $\forall p \in P, \forall s \in S\backslash P, s \not\leq p$.

**Theorem 2.6.** *If all complete $S$-posets are continuously complete, then the pomonoid $S$ is strongly indecomposable.*

*Proof.* Let $(T, I)$ be a strong decomposition for $S$. In each of the following cases, we construct a complete $S$-poset which is not continuously complete:

**Case 1**: $\forall t \in T, \forall i \in I, t \not\leq i$. Consider the poset $A = \mathbf{2} \times \mathbf{2} = \{\bot, a, b, \top\}$ with the actions given by: $as = a$ if $s \in T, as = \bot$ if $s \in I$; and $\bot, b, \top$ are fixed. Then $A$ is an $S$-poset. To see this, let $s, t \in S$. It suffices to show that $(as)t = a(st)$. There are 4 possible cases:

(i) Let $s, t \in T$. Then $st \in T$ and so $(as)t = at = a = a(st)$.

(ii) Let $s, t \in I$. Then $st \in I$ and so $(as)t = \bot t = \bot = a(st)$.

(iii) Let $s \in T,\ t \in I$. Then $st \in I$ and so $(as)t = at = \bot = a(st)$.

(iv) Let $s \in I,\ t \in T$. Then $st \in I$ and so $(as)t = \bot t = \bot = a(st)$ as required.

Also, since $\perp, \top$ are fixed, $x \leq y$ gives that $xs \leq ys$, for every $x, y \in A, s \in S$. Finally, let $s, t \in S$ and $s \leq t$. It suffices to verify that $as \leq at$. By the assumption, we get that $s, t \in T$ or $s, t \in I$ or $s \in I, t \in T$. If $s, t \in T$, $as = a = at$. If $s, t \in I, as = \perp = at$. If $s \in I, t \in T, as = \perp \leq a = at$. Since $(a \vee b)i = \top i = \top$, and $ai \vee bi = \perp \vee b = b$ for every $i \in I$, we conclude that $A$ is a complete $S$-poset which is not continuously complete.

**Case 2**: $\forall t \in T, \forall i \in I, i \not\leq t$. Consider the poset $B = \{\perp, a, b, c, \top\}$ as in the proof of Proposition 2.2, and the actions given by: $cs = c$ if $s \in T, cs = \top$ if $s \in I$; and $\perp, a, b, \top$ are fixed. Then $B$ is an $S$-poset. To see this, let $s, t \in S$. It suffices to show that $(cs)t = c(st)$. There are 4 possible cases:

(i) Let $s, t \in T$. Then $st \in T$ and so $(cs)t = ct = c = c(st)$.

(ii) Let $s, t \in I$. Then $st \in I$ and so $(cs)t = \top t = \top = c(st)$.

(iii) Let $s \in T,\ t \in I$. Then $st \in I$ and so $(cs)t = ct = \top = c(st)$.

(iv) Let $s \in I,\ t \in T$. Then $st \in I$ and so $(cs)t = \top t = \top = c(st)$ as required.

Also, since $\perp, a, b, \top$ are fixed, $x \leq y$ gives that $xs \leq ys$, for every $x, y \in B, s \in S$. Finally, let $s, t \in S$ and $s \leq t$. It suffices to verify that $cs \leq ct$. By the assumption, we get that $s, t \in T$ or $s, t \in I$ or $s \in T, t \in I$. If $s, t \in T$, $cs = c = ct$. If $s, t \in I, cs = \top = ct$. If $s \in T, t \in I, cs = c \leq \top = ct$. Since $(a \vee b)i = ci = \top$, and $ai \vee bi = a \vee b = c$ for every $i \in I$, we conclude that $B$ is a complete $S$-poset which is not continuously complete. □

**Corollary 2.7.** *Let $(S,=)$ be a pomonoid. If all complete $S$-posets are continuously complete, then $S$ has no prime ideal.*

**Theorem 2.8.** *Let $S$ be a nontrivial pomonoid. If all complete $S$-posets are continuously complete, then either the identity of $S$ is not externally adjoined (there exist $s, t \in S \setminus \{1\}$ with $st = 1$) or $\downarrow 1 \neq \{1\} \neq \uparrow 1$ (there exist $u, v \in S$ such that $u < 1 < v$).*

*Proof.* Let the identity of $S$ be externally adjoined and $\downarrow 1 = \{1\}$ or $\uparrow 1 = \{1\}$. Then $P = S \setminus \{1\}$ is a prime ideal of $S$ such that $\forall p \in P, p \not\leq 1$ or $\forall p \in P, 1 \not\leq p$. Thus $S$ is strongly decomposable, contradicting Theorem 2.6. □

**Corollary 2.9.** *Consider the nontrivial pomonoid $S$ with equality as its order. If all complete $S$-posets are continuously complete, then the identity of $S$ is not externally adjoined.*

**Corollary 2.10.** *Let $S$ be a nontrivial pomonoid. If all complete $S$-posets are continuously complete, then the identity of $S$ is neither the bottom nor the top element of $S$.*

*Proof.* Let 1 be the bottom element of $S$. We apply Theorem 2.8. If $st = 1$ for some $s, t \neq 1$ in $S$, then from $1 \leq s$ we get that $1 \leq t \leq st = 1$, and so $t = 1$ which is a contradiction. Also for $u, v \in S$, the case $u < 1 < v$ can not happen because 1 is the bottom element. Similarly, 1 is not the top element. □

**Corollary 2.11.** *Let $S$ be a nontrivial pomonoid. If all complete $S$-posets are continuously complete, then the zero of $S$ (if exists) is not externally adjoined.*

*Proof.* Let $S$ have the zero element 0 which is externally adjoined. Then, $P = \{0\}$ is a prime ideal of $S$. Since for all $s, t \neq 0$, $st \neq 0$, there exist no nonzero elements $s, t \in S$ with $s \leq 0 \leq t$. In fact, otherwise $st \leq 0t = 0 = s0 \leq st$ and so $st = 0$ which is a contradiction. This clearly implies that for every $s \in S \backslash P$, $s \not\leq 0$ or for every $s \in S \backslash P$, $0 \not\leq s$. Consequently, $S$ is strongly decomposable, which contradicts Theorem 2.6. □

The following example shows that the converses of Corollaries 2.9, 2.10 and 2.11 are not true in general.

**Example 2.12.** Consider the pomonoid $S = \{0, 1, s, t\}$ with the equality order and the operation defined as $s^2 = 1$, $t^2 = 0$, $st = ts = t$. Then $P = \{0, t\}$ is a prime ideal. So, by Corollary 2.7, there exists a complete $S$-poset which is not continuously complete.

Recall the following definition and lemma from [1].

**Definition 2.13.** [1] A *divisibility monoid* is a pomonoid $S$ in which $s \leq t$ is equivalent to $t \in Ss$, and also to $t \in sS$.

**Lemma 2.14.** [1] *In any divisibility monoid, $st = 1$ implies $s = t = 1$.*

Now, in view of Theorem 2.8 and Lemma 2.14, the following result is immediate.

**Proposition 2.15.** *Let $S$ be a nontrivial pomonoid. If all complete $S$-posets are continuously complete, then either $S$ is not a divisibility monoid or $\downarrow 1 \neq \{1\} \neq \uparrow 1$.*

**Remark 2.16.** For a nontrivial left simple, or right simple, or commutative, or idempotent pomonoid $S$, not all complete $S$-posets are continuously complete. To see this, consider the pomonoid $S = \{1, s\}$ with $s^2 = s$ and $1 \leq s$. Then $S$ satisfies all the mentioned properties. Also $S$ is strongly decomposable because $P = \{s\}$ is a prime ideal of $S$ and $s \not\leq 1$. So, by Theorem 2.6, not all complete $S$-posets are continuously complete.

## 3. Characterizing $S$ by continuous completeness

In this section, we give some sufficient conditions on $S$ under which continuous completeness coincides with completeness.

First, notice that by Lemma 4.1 of [8] we get

**Lemma 3.1.** *For a pogroup $S$, all complete $S$-posets are continuously complete.*

The converse of the above lemma is true if we have the additional condition that "for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$". To see this, first recall from [8] that a pomonoid $S$ which has no proper non-empty left (right) poideal is said to be *left* (*right*) *simple*.

**Lemma 3.2.** *A pomonoid $S$ is left (right) simple if and only if for all $s \in S$ there exists $x \in S$ such that $1 \leq xs$ $(1 \leq sx)$.*

Also, we see that:

**Lemma 3.3.** *Let $S$ be a left (right) simple pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then $S$ is a pogroup.*

*Proof.* Let $S$ be left simple and $s \in S$. Then there exist $t, u \in S$ such that $1 \leq ts$ and $1 \leq u(ts) = (ut)s$, by Lemma 3.2, and hence $s(ut) \leq 1$ by the hypothesis. On the other hand, again using the hypothesis, we have $1 \leq u(ts)$ implies $t(su) = (ts)u \leq 1$ and hence $1 \leq (su)t = s(ut)$. Consequently, $s(ut) = 1$ which implies that $S$ is a pogroup. A similar argument can be applied for the case where $S$ is a right simple pomonoid. □

**Theorem 3.4.** *Let $S$ be a pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then the following are equivalent:*

(i) *All complete $S$-posets are continuously complete.*
(ii) *$S$ is left simple.*
(iii) *$S$ is right simple.*
(iv) *$S$ is a pogroup.*

*Proof.* The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow from Lemma 3.3. Also, Lemma 3.1 gives (iv) $\Rightarrow$ (i). It remains to prove (i) $\Rightarrow$ (ii). Suppose that $S$ is not left simple. Then, by Lemma 3.2, there exists $s_0 \in S$ such that for all $t \in S$, $1 \not\leq ts_0$ and hence $s_0t \not\leq 1$ by the hypothesis. Now, applying Theorem 2.6, we claim that $S$ is strongly decomposable. Take $P = \{s \in S \mid \forall t \in S, st \not\leq 1\}$. Since $s_0 \in P$, $P$ is a nonempty, also it is clearly a proper subset of $S$. To show that $P$ is a prime ideal of $S$, take $p \in P, s \in S$. Then $sp, ps \in P$, since :

(a) if $sp \in S\backslash P$, then $(sp)t \leq 1$ for some $t \in S$. Using the assumption, we get that $1 \leq t(sp) = (ts)p$ and then $p(ts) \leq 1$. This means that $p \in S\backslash P$ which is a contradiction, and

(b) if $ps \in S\backslash P$, then $p(st) = (ps)t \leq 1$ for some $t \in S$ and so $p \in S\backslash P$, which is a contradiction.

Also, let $st \in P$ for some $s, t \in S$. If $s, t \in S\backslash P$, then $su \leq 1$ and $tv \leq 1$ for some $u, v \in S$. This implies $(st)(vu) = s(tv)u \leq s(1)u \leq 1$ which means $st \in S\backslash P$, a contradiction. Finally, let $p \leq s$ for some $p \in P, s \in S\backslash P$. Then, $st \leq 1$ for some $t \in S$. Since $p \leq s$, $pt \leq st \leq 1$. This implies that $p \in S\backslash P$ which is a contradiction. So, $\forall p \in P, \forall s \in S\backslash P, p \not\leq s$. Consequently, the pair $(S\backslash P, P)$ forms a strong decomposition for $S$, as claimed. □

**Corollary 3.5.** *Let $(S,=)$ be a pomonoid whose right invertible elements are invertible. Then, all complete $S$-posets are continuously complete if and only if $S$ is a pogroup.*

Now, using the notion of *residuation*, we find a sufficient condition for the equivalence of completeness and continuous completeness. First, recall the following definition from [6].

**Definition 3.6.** Let $A, B$ be posets. A monotone mapping $f : A \longrightarrow B$ is called *residuated* if there exists a (necessarily unique) monotone map $f^+ : B \longrightarrow A$, called the *residual* of $f$, such that $ff^+ \leq id_B$ and $id_A \leq f^+f$. In fact, the map $f^+$ is defined as

$$f^+(b) = max\{a \in A \mid f(a) \leq b\}, \text{ for } b \in B.$$

**Definition 3.7.** A pomonoid $S$ is called *strongly right residuated* if for every $s \in S$, the right translation mapping $\rho_s : S \longrightarrow S$, $\rho_s(x) = xs$, is residuated with the residual $\rho_s^+ = \rho_t$, for some $t \in S$.

By an easy verification, we have

**Lemma 3.8.** *A pomonoid $S$ is strongly right residuated if and only if for each $s \in S$ there exists a unique $t \in S$ such that $ts \leq 1 \leq st$. It then follows that $sts = s$ and $tst = t$.*

**Theorem 3.9.** *Let $S$ be a strongly right residuated pomonoid. Then all complete $S$-posets are continuously complete.*

*Proof.* Let $A$ be a complete $S$-poset, $X \subseteq A$ and $s \in S$. The inequality $\bigvee(Xs) \leq (\bigvee X)s$ always holds. For the converse, since $S$ is strongly right residuated, by Lemma 3.8, $ts \leq 1 \leq st$, for some $t \in S$. Now $1 \leq st$ implies $x \leq x(st) = (xs)t \leq (\bigvee(Xs))t$, for every $x \in X$. This implies that $\bigvee X \leq (\bigvee(Xs))t$ and so $(\bigvee X)s \leq (\bigvee(Xs))ts$. On the other hand, since $ts \leq 1$, $(\bigvee(Xs))ts \leq \bigvee(Xs)$. So, the equality holds and $A$ is continuously complete. □

**Theorem 3.10.** *Let $S$ be a pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then all complete $S$-posets are continuously complete if and only if $S$ is strongly right residuated.*

*Proof.* First notice that under the given condition, $S$ is strongly right residuated if and only if $S$ is right simple. Now the result follows from Theorem 3.4. $\square$

Finally, we give some more necessary and sufficient conditions on $S$ under which continuous completeness and completeness coincide.

**Corollary 3.11.** *Let $(S,=)$ be a commutative or a finite pomonoid. Then, all complete $S$-posets are continuously complete if and only if $S$ is a pogroup.*

*Proof.* Notice that in a commutative monoid, every right invertible element is invertible. Also, the same note is true in a finite monoid (since every map on a finite set is onto if and only if it is one-one, and every monoid $S$ is isomorphic to a submonoid of the monoid of all maps on $S$). So, by Corollary 3.5, the proof is complete. $\square$

**Definition 3.12.** Let $S$ be a pomonoid. An $S$-poset $A$ is called *residuated* if for all $s \in S$, $\rho_s : A \to A$, $\rho_s(a) = as$ for any $a \in A$, is residuated.

**Theorem 3.13.** *Let $S$ be a pomonoid and $A$ be a complete $S$-poset. Then $A$ is continuously complete if and only if it is residuated.*

*Proof.* Let $A$ be continuously complete and $s \in S$. We must prove that the mapping $\rho_s$ on $A$ is residuated. It suffices to show that for every $y \in A$ the set $X = \{x \in A \mid xs = \rho_s(x) \leq y\}$ is non-empty and has the top element. Since $A$ is (continuously) complete, it has the bottom element $\perp$ and $\perp s = (\bigvee \emptyset)s = \bigvee \emptyset s = \bigvee \emptyset = \perp \leq y$ which implies that $\perp \in X$ and so $X \neq \emptyset$. Also, $\bigvee X$ exists in $A$. We see that $\bigvee X \in X$, because $(\bigvee X)s = \bigvee(Xs) = \bigvee\{xs \mid x \in X\} \leq y$.

For the converse, let $A$ be residuated, $X \subseteq A$, and $s \in S$. It suffices to prove the nontrivial equality $(\bigvee X)s \leq \bigvee(Xs)$. By the hypothesis, $\rho_s$ is residuated and so $\rho_s\rho_s^+ \leq id \leq \rho_s^+\rho_s$. Now, for every $x \in X$, we have

$$x \leq \rho_s^+(\rho_s(x)) \leq \rho_s^+(\bigvee \rho_s(X))$$

and so $\bigvee X \leq \rho_s^+(\bigvee \rho_s(X))$. This implies

$$(\bigvee X)s = \rho_s(\bigvee X) \leq \rho_s\rho_s^+(\bigvee \rho_s(X)) \leq \bigvee \rho_s(X) = \bigvee(Xs)$$

as required. $\square$

By Theorems 3.4, 3.10, and 3.13, we get the following result.

**Proposition 3.14.** *Let $S$ be a pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then, the following are equivalent:*
(i) *All complete $S$-posets are continuously complete.*
(ii) *All complete $S$-posets are residuated.*
(iii) *$S$ is strongly right residuated.*

(iv) $S$ *is left simple.*
(v) $S$ *is right simple.*
(vi) $S$ *is a pogroup.*

**Corollary 3.15.** *Let* $(S, =)$ *be a pomonoid whose right invertible elements are invertible (such as commutative and finite monoids). Then, the following are equivalent:*
(i) *All complete* $S$*-posets are continuously complete.*
(ii) *All complete* $S$*-posets are residuated.*
(iii) $S$ *is a pogroup.*

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