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## REPRESENTATION OF METRIC JETS

by *Elisabeth BURRONI and Jacques PENON*

*We dedicate this article to Francis Borceux*

### Abstract

Guided by the heuristic example of the tangential  $Tf_a$  of a map  $f$  differentiable at  $a$  which can be canonically represented by the unique continuous affine map it contains, we extend, in this article, into a specific metric context, this property of representation of a metric jet. This yields a lot of relevant examples of such representations.

L'application affine continue qui est tangente, en un point  $a$  fixé, à une application  $f$  différentiable en ce point, peut être très naturellement considérée comme un représentant de la tangentielle  $Tf_a$  de  $f$  en  $a$ . Cet exemple sera notre guide heuristique pour trouver un contexte métrique spécifique dans lequel cette propriété de représentation d'un jet métrique soit possible. Au passage, on fournit de nombreux exemples pertinents de telles représentations.

Key words : differential calculus, Gateaux differentials, fractal maps, jets, metric spaces, categories

AMS classification : 58C25, 58C20, 28A80, 58A20, 54E35, 18D20

### INTRODUCTION

This article is the sequel of a paper published in TAC [6] ; most of the proofs of the statements given here can be found in the second chapter of a paper published in arXiv [5].

We recall that maps  $f, g : M \rightarrow M'$  (where  $M, M'$  are metric spaces) are tangent at  $a$  (not isolated in  $M$ ), what we denote  $f \asymp_a g$ , if  $f(a) = g(a)$  and  $\lim_{a \neq x \rightarrow a} \frac{d(f(x), g(x))}{d(x, a)} = 0$  ; a metric jet (in short, a jet) is an equivalence class for this relation  $\asymp_a$ , restricted to the set of the

maps which are locally lipschitzian at  $a$  (in short,  $LL_a$ ). We say that a map  $f$  is tangential at  $a$  (in short,  $Tang_a$ ) if it possesses a tangent at  $a$  which is  $LL_a$ ; then the jet composed of all the  $LL_a$  maps which are tangent to  $f$  at  $a$  is called the tangential of  $f$  at  $a$  and denoted  $Tf_a$ .

$\mathbb{L}\mathbb{L}$  and  $\mathbb{J}\text{et}$  are the cartesian categories whose objects are pointed metric spaces (for both) and morphisms  $(M, a) \rightarrow (M', a')$ , the  $LL_a$  maps  $f$  verifying  $f(a) = a'$  (for  $\mathbb{L}\mathbb{L}$ ), and the jets  $\varphi$  whose elements are forequoted maps which are tangent altogether at  $a$  (for  $\mathbb{J}\text{et}$ ). We denote  $q : \mathbb{L}\mathbb{L} \rightarrow \mathbb{J}\text{et}$  the canonical surjection which associates its tangential  $Tf_a$  to a  $LL_a$  map  $f : (M, a) \rightarrow (M', a')$ .

This paper takes up and develops two previous talks ([2] and [3]). Here, we propose a frame in which each jet can be canonically represented by one of its elements. This frame is the algebraico-metric structure of “ $\Sigma$ -contracting” metric space (equipped with a “central point” denoted  $\omega$ ); the morphisms between such spaces, called “ $\Sigma$ -homogeneous” maps, have a fundamental “ $\Sigma$ -uniqueness property” : two  $\Sigma$ -homogeneous maps which are tangent at  $\omega$  are equal.

Carrying on the analogy with the classical differential calculus, we are interested (in the  $\Sigma$ -contracting metric world) in maps  $f$  which are tangential at  $\omega$  and whose tangential  $Tf_\omega$  possesses a  $\Sigma$ -homogeneous element which can represent it; such a map is said “ $\Sigma$ -contactable” at  $\omega$ , the unique  $\Sigma$ -homogeneous  $LL_\omega$  element tangent to it at  $\omega$  being its “ $\Sigma$ -contact” at  $\omega$ . In many respects, the properties of “ $\Sigma$ -contactibility” are similar to those of differentiability, as, for instance, the search of extrema for a map taking its values in  $\mathbb{R}$  (section 5).

We will mainly be interested in two special cases, in the normed vector space (in short n.v.s.), which will provide many examples. The first one with  $\Sigma = \mathbb{R}_+$  (section 3), brings back the “old” interesting notion of maps which are differentiable in the sense of Gateaux [7]. The second one (section 4) immerses ourselves in the fractal universe. We finally notice that the notion of contactibility does not entirely exhaust the one of tangentiality, since there exist maps which are tangential (at a central point) and not contactable (at this point).

For general definitions in category theory (for instance cartesian or enriched categories), see [1].

**Acknowledgements :** It is a talk about Ehresmann's jets, given by Francis Borceux at the conference organised in Amiens in 2002 in honour of Andrée and Charles Ehresmann which has initiated our work. Since at that epoch we were interested, in our teaching, in what could be described uniquely with metric tools ... hence the idea of the metric jets!

## 1 Valued monoids, contracting spaces

Our aim, in this section, is to define the algebraico-metric notion of contracting spaces and prove they possess the above mentioned uniqueness property.

**Definition 1.1** *a valued monoid  $\Sigma$  is a monoid (its law of composition is denoted here multiplicatively) equipped with a particular element denoted  $0$ , and with a monoid homomorphism  $v : \Sigma \longrightarrow \mathbb{R}_+$  (where  $\mathbb{R}_+ = [0, +\infty[$ ), called the valuation of  $\Sigma$ , verifying the two conditions :*

- (1)  $\forall t \in \Sigma \quad (v(t) = 0 \iff t = 0)$ ,
- (2)  $\exists t \in \Sigma \quad (0 < v(t) < 1)$ .

*Thanks to (1), we have that  $0$  is an absorbing element in  $\Sigma$ .*

### Examples 1.2

1)  $\mathbb{R}$  and its multiplicative submonoids  $\mathbb{R}_+$  and  $[0, 1]$  are valued monoids, where  $0$  is their absorbing element (their valuation being the absolute value).

2) If  $r$  is a real number verifying  $0 < r < 1$ , we denote  $\mathbb{N}'_r$  the additive monoid  $\mathbb{N}'_r = \mathbb{N} \cup \{\infty\}$ , where  $\infty$  is its absorbing element, and whose valuation  $v_r : \mathbb{N}'_r \longrightarrow \mathbb{R}_+$  is given by  $v_r(n) = r^n$  if  $n \in \mathbb{N}$  and  $v_r(\infty) = 0$ .

### Definition 1.3

1) A morphism of valued monoids  $\sigma : \Sigma \longrightarrow \Sigma'$  is a monoid homomorphism verifying :  $\forall t \in \Sigma \quad (v(\sigma(t)) = v(t))$ .

2) A valued submonoid of  $\Sigma'$  is a valued monoid  $\Sigma$  verifying  $\Sigma \subset \Sigma'$  such that the canonical injection  $j : \Sigma \hookrightarrow \Sigma'$  is a morphism of valued monoids.

**Remark 1.4**

A valuation  $v : \Sigma \longrightarrow \mathbb{R}_+$  is itself a morphism of valued monoids.

**Definition 1.5**  $\Sigma$  being a valued monoid, a  $\Sigma$ -contracting metric space (in short a  $\Sigma$ -contracting space) is a metric space  $M$  pointed by  $\omega$  (said to be central), equipped with an external operation  $\Sigma \times M \longrightarrow M : (t, x) \mapsto t \star x$  which, in addition of the usual properties :

- (1)  $\forall x \in M \quad (1 \star x = x)$ ,
- (2)  $\forall t, t' \in \Sigma \quad \forall x \in M \quad (t' \star (t \star x) = (t'.t) \star x)$ ,

verifies also the following conditions :

- (3)  $0 \star \omega = \omega$ ,
- (4)  $\forall t \in \Sigma \quad \forall x, y \in M \quad (d(t \star x, t \star y) = v(t)d(x, y))$

The central point of  $\Sigma$ -contracting spaces will usually be denoted  $\omega$ .

**Remark 1.6** A  $\Sigma$ -contracting space  $M$  verifies the following properties : (1)  $\forall t \in \Sigma \quad (t \star \omega = \omega)$ , (2)  $\forall x \in M \quad (0 \star x = \omega)$ .

**Examples 1.7**

1) Let  $E$  be a n.v.s. ; fixing a point  $a \in E$ , the pointed n.v.s.  $(E, a)$  is denoted  $E_a$ . We make this  $E_a$  a  $\mathbb{R}$ -contracting space (with central point  $a$ ), setting, for  $t \in \mathbb{R}$  and  $x \in E$ ,  $t \star x = a + t(x - a)$ . This external operation on  $E_a$  is said to be "standard".

2) When  $\Sigma$  is a valued monoid whose valuation  $v : \Sigma \longrightarrow \mathbb{R}_+$  is injective, then  $\Sigma$  becomes itself a  $\Sigma$ -contracting space setting  $\omega = 0$  and, for  $s, t \in \Sigma$ ,  $t \star s = t.s$  and  $d(s, t) = |v(t) - v(s)|$ .

3) If  $M, M'$  are  $\Sigma$ -contracting spaces, then so is  $M \times M'$ .

4) Let  $\sigma : \Sigma \longrightarrow \Sigma'$  be a morphism of valued monoids ; then every  $\Sigma'$ -contracting space can be canonically equipped with a structure of  $\Sigma$ -contracting space, the new  $\Sigma$ -operation being :  $(t, x) \mapsto \sigma(t) \star x$ .

**Remarks 1.8**

1) If  $\Sigma$  is a valued monoid, every  $\mathbb{R}_+$ -contracting space  $M$  has also a “canonical” structure of  $\Sigma$ -contracting space (its external operation being  $: (t, x) \mapsto v(t) \star x$  for every  $t \in \Sigma$  and  $x \in M$ ). In particular, a pointed n.v.s.  $E_a$  has also a canonical structure of  $\Sigma$ -contracting space, its external operation being, for  $t \in \Sigma$  and  $x \in E$ ,  $t \star x = a + v(t)(x - a)$ .

2) We notice that we have two structures of  $\mathbb{R}$ -contracting space on  $E_a$  : the standard one of 1.7, and the above canonical one, whose external operation is  $t \star x = a + |t|(x - a)$ .

3) In a  $\Sigma$ -contracting space  $M \neq \{\omega\}$ ,  $\omega$  is never isolated in  $M$ .

**Definition 1.9**

A  $\Sigma$ -contracting space is revertible if, for every  $t \in \Sigma$  verifying  $t \neq 0$ , the map  $t \star (-) : M \rightarrow M$  is bijective. In this case, we set  $t \star^{-1} x = (t \star (-))^{-1}(x)$  for  $x \in M$ .

**Remark 1.10**

If  $\sigma : \Sigma \rightarrow \Sigma'$  is a morphism of valued monoids and  $M$  a revertible  $\Sigma'$ -contracting space, then  $M$  is a revertible  $\Sigma$ -contracting space.

**Examples and counter-examples 1.11**

1) The pointed n.v.s.  $E_a$  is a revertible  $\mathbb{R}_+$ (or  $\mathbb{R}$ )-contracting space for the standard structure; with  $t \star^{-1} x = t^{-1} \star x = a + t^{-1}(x - a)$  for every  $t \in \Sigma$  ( $t \neq 0$ ) and  $x \in E$ . Actually, for each valued monoid  $\Sigma$ ,  $E_a$  is a revertible  $\Sigma$ -contracting space for the canonical structure; with  $t \star^{-1} x = t^{-1} \star x = a + v(t)^{-1}(x - a)$  for every  $t \in \Sigma$  ( $t \neq 0$ ) and  $x \in E$ .

2)  $[0, 1]$  and  $\mathbb{N}'_r$  are  $\Sigma$ -contracting spaces with, respectively,  $\Sigma = [0, 1]$ ,  $\omega = 0$ , and  $\Sigma = \mathbb{N}'_r$ ,  $\omega = \infty$ . But none of them is revertible.

**Definition 1.12** *Let  $\Sigma$  be a valued monoid.*

1) A map  $h : M \rightarrow M'$  is  $\Sigma$ -homogeneous if  $M, M'$  are  $\Sigma$ -contracting spaces and  $h$  verifies  $: \forall t \in \Sigma \ \forall x \in M \quad (h(t \star x) = t \star h(x))$ .

2)  $M'$  being a  $\Sigma$ -contracting space and  $M$  a metric subspace of  $M'$ , then  $M$  is a  $\Sigma$ -contracting subspace of  $M'$  if  $\omega \in M$  and  $t \star x \in M$

for all  $t \in \Sigma$  and  $x \in M$ ; then the canonical injection  $M \hookrightarrow M'$  is  $\Sigma$ -homogeneous.

**Remarks 1.13**

1) A  $\Sigma$ -homogeneous map  $h : M \rightarrow M'$  verifies  $h(\omega) = \omega$ .

2) Let  $\sigma : \Sigma \rightarrow \Sigma'$  be a morphism of valued monoids,  $M$  and  $M'$  two  $\Sigma'$ -contracting spaces. Then, every  $\Sigma'$ -homogeneous map  $M \rightarrow M'$  is  $\Sigma$ -homogeneous; the inverse is true when  $\sigma$  is surjective.

**Proposition 1.14** For every  $\Sigma$ -homogeneous map  $h : M \rightarrow M'$ , we have the equivalence :  $h$   $k$ -lipschitzian  $\iff h$   $k$ - $LL_\omega$

Proof :  $r > 0$  being such that  $h|_{B(\omega,r)}$  is  $k$ -lipschitzian, we choose  $t \in \Sigma$  such that  $0 < v(t) < 1$ ; then, for each  $x, y \in M$ , there exists  $n \in \mathbb{N}$  verifying  $t^n \star x, t^n \star y \in B(\omega, r)$ . □

**Theorem 1.15** (of  $\Sigma$ -uniqueness)

Let  $h_1, h_2 : M \rightarrow M'$  be two  $\Sigma$ -homogeneous maps verifying  $h_1 \succ_\omega h_2$ ; then  $h_1 = h_2$ .

Proof : Let us take  $t \in \Sigma$  verifying  $0 < v(t) < 1$  and fix  $x \in M$ . We can assume that  $x \neq \omega$ . Let us set  $x_n = t^n \star x$  for each  $n \in \mathbb{N}$ . Then, we have  $x_n \neq \omega$  and  $\lim_n x_n = \omega$ , so that we can write :  $\frac{d(h_1(x_n), h_2(x_n))}{d(x_n, \omega)} = \frac{d(h_1(t^n \star x), h_2(t^n \star x))}{d(t^n \star x, t^n \star \omega)} = \frac{d(t^n \star h_1(x), t^n \star h_2(x))}{d(t^n \star x, t^n \star \omega)} = \frac{v(t^n)d(h_1(x), h_2(x))}{v(t^n)d(x, \omega)} = \frac{d(h_1(x), h_2(x))}{d(x, \omega)}$ . But, as  $\lim_n x_n = \omega$  and  $h_1 \succ_\omega h_2$ , this provides  $0 = \lim_n \frac{d(h_1(x_n), h_2(x_n))}{d(x_n, \omega)} = \frac{d(h_1(x), h_2(x))}{d(x, \omega)}$  which implies  $d(h_1(x), h_2(x)) = 0$ , i.e  $h_1(x) = h_2(x)$ . □

**Theorem 1.16** Let  $M, M'$  be  $\Sigma$ -contracting spaces with  $M'$  revertible,  $V$  a neighborhood of  $\omega$  in  $M$ , and maps  $f : V \rightarrow M', h : M \rightarrow M'$  such that  $h$  is  $\Sigma$ -homogeneous and verifies  $f \succ_\omega h|_V$ . Then, for all  $x \in M$ , we have  $h(x) = \lim_{0 \neq v(t) \rightarrow 0} t^{-1} \star f(t \star x)$ .

Proof : The above equality is clearly true for  $x = \omega$ . If  $x \neq \omega$ , we have  $\omega = \lim_{v(t) \rightarrow 0} t \star x$  (since  $d(t \star x, \omega) = d(t \star x, t \star \omega) = v(t)d(x, \omega)$ ), which insures that there exists  $\varepsilon > 0$  such that, for all

$t \in \Sigma$  verifying  $0 < v(t) < \varepsilon$ , we have  $t \star x \in V - \{\omega\}$ . Thus, for all these  $t$ , we can write :  $\frac{d(f(t \star x), h(t \star x))}{d(t \star x, \omega)} = \frac{d(f(t \star x), t \star h(x))}{d(t \star x, t \star \omega)} = \frac{d(t \star (t^{-1} f(t \star x)), t \star h(x))}{v(t) d(x, \omega)} = \frac{v(t) d(t^{-1} f(t \star x), h(x))}{v(t) d(x, \omega)} = \frac{d(t^{-1} f(t \star x), h(x))}{d(x, \omega)}$ . Since  $f \asymp_{\omega} h|_V$ , we have  $\lim_{0 \neq v(t) \rightarrow 0} \frac{d(t^{-1} f(t \star x), h(x))}{d(x, \omega)} = \lim_{0 \neq v(t) \rightarrow 0} \frac{d(f(t \star x), h(t \star x))}{d(t \star x, \omega)} = 0$ , which finally means that  $\lim_{0 \neq v(t) \rightarrow 0} t \star^{-1} f(t \star x) = h(x)$ .  $\square$

**Theorem 1.17** *Let  $M, M'$  be  $\Sigma$ -contracting spaces with  $M'$  revertible,  $V$  a neighborhood of  $\omega$  in  $M$ ,  $g : V \rightarrow M'$  a  $k$ - $LL_{\omega}$  map, and  $h : M \rightarrow M'$  a  $\Sigma$ -homogeneous map verifying  $g \asymp_{\omega} h|_V$ . Then,  $h$  is  $k$ -lipschitzian.*

*Proof* : Let  $W$  be a neighborhood of  $\omega$  in  $V$  such that  $g|_W$  is  $k$ -lipschitzian,  $x, y \in M$ , and  $t \in \Sigma$  such that  $0 < v(t) < 1$ ; then, there exists  $N \in \mathbb{N}$  such that  $t^n \star x, t^n \star y \in W$  for all  $n \geq N$ ; so that, for all these  $n$ , we have  $d(g(t^n \star x), g(t^n \star y)) \leq kd(t^n \star x, t^n \star y) = kv(t^n) d(x, y)$ , which provides  $d(t^n \star^{-1} g(t^n \star x), t^n \star^{-1} g(t^n \star y)) = (v(t^n))^{-1} d(g(t^n \star x), g(t^n \star y)) \leq kd(x, y)$ . Now,  $d$  being continuous, we obtain (doing  $n \rightarrow +\infty$ ) :  $d(h(x), h(y)) \leq kd(x, y)$ .  $\square$

**Corollary 1.18**  *$M, M'$  being as in 1.17, and  $h : M \rightarrow M'$  being a  $\Sigma$ -homogeneous map, we have the equivalences :*

$$h \text{ lipschitzian} \iff h \text{ } LL_{\omega} \iff h \text{ } Tang_{\omega}$$

**Counter-example 1.19** We give here an example of function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  which is  $\mathbb{R}$ -homogeneous and continuous (since it is  $LSL$  : see 1.20 below), but not lipschitzian (thus not  $Tang_0$ ). Consider the function  $f(x, y) = x \sin \frac{y}{x}$  if  $x \neq 0$  and  $f(0, y) = 0$  (see Figure 1 in [5]). This  $f$  is clearly  $\mathbb{R}$ -homogeneous  $\mathbb{R}_0^2 \rightarrow \mathbb{R}_0$ . We also notice that  $f$  is  $LSL$  (at every point) since it is differentiable on  $\mathbb{R}^* \times \mathbb{R}$ ; and it is clear at each point  $(0, a)$ . Now, if  $x \neq 0$ , we have  $\frac{\partial f}{\partial x}(x, y) = \sin \frac{y}{x} - \frac{y}{x} \cos \frac{y}{x}$  and thus, putting  $x_n = \frac{1}{n^2}$  and  $y_n = \frac{2\pi}{n}$ , we obtain  $\lim_n \frac{\partial f}{\partial x}(x_n, y_n) = -\infty$ , where  $\lim_n (x_n, y_n) = (0, 0)$ . Thus, this function  $f$  cannot be lipschitzian !



**Remarks 1.20**

1) For a map  $f : M \rightarrow M'$  between metric spaces; *LSL* means “locally semi-lipschitzian” at every point of  $M$ ; and “semi-lipschitzian” at  $a \in M$  means that, there exists a real  $k > 0$  such that, for all  $x \in M$ , we have  $d(f(x), f(a)) \leq kd(x, a)$ .

2) By the way, in 1.19, we have prove that :  $LSL_a \not\Rightarrow Tang_a$ . Thus, we cannot complete our equivalences of 1.18, adding the properties of being  $LSL_\omega$  or continuous at  $\omega$ ! Even though, for linear maps, all these properties are equivalent.

## 2 Representability and Contactibility

The  $\Sigma$ -uniqueness property allows to choose at most one canonical representative element in each jet between  $\Sigma$ -contracting spaces (pointed by their central point  $\omega$ ); hence the term of “ $\Sigma$ -representable” jets. The maps  $f$  which are tangential at  $\omega$  and whose tangential  $Tf_\omega$  is  $\Sigma$ -representable are called “ $\Sigma$ -contactable” at  $\omega$ .

**Remarks 2.1**

1) Let  $\Sigma$  be a valued monoid; a map  $h : M \rightarrow M'$  which is  $\Sigma$ -homogeneous and lipschitzian will be called  $\Sigma$ -Lhomogeneous.

2) A  $\Sigma$ -Lhomogeneous map is a  $\Sigma$ -homogeneous which is  $LL_\omega$ .

We denote  $\Sigma\text{-Contr}$ , the category whose objects are the  $\Sigma$ -contracting spaces and whose morphisms are the  $\Sigma$ -Lhomogeneous maps (it is a suitable world for guarantying the  $\Sigma$ -uniqueness property). When  $\sigma : \Sigma \rightarrow \Sigma'$  is a morphism of valued monoids, there exists a canonical functor  $\hat{\sigma} : \Sigma'\text{-Contr} \rightarrow \Sigma\text{-Contr}$ . For every valued monoid  $\Sigma$ , we also have another canonical functor  $U : \Sigma\text{-Contr} \rightarrow \mathbb{L}\mathbb{L}$  defined by  $U(M) = (M, \omega)$  and  $U(h) = h$ ; then we call  $J$  the following composite :  $\Sigma\text{-Contr} \xrightarrow{U} \mathbb{L}\mathbb{L} \xrightarrow{q} \text{Jet}$  (refer to the introduction for  $q : \mathbb{L}\mathbb{L} \rightarrow \text{Jet}$ ).

**Proposition 2.2**  *$\Sigma\text{-Contr}$  is a cartesian category and the previous functors  $U$ , and then  $J$ , are strict morphisms of cartesian categories.*

Now, since for each  $M, M' \in |\Sigma\text{-Contr}|$ , the canonical map :  $\Sigma\text{-Contr}(M, M') \xrightarrow{\text{can}} \mathbb{J}\text{et}(JM, JM')$ , defined by  $\text{can}(h) = J(h) = \text{Th}_\omega$ , is injective (thanks to the  $\Sigma$ -uniqueness property), we can equip  $\Sigma\text{-Contr}(M, M')$  with the distance  $d(h, h') = d(J(h), J(h'))$  (we recall (see [6]) that the category  $\mathbb{J}\text{et}$  is enriched in  $\text{Met}$ , the cartesian category whose objects are the metric spaces and morphisms, the *LSL* maps).

**Proposition 2.3**

1) *The cartesian category  $\Sigma\text{-Contr}$  is enriched in  $\text{Met}$ .*

2) *For each  $M, M_0, M_1 \in |\Sigma\text{-Contr}|$ , the following canonical map :  $\Sigma\text{-Contr}(M, M_0 \times M_1) \xrightarrow{\text{can}} \Sigma\text{-Contr}(M, M_0) \times \Sigma\text{-Contr}(M, M_1)$  is an isometry.*

**Proposition 2.4** *Let  $h, h' \in \Sigma\text{-Contr}(M, M')$ ; then, we have  $d(h, h') = \sup\{C(x) \mid x \in B'(\omega, 1)\}$ , where  $C(x) = \frac{d(h(x), h'(x))}{d(x, \omega)}$  if  $x \neq \omega$ ,  $C(\omega) = 0$ .*

**Proposition 2.5** *Let  $M, M'$  be  $\Sigma$ -contracting spaces where  $M'$  is revertible, and  $h : M \rightarrow M'$  a  $\Sigma$ -Lhomogeneous map. Let us set  $k = \sup\{\frac{d(h(x), h(y))}{d(x, y)} \mid x, y \in B'(\omega, 1); x \neq y\}$ . Then :*

1)  *$h$  is  $k$ -lipschitzian,*

2)  *$\rho(\text{Th}_\omega) = k$  (if  $\varphi$  is a jet, its lipschitzian ratio  $\rho(\varphi) = \inf K(\varphi)$  where  $K(\varphi) = \{k > 0 \mid \exists f \in \varphi, f \text{ is } k\text{-}LL_a\}$ ; see [6]).*

Proof : Come from 1.17. □

**Definition 2.6** *Consider two  $\Sigma$ -contracting spaces  $M, M'$  and a jet  $\varphi : (M, \omega) \rightarrow (M', \omega)$ . We say that :  $\varphi$  is  $\Sigma$ -representable if there exists a  $\Sigma$ -Lhomogeneous element  $h : M \rightarrow M'$  verifying  $J(h) = \varphi$  (i.e.  $\text{Th}_\omega = \varphi$ ). Thanks to the uniqueness theorem, such a  $h$  is unique, and may thus be called the  $\Sigma$ -representative element of the jet  $\varphi$ .*

**Remark 2.7** *The  $\Sigma$ -representable jets are stable under composition, and pairs (and thus products).*

We now call  *$\Sigma$ -contracting domain* a pair  $(M, V)$  where  $M$  is a  $\Sigma$ -contracting space and  $V$  a neighborhood of  $\omega$  in  $M$ . Besides,  $f : (M, V) \rightarrow (M', V')$  is said to be a centred map if  $(M, V)$  and  $(M', V')$  are  $\Sigma$ -contracting domains, and  $f : V \rightarrow V'$  verifies  $f(\omega) = \omega$ .

**Definition 2.8** Let  $f : (M, V) \longrightarrow (M', V')$  be a centred map. We say that  $f$  is  $\Sigma$ -contactable if  $f : V \longrightarrow V'$  is tangentiabile at  $\omega$  and if the following composite jet is  $\Sigma$ -representable :

$$(M, \omega) \xrightarrow{\sim} (V, \omega) \xrightarrow{Tf_\omega} (V', \omega) \xrightarrow{\sim} (M', \omega)$$

We denote  $K_\Sigma f : M \longrightarrow M'$  (or merely  $Kf$  if none ambiguity about  $\Sigma$ ) the unique representative element of the above composite jet; and we call it the  $\Sigma$ -contact of  $f$ .

**Remarks 2.9**

1) In other words,  $f$  is  $\Sigma$ -contactable if there exists a  $\Sigma$ -Lhomogeneous  $h : M \longrightarrow M'$  such that  $f \succ_\omega h|_V$  (where, here,  $f$  is seen as a map  $V \longrightarrow M'$ ). In that case,  $h = K_\Sigma f$ ; and  $K_\Sigma f \in \Sigma\text{-Contr}(M, M')$ .

2) Let  $\sigma : \Sigma \longrightarrow \Sigma'$  be a morphism of valued monoids. If  $f$  is a  $\Sigma'$ -contactable centred map,  $f$  is also  $\Sigma$ -contactable, with  $K_\Sigma f = K_{\Sigma'} f$ .

3) Let  $E, E'$  be n.v.s.,  $U$  an open subset of  $E$ ,  $a \in U$ ,  $f : U \longrightarrow E'$  a map. If  $f$  is differentiable at  $a$ , then  $f : (E_a, U) \longrightarrow (E'_{f(a)}, E')$  is standard  $\mathbb{R}$ -contactable with  $K_{\mathbb{R}} f(x) = f(a) + df_a(x-a)$ , its continuous affine tangent at  $a$ ; and, for every valued monoid  $\Sigma$ , it is even canonically  $\Sigma$ -contactable with  $K_\Sigma f = K_{\mathbb{R}} f$  written as above.

**Proposition 2.10** Let  $f : (M, V) \longrightarrow (M', V')$  be a centred  $\Sigma$ -contactable map. Then, for all  $x \in M$ ,  $Kf(x) = \lim_{0 \neq v(t) \rightarrow 0} t \overset{-1}{\star} f(t \star x)$ .

**Proposition 2.11** Here, for lightening, we omit the surrounding  $\Sigma$ -contracting spaces of the several neighborhoods of  $\omega$ .

1) Let  $f : V \longrightarrow V'$ ,  $g : V' \longrightarrow V''$  be two centred maps. If  $f$  and  $g$  are  $\Sigma$ -contactable, so is  $g.f$ , with  $K(g.f) = Kg.Kf$ .

2) Let  $f_0 : V \longrightarrow V_0$ ,  $f_1 : V \longrightarrow V_1$  be two centred maps. If  $f_0$  and  $f_1$  are  $\Sigma$ -contactable, so is the pair  $(f_0, f_1) : V \longrightarrow V_0 \times V_1$ , with  $K(f_0, f_1) = (Kf_0, Kf_1)$ .

3) Let  $f_0 : V_0 \longrightarrow V'_0$ ,  $f_1 : V_1 \longrightarrow V'_1$  be two centred maps. If  $f_0$  and  $f_1$  are  $\Sigma$ -contactable, so is the product  $f_0 \times f_1 : V_0 \times V_1 \longrightarrow V'_0 \times V'_1$ , with  $K(f_0 \times f_1) = Kf_0 \times Kf_1$ .

In the n.v.s. frame, we can define, with the help of an isometric translation, the analog  $kf_a$  of the differential at  $a$   $df_a$ , for the contact  $Kf$  (this one generalizing the continuous affine tangent to  $f$  at  $a$ ). More precisely, we recall that, for every n.v.s.  $E$  and  $a \in E$ , the pointed n.v.s.  $E_a$  is a  $\Sigma$ -contracting space for any valued monoid  $\Sigma$ . Now, if  $f : (E_a, U) \longrightarrow (E'_{f(a)}, U')$  is a  $\Sigma$ -contactable map (in the sens of 2.8 and 2.9), then we say that  $f$  is  $\Sigma$ -contactable at  $a$ , its  $\Sigma$ -contact at  $a$  being  $k_\Sigma f_a = \Theta_{af(a)}^{-1}(Kf_\Sigma)$ , in short  $kf_a = \Theta_{af(a)}^{-1}(Kf)$ ; where  $\Theta_{aa'} : \Sigma\text{-Contr}(E_0, E'_0) \longrightarrow \Sigma\text{-Contr}(E_a, E'_{a'})$  is the isometric translation defined by  $\Theta_{aa'}(h)(x) = a' + h(x - a)$ . The  $\Sigma$ -Lhomogeneous map  $kf_a : E_0 \longrightarrow E'_0$  is thus the translate at 0 of the  $\Sigma$ -Lhomogeneous map  $Kf : E_a \longrightarrow E'_{f(a)}$ , the  $\Sigma$ -contact of  $f$ .

The formulas of 2.11 can then be rewritten in this context for the  $kf_a$ , absolutely similar to those of  $df_a$  (the  $kf_a$  are stable under composition, pairs and products, with :  $k(g.f)_a = kg_{f(a)}kf_a$ ,  $k(f_1, f_2)_a = (kf_{1a}, kf_{2a})$  and  $k(f_1 \times f_2)_{(a_1, a_2)} = kf_{1a_1} \times kf_{2a_2}$ ).

**Remarks 2.12**

1) Let  $E, E'$  two n.v.s.,  $U$  an open subset of  $E$ ,  $a \in U$ ; then  $f : U \longrightarrow E'$  differentiable at  $a \implies f : (E_a, U) \longrightarrow (E'_{f(a)}, E')$   $\Sigma$ -contactable at  $a$  for any  $\Sigma$ ; with  $k_\Sigma f_a = df_a$ . More precisely, if  $f$  is  $\Sigma$ -contactable at  $a$ , then  $f$  is differentiable at  $a$  iff  $k_\Sigma f_a$  is linear.

2)  $f : (\mathbb{R}_a, U) \longrightarrow (E'_{f(a)}, U')$  is standard  $\mathbb{R}$ -contactable at  $a$  iff  $f : U \longrightarrow U'$  is differentiable at  $a$  with  $k_{\mathbb{R}} f_a = df_a$ . It is not always the case : see 2)below.

3) We prove here that standard  $\mathbb{R}$ -contactable at  $a \not\Rightarrow$  differentiable at  $a$  : consider the function  $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$  defined by  $f(0, 0) = 0$  and  $f(x, y) = \frac{xy^2}{x^2+y^2}$  if  $(x, y) \neq (0, 0)$ . This  $f$  is differentiable on  $\mathbb{R}^2 - \{(0, 0)\}$ , and, since for  $(x, y) \neq (0, 0)$ ,  $\|df_{(x,y)}\|$  is bounded,  $f$  is lipschitzian on  $\mathbb{R}^2$ . Besides, it is obviously  $\mathbb{R}$ -homogeneous, so that  $f$  is standard  $\mathbb{R}$ -contactable at 0, with  $k_{\mathbb{R}} f_0 = f$ . However, since  $f$  is not linear, it cannot be differentiable at 0 (see Figure 5 in [5]) !

**Proposition 2.13** *If  $f : (E_a, U) \longrightarrow (E'_{f(a)}, U')$  is  $\Sigma$ -contactable at  $a$ , we have :*

$$1) kf_a(x) = \lim_{0 \neq v(t) \rightarrow 0} \frac{f(a+v(t)x) - f(a)}{v(t)} \text{ for all } x \in E,$$

2)  $\rho(Tf_a) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x, y \in B'(0, 1), x \neq y\}$ , where  $h = kf_a$  is  $\rho(Tf_a)$ -lipschitzian (see 2.5).

### 3 G-differentiability

Here  $\Sigma = \mathbb{R}_+$ . René Gateaux<sup>1</sup>, defined maps (said “differentiable in the sense of Gateaux”) which are very close to our  $\mathbb{R}_+$ -contactable maps : the main difference being the fact that we replace his continuous maps (see Bouligand in [7]) by lipschitzian maps. In homage to Gateaux, we choose “G-differentiable” for “ $\mathbb{R}_+$ - contactable”<sup>2</sup>.

$E$  and  $E'$  being two n.v.s., we merely write  $\mathbb{H}om_+(E, E')$  for the n.v.s. that we should denote  $\mathbb{R}_+$ -Contr( $E_0, E'_0$ ); we thus recall that its elements are the  $\mathbb{R}_+$ -Lhomogeneous maps  $h : E_0 \rightarrow E'_0$ , i.e the maps  $h : E \rightarrow E'$  which are lipschitzian and verify  $h(tx) = th(x)$  for all  $t \in \mathbb{R}_+$  and  $x \in E$ .

#### Examples 3.1

1) Standard  $\mathbb{R}$ -Lhomogeneous implies  $\mathbb{R}_+$ -Lhomogeneous; in particular, the continuous linear maps from  $E$  to  $E'$  are in  $\mathbb{H}om_+(E, E')$ .

2) Let  $E$  be a n.v.s.; then every norm  $N$  on  $E$ , which is equivalent to the given norm  $\| \cdot \|$  on  $E$ , is in  $\mathbb{H}om_+(E, \mathbb{R})$ .

3) The maps Max, Min :  $\mathbb{R}^n \rightarrow \mathbb{R}$  are in  $\mathbb{H}om_+(\mathbb{R}^n, \mathbb{R})$ , since they are 1-lipschitzian.

#### Proposition 3.2

1) Let  $h \in \mathbb{H}om_+(E, E')$ , where  $E \neq \{0\}$ ; we have  $\|h\| = \sup\{\|h(x)\| \mid \|x\| = 1\}$ .

2) Let  $h \in \mathbb{H}om_+(E, E')$ , then, for all  $\varepsilon > 0$ , we have  $\rho(Th_0) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x \neq y; x, y \in C(\varepsilon)\}$ , where  $C(\varepsilon) = \{x \in E \mid 1 - \varepsilon < \|x\| < 1 + \varepsilon\}$ .

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1. The young French René Gateaux was one of the first victims of the first world war, he was twenty five years old when he died on the third of october 1914.

2. it is shorter than “lipschitzian Gateaux-differentiable” which would be more convenient.

*Proof* : 2) Let us set  $r(\varepsilon) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x \neq y; x, y \in C(\varepsilon)\}$ . It is clear that  $r(\varepsilon) \leq \rho(\text{Th}_0)$  since  $h$  is  $\rho(\text{Th}_0)$ -lipschitzian (by 2.12). Conversely, take  $a \in E, a \neq 0$  and let us set  $\varepsilon' = \|a\|\varepsilon$ . Then, for every  $x \in B(a, \varepsilon')$ , we have  $\frac{x}{\|a\|}, \frac{a}{\|a\|} \in C(\varepsilon)$ . Then  $r(\varepsilon) \geq \frac{\|h(\frac{x}{\|a\|})-h(\frac{a}{\|a\|})\|}{\|\frac{x}{\|a\|}-\frac{a}{\|a\|}\|} = \frac{\|h(x)-h(a)\|}{\|x-a\|}$  if  $x \neq a$ ; so that  $h$  is  $r(\varepsilon)$ - $LSL_a$ . In particular, for every  $a \in B'(0, 1) - \{0\}$ ,  $h$  is  $r(\varepsilon)$ - $LSL_a$ , with  $B'(0, 1)$  convex. Thus,  $h|_{B'(0,1)}$  is  $r(\varepsilon)$ -lipschitzian (see section 1 in [6]), so that  $\rho(\text{Th}_0) \leq r(\varepsilon)$ .  $\square$

**Proposition 3.3** *Let  $E$  be a n.v.s.; then the following map is a linear isometry :  $\text{can} : \mathbb{H}\text{om}_+(\mathbb{R}, E) \longrightarrow E \times E : h \mapsto (-h(-1), h(1))$ .*

**Definition 3.4** *A centred map  $f : (E_a, U) \longrightarrow (E'_{f(a)}, U')$  is said to be G-differentiable at  $a$  if it is  $\mathbb{R}_+$ -contactable at  $a$ ; in this case,  $K_+f$  and  $k_+f_a$  will respectively merely denote  $K_{\mathbb{R}_+}f$  and  $k_{\mathbb{R}_+}f_a$ .*

**Remarks 3.5** In the following cases,  $E$  and  $E'$  are n.v.s.,  $f : U \longrightarrow E'$  with  $U$  an open subset of  $E$ , and  $a \in U$ .

1) If  $f$  is differentiable at  $a$ , then  $f$  is G-differentiable at  $a$  with  $k_+f_a = df_a$ . More precisely, if  $f$  is G-differentiable at  $a$ , then  $f$  is differentiable at  $a$  iff  $k_+f_a$  is linear.

2) If  $f$  is G-differentiable at  $a$ , then  $f$  is standard  $\mathbb{R}$ -contactable at  $a$  iff  $k_+f_a$  is standard  $\mathbb{R}$ -homogeneous with  $k_+f_a = k_{\mathbb{R}}f_a$ .

3) If  $f$  is G-differentiable at  $a$ , then  $f$  is  $\Sigma$ -contactable at  $a$  for all valued monoid  $\Sigma$ , with  $k_{\Sigma}f_a = k_+f_a$ .

**Examples 3.6** The following examples are all G-differentiable :

1) The norm function  $\vartheta : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto |x|$ , which verifies  $k_+\vartheta_0 = \vartheta$  and, for  $a \neq 0, k_+\vartheta_a = d\vartheta_a = \text{sign}(a)Id_{\mathbb{R}}$  where  $\text{sign}(a) = \frac{a}{|a|}$ .

2) The euclidian norm function  $N^2 : \mathbb{R}^n \longrightarrow \mathbb{R} : x \mapsto \|x\|_2$ , which verifies  $k_+N^2_0 = N^2$  and, for  $a \neq 0, k_+N^2_a = dN^2_a$ .

3) The functions Max, Min :  $\mathbb{R}^n \longrightarrow \mathbb{R}$ , which verify :

$k_+\text{Max}_a(x) = \text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i)$  and  $k_+\text{Min}_a(x) = \text{Min}_{i \in \underline{\mathcal{I}}(a)}(x_i)$ , where  $\bar{\mathcal{I}}(a) = \{i \in \{1, \dots, n\} \mid a_i = \text{Max}(a)\}$  and  $\underline{\mathcal{I}}(a) = \{i \in \{1, \dots, n\} \mid a_i = \text{Min}(a)\}$ ; which provides  $k_+\text{Max}_0 = \text{Max}$  and  $k_+\text{Min}_0 = \text{Min}$ .

4) The product norm function  $N^\infty : \mathbb{R}^n \longrightarrow \mathbb{R} : x \mapsto \|x\|_\infty$ , which verifies  $k_+ N_0^\infty = N^\infty$  and, for  $a \neq 0$ ,  $k_+ N_a^\infty(x) = \text{Max}_{i \in \bar{\mathcal{I}}(|a|)}(\text{sign}(a_i)x_i)$ , where  $|a| = (|a_1|, \dots, |a_n|)$ .

5) The norm function  $N^1 : \mathbb{R}^n \longrightarrow \mathbb{R} : x \mapsto \|x\|_1$ , which verifies  $k_+ N_a^1(x) = \sum_{i \in \mathcal{I}_0(a)} |x_i| + \sum_{i \notin \mathcal{I}_0(a)} \text{sign}(a_i)x_i$ , where  $\mathcal{I}_0(a) = \{i \in \{1, \dots, n\} \mid a_i = 0\}$ .

Proof : First, all the above functions  $h$  being  $\mathbb{R}_+$ -Lhomogeneous, they are all G-differentiable at 0 with  $k_+ h_0 = h$ .

3) It comes from the equality  $\text{Max}(a + x) = \text{Max}(a) + \text{Max}_{i \in \bar{\mathcal{I}}(a)} x_i$ , which is true for all  $a$ , locally in a neighborhood of 0. Indeed :

- If  $\bar{\mathcal{I}}(a) = \{1, \dots, n\}$  (i.e, if  $a$  is a constant  $n$ -uplet), then, for all  $x \in \mathbb{R}^n$ , we have  $\text{Max}(a + x) = \text{Max}(a) + \text{Max}(x) = \text{Max}(a) + \text{Max}_{i \in \bar{\mathcal{I}}(a)} x_i$ .

- If  $\bar{\mathcal{I}}(a) \neq \{1, \dots, n\}$ . Let  $j \in \bar{\mathcal{I}}(a)$  such that  $\text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i) = x_j$ ; we can write :  $\text{Max}(a) + \text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i) = a_j + x_j \leq \text{Max}(a + x)$ . Conversely, we set  $r = \frac{1}{2} \text{Min}_{i \notin \bar{\mathcal{I}}(a)}(\text{Max}(a) - a_i)$ ; then  $r > 0$ . Let  $V$  be the open ball  $B_\infty(0, r)$  (for the product norm  $\| \cdot \|_\infty$ ); then, for  $x \in V$ , we have :

- if  $j \notin \bar{\mathcal{I}}(a)$ ,  $x_j - \text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i) \leq |x_j| + |\text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i)| \leq 2\|x\|_\infty < 2r \leq \text{Max}(a) - a_j$ , and thus again  $a_j + x_j \leq \text{Max}(a) + \text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i)$ ,

- if  $j \in \bar{\mathcal{I}}(a)$ ,  $a_j + x_j = \text{Max}(a) + x_j \leq \text{Max}(a) + \text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i)$ ; finally, for all  $x \in V$ , we have  $\text{Max}(a + x) \leq \text{Max}(a) + \text{Max}_{i \in \bar{\mathcal{I}}(a)}(x_i)$ ; hence the result. Same for Min.

4) Since  $N^\infty = \text{Max}.\vartheta^n$ , (where  $\vartheta^n = \vartheta \times \dots \times \vartheta$ ,  $n$  times), the function  $N^\infty$  is G-differentiable by composition (section 2).

5) Since  $N^1 = \sigma.\vartheta^n$ , where  $\sigma$  is the addition of  $\mathbb{R}^n$ ,  $N^1$  is G-differentiable still by composition. □

### Remarks 3.7

1) We are giving here, for each G-differentiable function  $h$  studied in 3.6, the domain  $D(h)$  on which  $h$  is differentiable.

a)  $D(\vartheta) = \mathbb{R}^* = \mathbb{R} - \{0\}$ .

b)  $D(N^2) = (\mathbb{R}^*)^n$ .

c)  $D(\text{Max}) = \{a \in \mathbb{R}^n \mid \exists! i \leq n (a_i = \text{Max}(a))\}$ . For instance, for  $n = 2$ ,  $D(\text{Max}) = \Delta^c$  where  $\Delta$  is the diagonal.

d) Here  $D(N^\infty) = \{a \in \mathbb{R}^n \mid \exists! i \leq n \ (|a_i| = \|a\|_\infty)\}$ . For  $n = 2$ ,  $D(N^\infty) = (\{a \in \mathbb{R}^2 \mid |a_1| = |a_2|\})^c$ .

e) For  $N^1$ , we have  $D(N^1) = \{a \in \mathbb{R}^n \mid \forall i \leq n \ a_i \neq 0\}$ .

2) Of course, there exist  $G$ -differentiable maps which are not  $\mathbb{R}_+$ -homogeneous : 3.6 gives a lot of such examples. Indeed, a translate  $g(x) = f(a + x)$  of a  $\mathbb{R}_+$ -homogeneous map  $f$  is not necessarily still  $\mathbb{R}_+$ -homogeneous, although, in our examples, such a translate remains  $G$ -differentiable (by composition).

**Proposition 3.8** *Let  $f : (\mathbb{R}_a, U) \longrightarrow (E'_{f(a)}, U')$  be a centred map. Then  $f$  is  $G$ -differentiable iff  $f$  admits left and right derivatives at  $a$ . In this case, referring to 3.3 for the linear isometry  $can$ , we have  $k_+f_a = can^{-1}(f'_l(a), f'_r(a))$ .*

### Continuously $G$ -differentiable maps

Our aim here is to prove that, in finite dimension, every *continuously  $G$ -differentiable* map  $f$  (i.e  $G$ -differentiable such that  $k_+f$  is continuous) is in fact of class  $C^1$ . We need some preliminary results.

**Proposition 3.9** *Let  $U$  be an open subset of  $\mathbb{R}$  and  $f : U \longrightarrow \mathbb{R}$  a continuous function admitting left and right derivatives at every point of  $U$  and such that the functions  $f'_l, f'_r : U \longrightarrow \mathbb{R}$  are continuous at  $a \in U$ . Then  $f'_l(a) = f'_r(a)$ , so that  $f$  is derivable at  $a$ .*

Proof : We need the following well-known lemma :

#### Lemma 3.10

1) *Let  $f : [a, b] \longrightarrow \mathbb{R}$  be a continuous function which admits a right derivative at every point of  $]a, b[$ , and  $k \in \mathbb{R}$ . Then,*

a) *If, for all  $t \in ]a, b[$   $f'_r(t) \leq k$ , then  $f(b) - f(a) \leq k(b - a)$ ,*

b) *If, for all  $t \in ]a, b[$   $f'_r(t) \geq k$ , then  $f(b) - f(a) \geq k(b - a)$ .*

2) *Same statements with  $f'_l$  instead of  $f'_r$ .*

We come back to the proof of 3.9 : Let  $\varepsilon > 0$ ; let us prove that  $f'_l(a) \leq f'_r(a) + \varepsilon$ . Since  $f'_r$  is continuous at  $a$ , there exists  $\eta > 0$  such



that  $f'_r(x) < f'_r(a) + \varepsilon$  for all  $x \in ]a - \eta, a + \eta[ \subset U$ . Let  $k = f'_r(a) + \varepsilon$ . Then, 3.10 provides that, for all  $x \in \mathbb{R}$  verifying  $a - \eta < x < a$ , we have  $f(a) - f(x) \leq k(a - x)$ , i.e.  $\frac{f(a) - f(x)}{a - x} \leq k$ , which gives  $f'_l(a) \leq k$  (doing  $x \rightarrow a-$ ); hence  $f'_l(a) \leq f'_r(a)$ , doing  $\varepsilon \rightarrow 0$ . Same for the reverse inequality.  $\square$

**Corollary 3.11** *Let  $f : (E, U) \rightarrow (\mathbb{R}, U')$  be a  $G$ -differentiable map, such that the map  $k_+ f : U \rightarrow \mathbb{H}om_+(E, \mathbb{R}) : x \mapsto k_+ f_x$  is continuous at  $a \in U$ . Then, for all  $v \in E$ , the directionnal derivative at  $a \frac{\partial f}{\partial v}(a) = \lim_{0 \neq t \rightarrow 0} \frac{f(a+tv) - f(a)}{t}$  exists in  $\mathbb{R}$ .*

**Theorem 3.12** *We assume that  $E$  and  $E'$  are n.v.s. of finite dimensions and that  $f : (E, U) \rightarrow (E', U')$  is a centred map which is continuously  $G$ -differentiable. Then,  $f$  is of class  $C^1$ .*

Proof: Begin first with  $E' = \mathbb{R}$ .  $\square$

## 4 Fractality and neo-fractality

Here  $\Sigma = \mathbb{N}'_r$  (see section 1). The interest of this particular case is to speak of fractality.

As in section 3, we remain in the n.v.s. context. Let us fix a real number  $0 < r < 1$ . Now,  $E, E'$  being two n.v.s., we specify (referring to 1.2 and 1.11 for  $\mathbb{N}'_r$ ) that the  $\mathbb{N}'_r$ -Lhomogeneous maps  $h : E_0 \rightarrow E'_0$  are the maps  $h : E \rightarrow E'$  which are lipschitzian and which satisfy the following fractality property :  $h(rx) = rh(x)$  for all  $x \in E$ ; such maps will be called “ $r$ -Lfractal”. Thus, we merely write  $\mathbb{F}rac_r(E, E')$  for the n.v.s. which we should denote  $\mathbb{N}'_r\text{-Contr}(E_0, E'_0)$  : see section 2.

### Examples 4.1

- 1) The  $\mathbb{R}_+$ -Lhomogeneous maps  $E_0 \rightarrow E'_0$  are  $r$ -Lfractal.
- 2) Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(0) = 0$  and  $f(x) = x \sin \log |x|$  for all  $x \neq 0$ . Then,  $f$  is  $r$ -Lfractal for  $r = e^{-2\pi}$ .
- 3) More generally, for  $p \in \{1, 2, \infty\}$ , the map  $f^p : \mathbb{R}^n \rightarrow \mathbb{R}^n : x \mapsto \lambda_p(x)x$ , where  $\lambda_p : \mathbb{R}^n \rightarrow \mathbb{R}$  is the function defined by  $\lambda_p(0) = 0$  and  $\lambda_p(x) = \sin \log \|x\|_p$  for  $x \neq 0$ . Then  $f^p$  is  $r$ -Lfractal for  $r = e^{-2\pi}$ .

4)  $\mathbb{K}$  being the triadic Cantor set, let  $K_\infty = \cup_{n \in \mathbb{N}} 3^n \mathbb{K}$  and  $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto d(x, K_\infty)$ . Then  $g$  is  $\frac{1}{3}$ -Lfractal.

Proof :

2) is a particular case of 3) ... see Fig 1.

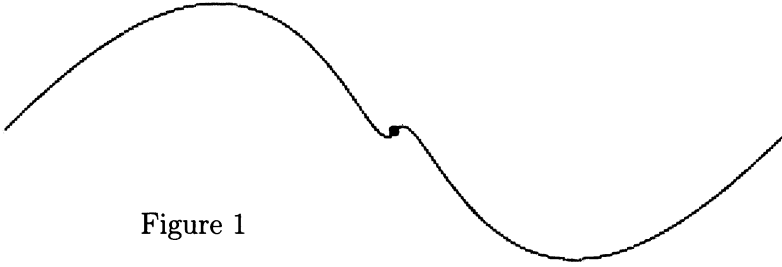


Figure 1

3) The fact that  $f^p$  is lipschitzian, comes from the formulas  $k_+ f_x^p(y) = (\sin \log \|x\|_p) y + \frac{\cos \log \|x\|_p}{\|x\|_p} k_+ N_x^p(y) x$  and  $|k_+ N_x^p(y)| \leq \|y\|_p$  (referring to the examples of 4.6). Using now 3.11, we obtain that  $f^p$  is 2-lipschitzian.

4)  $g$  is 1-lipschitzian. Besides, as  $\frac{1}{3} K_\infty = K_\infty$ , we have  $g(\frac{1}{3} x) = d(\frac{1}{3} x, K_\infty) = d(\frac{1}{3} x, \frac{1}{3} K_\infty) = \frac{1}{3} d(x, K_\infty) = \frac{1}{3} g(x)$ . See Fig 2<sup>3</sup>

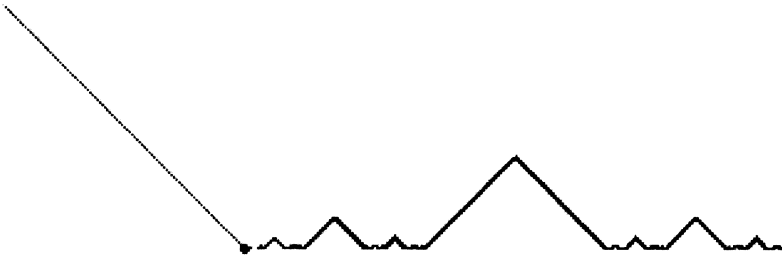


Figure 2

□

**Proposition 4.2** *Let  $h \in \text{Frac}_r(E, E')$ .*

1) *If  $E \neq \{0\}$ , we have :  $\|h\| = \sup\{\frac{\|h(x)\|}{\|x\|} \mid r < \|x\| \leq 1\}$ .*

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3. We could call  $g$  the “Giseh” function, if, as Napoleon, we gaze at the Giseh pyramides diminishing at the horizon !

2) For every  $\varepsilon > 0$ , we have  $\rho(\text{Th}_0) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x \neq y, x, y \in C(r, \varepsilon)\}$ , with  $C(r, \varepsilon) = \{x \in E \mid r < \|x\| < 1 + \varepsilon\}$ .

Proof: 2) Let us put  $R(\varepsilon) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x \neq y, x, y \in C(r, \varepsilon)\}$ . Clearly,  $R(\varepsilon) \leq \rho(\text{Th}_0)$  since  $h$  is  $\rho(\text{Th}_0)$ -lipschitzian (by 2.12). Let us now show that  $\rho(\text{Th}_0) \leq R(\varepsilon)$ . Let  $a \in E$  verifying  $0 < \|a\| \leq 1$  and  $n \in \mathbb{N}$  such that  $r^{n+1} < \|a\| \leq r^n$ , i.e  $r < \|r^{-n}a\| \leq 1$ . Let us put  $\varepsilon' = r^n \text{Min}\{\varepsilon, r^{-n}\|a\| - r\}$  and let  $x \in B(a, \varepsilon')$ . Then,  $r^{-n}a, r^{-n}x \in C(r, \varepsilon)$ ; so that, if  $x \neq a$ , we have  $R(\varepsilon) \geq \frac{\|h(r^{-n}x)-h(r^{-n}a)\|}{\|r^{-n}x-r^{-n}a\|} = \frac{\|h(x)-h(a)\|}{\|x-a\|}$ , which proves that  $h$  is  $R(\varepsilon)$ -LSL $_a$ . This being true for every  $a \in B'(0, 1) - \{0\}$ , we deduce that the restriction  $h|_{B'(0,1)}$  is  $R(\varepsilon)$ -lipschitzian (see section 1 in [5]) which finally implies that  $\rho(\text{Th}_0) \leq R(\varepsilon)$ .  $\square$

**Definition 4.3** A centred map  $f(E_a, U) \longrightarrow (E'_{f(a)}, U')$  is said to be  $r$ -neo-fractal at  $a \in U$  if  $f$  is  $\mathbb{N}'_r$ -contactable at  $a$ ; in this case,  $k_r f_a$  will merely denote  $k_{\mathbb{N}'_r} f_a$ .

**Remarks 4.4** In each case,  $E$  and  $E'$  are n.v.s. and  $f : U \longrightarrow E'$  where  $U$  is an open subset of  $E$ , and  $a \in U$ .

1) When  $f$  is G-differentiable at  $a$ , it is also  $r$ -neo-fractal for every  $0 < r < 1$ , and we have  $k_r f_a = k_+ f_a$ .

2) Every  $r$ -fractal map  $h : E \longrightarrow E'$  is  $r$ -neo-fractal at 0 and we have  $k_r h_0 = h$ .

3) Every  $r$ -neo-fractal map at  $a$  is tangential at  $a$ .

4) When  $f$  is  $r$ -neo-fractal at  $a$ , then  $f$  is G-differentiable at  $a$  iff  $k_r f_a \in \mathbb{H}\text{om}_+(E, E')$ ; in this case,  $k_+ f_a = k_r f_a$ .

### Examples 4.5

We consider successively the examples 2),3),4) already studied in 4.1 :

1) The function  $f$  is  $e^{-2\pi}$ -neo-fractal at 0, and differentiable on  $\mathbb{R}^*$ .

2) For  $p \in \{1, 2, \infty\}$ , the map  $f^p$  is  $e^{-2\pi}$ -neo-fractal at 0, and G-differentiable at every  $x \neq 0$ .

3) The Giseh function  $g$  is G-differentiable at every  $x \notin K_\infty$ . Furthermore, if we denote  $K_\infty^+$  and  $K_\infty^-$  the subsets of  $K_\infty$  defined, for  $x \in K_\infty$ , by :  $x \in K_\infty^+ \iff \exists \varepsilon > 0 \ (]x - \varepsilon, x[ \cap K_\infty = \emptyset)$ ,

$$x \in K_\infty^- \iff \exists \varepsilon > 0 \ (]x, x + \varepsilon[ \cap K_\infty = \emptyset),$$

then  $g$  is  $\frac{1}{3}$ -neo-fractal at every point of  $K_\infty^+ \cup K_\infty^-$ , and we have : for  $a \in K_\infty^+$ ,  $k_{\frac{1}{3}}g_a = g$ ; and for  $a \in K_\infty^-$ ,  $k_{\frac{1}{3}}g_a = g_-$ , where  $g_-(x) = g(-x)$ .

Proof : 1), 2) Use 4.4.

3) For the Giseh function, we verify that, in the neighborhood of  $a$ , we have  $g(x) = g(x - a)$  if  $a \in K_\infty^+$ , and  $g(x) = g(a - x)$  if  $a \in K_\infty^-$  ... for a detailed proof, see section 5 in [5].  $\square$

**Remarks 4.6** In the previous examples, we notice that :

1) a) Thanks to 4.4, we see that  $f$  is  $e^{-2\pi}$ -neo-fractal at 0, but not G-differentiable at 0. Same remark for the  $f^p$  where  $p \in \{1, 2, \infty\}$ .

b)  $g$  is not G-differentiable at all  $x \in K_\infty^+ \cup K_\infty^-$ , although it is  $\frac{1}{3}$ -neo-fractal at these points.

2) a) Of course, there exist neo-fractal maps which are not Lfractal : we have just, as in 3.7, to translate our previous examples at every point where they are neo-fractal.

b) As for the function  $f$  of 4.5 which remains differentiable at 0, although no more Lfractal, we obtain a convincing example considering the function  $x^2 + f(x)$  ... So guided, we can find a lot of other good examples of neo-fractal maps which are not Lfractal.

### Construction of fractal functions

Let  $s$  and  $T$  be strictly positive real numbers and  $f : \mathbb{R} \rightarrow \mathbb{R}$  a  $T$ -periodic and  $s$ -lipschitzian function which admits a right derivative at every point (we have  $|f'_r(x)| \leq s$  for all  $x \in \mathbb{R}$ ); in particular,  $f$  is bounded on  $\mathbb{R}$ . Then, we associate to  $f$  the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\varphi(0) = 0$  and, for  $x \neq 0$ ,  $\varphi(x) = xf(\log|x|)$ . Then  $\varphi$  admits a right derivative at every points of  $]0, +\infty[$  which is bounded, thus lipschitzian on  $\mathbb{R}$ . Clearly,  $\varphi$  is  $r$ -Lfractal.

Let us consider now the set  $\mathcal{P}_T$  of the  $T$ -periodic functions  $\mathbb{R} \rightarrow \mathbb{R}$  which are lipschitzian and which admit a right derivative at every point. Then,  $\mathcal{P}_T$  has a structure of vectorial subspace of  $\mathbb{R}^{\mathbb{R}}$ . The previous construction provides a map  $j : \mathcal{P}_T \rightarrow \text{Frac}_r(\mathbb{R}, \mathbb{R})$ , where  $j(f)$  is the

function  $\varphi$  associated to  $f$  as above. This map is clearly linear and injective. By composition, we have an injective linear map :

$$\mathcal{P}_T \xrightarrow{j} \text{Frac}_r(\mathbb{R}, \mathbb{R}) \xrightarrow{J} \text{Jet}((\mathbb{R}, 0), (\mathbb{R}, 0)) \xrightarrow{\text{can}} \text{Jet}_{\text{free}}(\mathbb{R}, \mathbb{R})$$

which provides the following statement (for the free jets, refer to [6]) :

**Proposition 4.7** *The space  $\text{Jet}_{\text{free}}(\mathbb{R}, \mathbb{R})$  is thus a vectorial space of infinite dimension.*

### Summary

The tangentiability maps have made in evidence new classes of maps. We give a recapitulative diagram of the various implications mentionned all along this paper :

$$\begin{array}{ccccccc}
 C^1 & \xrightarrow{1} & \text{Diff}_a & \xrightarrow{2} & \mathbb{R}\text{-cont}_a & \xrightarrow{2'} & G\text{-diff}_a & \xrightarrow{3} & r\text{-neofr}_a \\
 & \searrow 5 & & & & & & \swarrow 4 & \\
 & & LL_a & \xrightarrow{6} & \text{Tang}_a & \xrightarrow{7} & LSL_a & \xrightarrow{8} & C_a^0
 \end{array}$$

Where, here,  $\text{Diff}_a$ ,  $G\text{-diff}_a$  and  $r\text{-neofr}_a$  and  $\mathbb{R}\text{-cont}_a$  stand respectively for differentiable, G-differentiable ,  $r$ -neo-fractal and standard  $\mathbb{R}$ -contactable at  $a$  for all of them ( $C^1$  means “of class  $C^1$ ”, and  $C_a^0$  means continuous at  $a$ ). In the above diagram, every inverse implication is false ; we give counter-examples below.

### Counter-examples 4.8

In each case (except for 2), we denote  $f : \mathbb{R} \rightarrow \mathbb{R}$  the given counter-example (here  $a = 0$  and  $f(0) = 0$ ; the “number”  $i$ ) corresponding to a counter-example to the  $i^{\text{iet}}h$  above implication).

- 1)  $f(x) = x^2 \sin \frac{1}{x}$  (well-known),
- 2) see 2.12,
- 2')  $f(x) = |x|$  (see 3.6 and 3.7),
- 3)  $f(x) = x \sin(\log |x|)$  (see example 4.6 and 4.7),
- 4)  $f(x) = x \sin(\log |\log |x||)$  if  $x \neq 0$  [4].
- 5)  $f(x) = |x|$  (lipschitzian but not differentiable at 0).
- 6)  $f(x) = x^2 \sin \frac{1}{x^2}$  (see section 3 in [5]),

7)  $f(x) = x \sin \frac{1}{x}$  (same as for 6)),

8)  $f(x) = x^{\frac{1}{3}}$  (same as for 6)).

**Remarks 4.9** We complete the above diagram of implications, adding the following diagram (where  $\mathbb{R}$ -Lhom,  $\mathbb{R}_+$ -Lhom and  $r$ -Lfrac stand for standard  $\mathbb{R}$ -Lhomogeneous,  $\mathbb{R}_+$ -Lhomogeneous and  $r$ -Lfractal) :

$$\begin{array}{ccccc}
 \mathbb{R}\text{-Lhom} & \implies & \mathbb{R}_+\text{-Lhom} & \implies & r\text{-Lfrac} \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \mathbb{R}\text{-cont}_0 & \implies & G\text{-diff}_0 & \implies & r\text{-neofr}_0
 \end{array}$$

The inverse implications are false : refer to 2') and 3) in 4.8 for the horizontal non-implications, and to 3.7 and 4.6 for the vertical ones.

## 5 Local extrema

In this last section, we present nice generalisations of classical theorems about extrema of functions taking their values in  $\mathbb{R}$ . In particular, we give a sufficient condition for having an extremum which only needs hypotheses at order 1! In 5.1 and 5.2,  $\Sigma$  is a valued monoid.

**Theorem 5.1** *Let  $f : (M, U) \longrightarrow (\mathbb{R}_b, \mathbb{R})$  be a centred  $\Sigma$ -contactable function which admits a local minimum at  $\omega \in U$ ; then  $K_\Sigma f$  admits a global minimum at  $\omega$ .*

*Proof* : We recall (see 1.8 and 1.11) and that,  $\mathbb{R}_b$  is a revertible  $\Sigma$ -contracting space for the canonical structure, with  $t \star y = v(t)(y - b) + b$  (for every  $t \in \Sigma$  and  $y \in \mathbb{R}$ ), and  $t \overset{-1}{\star} y = t^{-1} \star y = \frac{y-b}{v(t)} + b$  (if  $t \neq 0$ ). If  $h = K_\Sigma f$ , we have  $h(x) = \lim_{0 \neq v(t) \rightarrow 0} t \overset{-1}{\star} f(t \star x) = \lim_{0 \neq v(t) \rightarrow 0} \frac{f(t \star x) - b}{v(t)} + b$ , for every  $x \in M$ . Since  $f$  admits a local minimum at  $\omega$ , we have  $f(x) \geq f(\omega) = b$  on a neighborhood  $V$  of  $\omega$  in  $U$ . Fixing  $x \in M$ , and since  $\lim_{v(t) \rightarrow 0} t \star x = \omega$ , there exists  $\varepsilon > 0$  such that, for all  $t \in \Sigma$ ,  $0 < v(t) < \varepsilon \implies t \star x \in V$ . So, when  $0 < v(t) < \varepsilon$ , we have  $f(t \star x) \geq b$ , which implies  $t \overset{-1}{\star} f(t \star x) = \frac{f(t \star x) - b}{v(t)} + b \geq b$ . Doing  $v(t) \rightarrow 0$ , we obtain  $h(x) \geq b = h(\omega)$ . □

**Corollary 5.2** *Let  $f : (E_a, U) \longrightarrow (\mathbb{R}_{f(a)}, \mathbb{R})$  a centred  $\Sigma$ -contactable function which admits a local minimum at  $a \in U$ ; then  $k_\Sigma f_a$  admits a global minimum at 0.*

Proof: It, comes from the fact that, for all  $x \in E$ , we have  $k_\Sigma f_a(x) = K_\Sigma f(x + a) - f(a) \geq 0 = k_\Sigma f_a(0)$ , where  $K_\Sigma f : E_a \longrightarrow \mathbb{R}_{f(a)}$ .  $\square$

**Remark 5.3** This gives back the well-known result of the differentiable case : “  $f$  admits a local minimum at  $a \implies a$  is a critical point of  $f$ ” since there exists a unique continuous linear function  $E \longrightarrow \mathbb{R}$  which admits a global minimum at 0 : the null function.

**Theorem 5.4** *Let  $f : (M, U) \longrightarrow (\mathbb{R}_b, \mathbb{R})$  a  $\mathbb{R}_+$ -contactable centred function. If  $M$  is a Daniel space (i.e a metric space in which every closed and bounded subset is compact) and if  $K_\Sigma f : M \longrightarrow \mathbb{R}_b$  admits a strict global minimum at  $\omega$ , then  $f$  admits a strict local minimum at  $\omega$ .*

Proof: We can suppose  $M \neq \{\omega\}$  and set  $S = \{x \in M \mid d(x, \omega) = 1\}$ . Then  $S$  is a non empty compact (if  $x \in M - \{\omega\}$ , we have  $\frac{1}{d(x, \omega)} \star x \in S$ ). Since  $h = K_\Sigma f$  is continuous,  $h$  reaches its inferior bound at  $x_0 \in S$ , so that  $h(x) \geq h(x_0) > b$  for all  $x \in S$ . Consider  $\varepsilon = h(x_0) - b > 0$ . Since  $f \succ_\omega h|_U$ , there exists  $\eta > 0$  such that  $B(\omega, \eta) \subset U$  and verifying the implication :  $0 < d(x, \omega) < \eta \implies |f(x) - h(x)| < \varepsilon d(x, \omega)$  for all  $x \in M$ . Let us fix  $x \in B(\omega, \eta) - \{\omega\}$ ; it verifies  $f(x) > h(x) - \varepsilon d(x, \omega)$ . If  $y = \frac{1}{d(x, \omega)} \star x$ , we have  $y \in S$ , so that  $h(y) \geq h(x_0)$  which implies  $h(y) - b - \varepsilon \geq h(x_0) - b - \varepsilon = 0$ . Hence (since  $h : M \longrightarrow \mathbb{R}_b$  is  $\mathbb{R}_+$ -homogeneous, where  $\mathbb{R}_+$  is a quasi-group)  $h(x) - \varepsilon d(x, \omega) = h(d(x, \omega) \star y) - \varepsilon d(x, \omega) = d(x, \omega)(h(y) - b) + b - \varepsilon d(x, \omega) = d(x, \omega)(h(y) - b - \varepsilon) + b \geq b$ . Thus, for all  $x \in B(\omega, \eta) - \{\omega\}$ , we have  $f(x) > h(x) - \varepsilon d(x, \omega) \geq b = f(\omega)$ .  $\square$

**Corollary 5.5** *Let  $f : (E_a, U) \longrightarrow (\mathbb{R}_{f(a)}, \mathbb{R})$  a centred  $G$ -differentiable map (where  $E$  is a n.v.s. of finite dimension), such that  $k_+ f_a > 0$  (i.e verifying  $k_+ f_a(x) > 0$  for every  $x \in E - \{a\}$ ). Then,  $f$  admits a strict local minimum at  $a$ .*

Proof: For all  $x \neq a$ , we have  $K_+ f(x) = f(a) + k_+ f_a(x - a) > f(a)$ .

**Remark 5.6** This theorem has not its equivalent, at order 1, in differential calculus, since a linear function cannot have a strict minimum. It is rather inspired by theorems giving sufficient conditions, at order 2, for the existence of extrema.

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