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ON BOUNDEDNESS AND SMALL-ORTHOGONALITY CLASSES

Dedicated to Jiří Adámek on the occasion of his sixtieth birthday

by Lurdes SOUSA

Abstract

Une caractérisation des catégories localement bornées et un critère pour identifier les sous-catégories α -orthogonales dans ces catégories (pour un cardinal régulier α) sont donnés.

1 Introduction

In [11], P. Gabriel and F. Ulmer proved that in locally presentable categories the orthogonal subcategory $\mathcal{N}^\perp$ is reflective for any set $\mathcal{N}$ of morphisms. The key point of the proof is the fact that for any object of the base category there is some infinite regular cardinal α such that the object is α -small, where α -smallness means α -presentability. In [10] and [15], P. Freyd and M. Kelly gave a generalization of this property for a wider range of categories, using a different concept of smallness for objects: boundedness. They showed that in a locally bounded category (as defined in [14] and [17]) the subcategory of all objects orthogonal to a set of morphisms is reflective. (In fact they went further: they proved that $\mathcal{N}^\perp$ is reflective for every class $\mathcal{N}$ which is the union of a set of morphisms with a class of epimorphisms.)

In a cocomplete category $\mathcal{A}$ an object A is said to be α -bounded if the hom-functor $\mathcal{A}(A, -)$ preserves α -directed unions. A locally bounded category (see [14]) is a complete and cocomplete category $\mathcal{A}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$ and an $\mathcal{E}$ -generator $\mathcal{G}$ such that (i) $\mathcal{A}$ has $\mathcal{E}$ -cointersections and (ii) there is a regular cardinal α such that each object of $\mathcal{G}$ is α -bounded. We call these categories *locally α -bounded* when they are $\mathcal{E}$ -cowellpowered and α is a regular cardinal which fits the condition (ii). Locally presentable categories and epi-reflective subcategories of the category of topological spaces are examples of locally bounded categories. We

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show that a cocomplete and cowellpowered category is locally bounded precisely when there is a regular cardinal α and a set $\mathcal{H}$ of α -bounded objects such that any object A of $\mathcal{A}$ is an α -directed union of objects of $\mathcal{H}$. This characterization will be useful in the study of small-orthogonality classes, that is, subcategories of the form $\mathcal{N}^\perp$ for $\mathcal{N}$ a set of morphisms.

In [13] the α -orthogonality classes of a locally α -presentable category were proved to be exactly the subcategories closed under limits and α -directed colimits, for all uncountable regular cardinals α . (Recall that, following [4], an α -orthogonality class is a subcategory of the form $\mathcal{N}^\perp$ for some set $\mathcal{N}$ of morphisms whose domains and codomains are α -presentable.) This characterization does not work for $\alpha = \aleph_0$, as was shown in [20] and [12]. A description of the $\aleph_0$ -orthogonality classes in locally finitely presentable categories in terms of closure properties was given in [5]: they are the subcategories $\mathcal{A}$ closed under products, directed colimits and $\mathcal{A}$ -pure subobjects. In the context of locally bounded categories we shall adopt the terminology *α -orthogonality class* as expected: the meaning is as in [4], just replacing “presentable” by “bounded”. The aim of this paper is to characterize the reflective subcategories of locally bounded categories which are small-orthogonality classes. In cowellpowered locally bounded categories a subcategory is a small-orthogonality class iff it is an α -orthogonality class for some α . We are going to restrict ourselves to reflective subcategories whose reflector preserves $\mathcal{M}$ -monomorphisms. For example, reflective subcategories of **Top** whose closure under subspaces is the category **Top**₀ of T_0 spaces have an $\mathcal{M}$ -preserving reflector, for $\mathcal{M} = \{\text{embeddings}\}$. Also the reflector from the category **Norm** of normed spaces and linear contractions into its subcategory **Ban** of Banach spaces preserves embeddings. In [18] Ringel studied the properties of $\mathcal{M}$ -preserving reflectors for $\mathcal{M}$ the class of monomorphisms. We show that, in locally α -bounded categories, a reflective subcategory with an $\mathcal{M}$ -preserving reflector is an α -orthogonality class iff it is closed under α -directed unions and α - $\mathcal{B}$ -neat subobjects. (The notion of α - $\mathcal{B}$ -neat morphism is parallel to the one of α - $\mathcal{B}$ -pure morphism, used in [5]: If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, a morphism $f : A \rightarrow B$ of $\mathcal{A}$ is said to be α - $\mathcal{B}$ -neat provided that, if we have morphisms e , u and v such that $f \cdot u = v \cdot e$ and e is a $\mathcal{B}$ -epimorphism, then there exists a morphism u' such that $u' \cdot e = u$.) For instance, the category **Top**₀ is an $\aleph_0$ -orthogonality class of **Top**, but the category **Sob** of sober spaces is not an $\aleph_0$ -orthogonality class of **Top**₀. The category **Ban** is an $\aleph_1$ -orthogonality class of **Norm**.

2 Properties of locally bounded categories

Let $\mathcal{A}$ be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ (where proper means that $\mathcal{E}$ and $\mathcal{M}$ consist of epimorphisms and monomorphisms, respectively). Recall that $\mathcal{E}$ and $\mathcal{M}$ determine each other: $\mathcal{E} = \mathcal{M}^\uparrow$ and $\mathcal{M} = \mathcal{E}^\downarrow$ ([10]).

A set $\mathcal{G}$ is said to be an $\mathcal{E}$ -generator of $\mathcal{A}$ if for each object A there is some subset $\{G_i, i \in I\}$ of $\mathcal{G}$ and an $\mathcal{E}$ -morphism $e : \coprod_{i \in I} G_i \rightarrow A$. (A detailed study of $\mathcal{E}$ -generators is made in, e.g., [6] and [7].)

Let $m_i : A_i \rightarrow A$, $i \in I$, be a diagram in $\mathcal{A}$ with all $m_i \in \mathcal{M}$. The $\mathcal{M}$ -union (or just *union*) of $(m_i)_{i \in I}$ is the supremum of $(m_i)_{i \in I}$, up to isomorphism, in the class of all $\mathcal{M}$ -subobjects of A . It coincides with the $\mathcal{M}$ -part $m : B \rightarrow A$ of the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism $\coprod_{i \in I} A_i \rightarrow A$. We shall often write $\cup_{i \in I} m_i = m$ or $\cup_{i \in I} A_i = B$ for short.

Let α be an infinite regular cardinal. An object A is said to be α -bounded if the hom-functor $\mathcal{A}(A, -)$ preserves α -directed unions.

2.1. Definition (1) ([14], [17]) A category $\mathcal{A}$ is said to be *locally bounded* if it is cocomplete, has a proper factorization system $(\mathcal{E}, \mathcal{M})$, and there is an infinite regular cardinal α such that:

- (i) $\mathcal{A}$ has $\mathcal{E}$ -cointersections;
- (ii) $\mathcal{A}$ has an $\mathcal{E}$ -generator all of whose objects are α -bounded.

(2) By a *locally α -bounded category with respect to $\mathcal{M}$* we shall mean a category under the conditions of (1), for a given α , which moreover is $\mathcal{E}$ -cowellpowered. The reference to $\mathcal{M}$ will often be omitted.

2.2. Remark Every locally bounded category is complete. In [14] and [17], the authors include completeness in the definition of locally bounded category. However the completeness comes for free, since any $\mathcal{E}$ -cocomplete category with an $\mathcal{E}$ -generator is complete. This follows from the fact that any such category is total (see [7]), that is, the Yoneda embedding $\mathcal{A} \hookrightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$ has a left adjoint ([16]); and any total category is complete and $\mathcal{M}$ -complete (see [7] and [8]).

2.3. Examples (1) Every locally presentable category is locally bounded with respect to monomorphisms, and also with respect to strong monomorphisms (see [10] and [2]).

(2) The category **Top** of topological spaces is locally $\aleph_0$ -bounded with respect to strong monomorphisms (= embeddings). And every epi-reflective subcategory

of **Top** is locally $\aleph_0$ -bounded with respect to embeddings. More generally, any $\mathcal{E}$ -reflective subcategory $\mathcal{B}$ of a locally α -bounded category with respect to $\mathcal{M}$ is also locally α -bounded with respect to $\mathcal{M} \cap \text{Mor}(\mathcal{B})$ ([10], [2]).

(3) Any topological category over **Set** (see [3]) is locally $\aleph_0$ -bounded with respect to strong monomorphisms.

(4) The category **Ban** of Banach spaces and linear contractions is locally $\aleph_1$ -bounded ([14], [17]).

2.4. Remark The following properties are easily verified:

(i) In a locally bounded category, for every object A there is an infinite regular cardinal α such that A is α -bounded ([10], 3.1.2).

(ii) In a cocomplete category if β and γ are regular cardinals such that $\beta \leq \gamma$, then every β -bounded object is also γ -bounded; consequently, the fulfillment of 2.1 for $\alpha = \beta$ ensures that it also holds for $\alpha = \gamma$.

2.5. Lemma *In a cocomplete category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ any $\mathcal{E}$ -quotient of an α -bounded object is α -bounded.*

Proof Let B be α -bounded, let $e : B \rightarrow E$ belong to $\mathcal{E}$ and let

$$C_i \xrightarrow{n_i} C \quad (i \in I)$$

be an α -directed $\mathcal{M}$ -union, that is, $1_C = \cup_{i \in I} n_i$. Given $f : E \rightarrow C$, there are some i and some morphism $f' : B \rightarrow C_i$ such that $f \cdot e = n_i \cdot f'$. Then, since $n_i \in \mathcal{M}$ and $e \in \mathcal{M}^\uparrow$, there exists $f'' : E \rightarrow C_i$ such that $f = n_i \cdot f''$. $\square$

2.6. Remark The property stated in Lemma 2.5 is in contrast to the case of α -presentability: a quotient of an α -presentable object is not necessarily α -presentable (see Remark 1.3 of [4]).

2.7. Lemma *In a cocomplete category with a proper $(\mathcal{E}, \mathcal{M})$ factorization system:*

- (i) *any α -small colimit of α -bounded objects is α -bounded;*
- (ii) *any α -small union of α -bounded objects is α -bounded.*

Proof (i) We are going to prove the statement for the particular case of coproducts. Then the result follows for colimits taking into account Lemma 2.5 and the fact that $\mathcal{M} \subseteq \text{Mono}$ implies that $\text{RegEpi} \subseteq \mathcal{E}$.

Let A_k ($k \in K$) be an α -small set of α -bounded objects. Let $c_i : C_i \rightarrow C$ ($i \in I$) be an α -directed union, and consider a morphism $d : \coprod_{k \in K} A_k \rightarrow C$. Since every A_k is α -bounded, there are morphisms $f_k : A_k \rightarrow C_{i_k}$ such that $d \cdot \nu_k = c_{i_k} \cdot f_k$ for all k (where ν_k are the injections of the coproduct). Since K is α -small and

I is α -directed, there is some $i \in I$ such that $i_k \leq i$, $k \in K$. Then, putting $g_k = (A_k \xrightarrow{f_k} C_{i_k} \longrightarrow C_i)$, we obtain $c_i \cdot g_k = d \cdot \nu_k$. Let $h : \coprod A_k \rightarrow C_i$ be the morphism determined by the morphisms g_k and the universality of the coproduct. Then we have $d = c_i \cdot h$.

(ii) Let $m_k : A_k \rightarrow A$ ($k \in K$) be a union (not necessarily α -directed) with K α -small and all A_k α -bounded. Let $c_i : C_i \rightarrow C$ ($i \in I$) be an α -directed union, and consider a morphism $f : A \rightarrow C$. Since $1_A = \cup_{k \in K} m_k$, the induced canonical morphism $e : \coprod A_k \rightarrow A$ belongs to $\mathcal{E}$. Put

$$d = f \cdot e$$

and let i and $h : \coprod A_k \rightarrow C_i$ be obtained as in (i). Then, we have the following commutative diagram:

$$\begin{array}{ccc} \coprod A_k & \xrightarrow{e} & A = \cup A_k \\ h \downarrow & & \downarrow f \\ C_i & \xrightarrow{c_i} & C \end{array}$$

By the diagonal fill-in property, there exists a morphism $t : A \rightarrow C_i$ such that $c_i \cdot t = f$. $\square$

2.8. Theorem *Let $\mathcal{A}$ be a cocomplete and $\mathcal{E}$ -cowellpowered category with a proper factorization system $(\mathcal{E}, \mathcal{M})$. The following conditions are equivalent:*

(i) $\mathcal{A}$ is locally α -bounded with respect to $\mathcal{M}$.

(ii) There is a set $\mathcal{H}$ of α -bounded objects such that any object of $\mathcal{A}$ is an α -directed $\mathcal{M}$ -union of objects of $\mathcal{H}$.

Proof (ii) $\Rightarrow$ (i): It is clear that if $\mathcal{H}$ is a set as in (ii), then it is an $\mathcal{E}$ -generator of $\mathcal{A}$. In fact, given $A \in \mathcal{A}$, let $H_i \xrightarrow{m_i} A$ ($i \in I$) be an α -directed $\mathcal{M}$ -union, with all H_i in $\mathcal{H}$. This means exactly that the induced canonical morphism $\coprod H_i \rightarrow A$ belongs to $\mathcal{E}$.

(i) $\Rightarrow$ (ii): Let $\mathcal{G}$ be an $\mathcal{E}$ -generator of $\mathcal{A}$ with all objects α -bounded. The class of objects

$$\mathcal{H} = \{ \mathcal{E}\text{-quotients of } \alpha\text{-small coproducts of objects of } \mathcal{G} \}$$

is essentially small, because $\mathcal{G}$ is small and $\mathcal{A}$ is $\mathcal{E}$ -cowellpowered. Moreover, from 2.5 and 2.7, the objects of $\mathcal{H}$ are α -bounded. We show that $\mathcal{H}$ fulfils (ii).

Let $A \in \mathcal{A}$, and let

$$\{f_i : G_i \rightarrow A, i \in I\} = \bigcup_{G \in \mathcal{G}} \mathcal{A}(G, A).$$

Let

$$\mathcal{J} = \{J \subseteq I : J \text{ is } \alpha\text{-small}\}.$$

Consider the following commutative diagram

$$\begin{array}{ccccc}
 G_j & & & & \\
 \nu_j^J \downarrow & \searrow f_j & & & \\
 G_J & \xrightarrow{e_J} & Q_J & \xrightarrow{m_J} & A \\
 \nu_j^K \downarrow & & \downarrow d_J^K & \nearrow m_K & \\
 G_K & \xrightarrow{e_K} & Q_K & &
 \end{array}$$

where:

- $G_J = \coprod_{j \in J} G_j$ and the morphisms ν_j^J are the corresponding injections;
- for each $J \subseteq K$, $\nu_j^K : G_J \rightarrow G_K$ is the obvious canonical morphism;
- $f_j : G_j \rightarrow A$ is the morphism determined by $f_j, j \in J$;
- $m_J \cdot e_J$ is the $(\mathcal{E}, \mathcal{M})$ factorization of $f_j : G_J \rightarrow A$;
- for each $J \subseteq K$, $d_J^K : Q_J \rightarrow Q_K$ is the morphism given by the diagonal fill-in property applied to the equality $(m_K \cdot e_K) \cdot \nu_j^K = m_J \cdot e_J$.

For $\mathcal{J}$ equipped with the inclusion order, both the diagrams

$$(\nu_j^K : G_J \rightarrow G_K)_{J \subseteq K, J, K \in \mathcal{J}} \quad \text{and} \quad (d_J^K : Q_J \rightarrow Q_K)_{J \subseteq K, J, K \in \mathcal{J}}$$

are α -directed. Moreover the colimit of the former one is $\coprod_{i \in I} G_i$. Let $\gamma_J : Q_J \rightarrow C = \text{Colim } Q_J$ be the colimit cocone of the latter one. Then there is a morphism $e : \coprod_{i \in I} G_i \rightarrow C$ making the left-hand square of the following diagram commutative.

$$\begin{array}{ccccc}
 G_J & \xrightarrow{e_J} & Q_J & \xrightarrow{m_J} & A \\
 \nu_J \downarrow & & \downarrow \gamma_J & & \uparrow m' \\
 \coprod_{i \in I} G_i & \xrightarrow{e} & C & \xrightarrow{e'} & \cup_{J \in \mathcal{J}} Q_J
 \end{array}$$

The morphism e belongs to $\mathcal{E}$, since all e_J do. Let $m' \cdot e'$ be the $(\mathcal{E}, \mathcal{M})$ factorization of the canonical morphism from C to A determined by the morphisms m_J . By hypothesis, $m' \cdot (e' \cdot e) : \coprod_{i \in I} G_i \rightarrow A$ belongs to $\mathcal{E}$ (because $\mathcal{G}$ is an $\mathcal{E}$ -generator). Consequently, m' lies in $\mathcal{E}$, and, since it also belongs to $\mathcal{M}$, is an isomorphism, that is, A is an union of the $\mathcal{M}$ -subobjects

$$m_J : Q_J \rightarrow A, \quad J \in \mathcal{J}.$$

□

2.9. Corollary *A locally bounded category is $\mathcal{E}$ -cowellpowered iff for every regular infinite cardinal β the class of all β -bounded objects is essentially small.*

Proof Let $\mathcal{A}$ be locally α -bounded. Without loss of generality we assume that $\beta \geq \alpha$. Then $\mathcal{A}$ is also locally β -bounded and has a set $\mathcal{H}$ of β -bounded objects such that any object of $\mathcal{A}$ is an $\mathcal{M}$ -union of objects of $\mathcal{H}$. Given a β -bounded object A let $m_i : H_i \rightarrow A$ ($i \in I$) be that existing union. The β -boundedness of A implies the equality $m_i \cdot t = 1_A$ for some $t : A \rightarrow H_i$. But then $A \simeq H_i$.

Conversely, let $\mathcal{A}$ be a category fulfilling the conditions of 2.1(1), and such that for every regular infinite cardinal β the class of all β -bounded objects is essentially small. Given an object X of $\mathcal{A}$, there is some regular infinite cardinal β such that X is β -bounded (see 2.4(i)). Consequently, by 2.5, the class of $\mathcal{E}$ -quotients of X has a representative set. $\square$

3 Small-orthogonality classes

In this section we study the following problem: When is a reflective subcategory¹ $\mathcal{B}$ of a locally bounded category $\mathcal{A}$ a *small-orthogonality class*, i.e., a category of the form $\mathcal{N}^\perp$, for $\mathcal{N}$ a set of morphisms? In this study we restrict ourselves to the particular case of the reflector $R : \mathcal{A} \rightarrow \mathcal{B}$ preserving $\mathcal{M}$ -monomorphisms. More precisely, we characterize those reflective subcategories of a locally α -bounded category with an $\mathcal{M}$ -preserving reflector which are of the form $\mathcal{N}^\perp$ with all morphisms of $\mathcal{N}$ having α -bounded domains and codomains.

In the case of locally presentable categories the subcategories of the form $\mathcal{N}^\perp$ for $\mathcal{N}$ a set of morphisms with α -presentable domains and codomains were characterized in [13] and [5] (see Introduction).

Throughout this section by an α -orthogonality class of a locally bounded category we shall mean a subcategory of the form $\mathcal{N}^\perp$ for some set $\mathcal{N}$ whose all morphisms have α -bounded domains and codomains. We borrow this terminology from [4] using boundedness instead of presentability.

3.1. Remark Recall that, for a subcategory $\mathcal{B}$ of $\mathcal{A}$, a morphism $g : C \rightarrow D$ of $\mathcal{A}$ is said to be a $\mathcal{B}$ -epimorphism if for any pair of morphisms $a, b : D \rightarrow B$ with $B \in \mathcal{B}$, the equality $a \cdot g = b \cdot g$ implies $a = b$.

Let $\mathcal{A} = \mathbf{Top}$. If $\mathcal{B} = \mathbf{Haus}$ the $\mathcal{B}$ -epimorphisms are just the dense morphisms of $\mathbf{Top}$. If $\mathcal{B} = \mathbf{Top}_0$ the $\mathcal{B}$ -epimorphisms are the b -dense morphisms, i.e., the continuous maps $f : X \rightarrow Y$ such that $\{y\} \cap H \cap f(X) \neq \emptyset$ for each $y \in Y$ and

¹Throughout this paper all subcategories are assumed to be full and isomorphism-closed.

each open set H of Y containing y . More generally, if $\mathcal{A}$ has equalizers and a proper factorization system $(\mathcal{E}, \mathcal{M})$, then for any subcategory $\mathcal{B}$ of $\mathcal{A}$ the $\mathcal{B}$ -epimorphisms are the morphisms which are dense with respect to the regular closure operator induced in $\mathcal{A}$ by $\mathcal{B}$ ([9]).

If $\mathcal{B}$ is reflective in $\mathcal{A}$ it is easy to see that the $\mathcal{B}$ -epimorphisms are just those morphisms of $\mathcal{A}$ whose image by the reflector is an epimorphism in $\mathcal{B}$.

3.2. Definition Let $\mathcal{A}$ be a locally bounded category and let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. A morphism $f : A \rightarrow B$ of $\mathcal{A}$ is said to be α - $\mathcal{B}$ -neat provided that in each commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ u \downarrow & & \downarrow v \\ A & \xrightarrow{f} & B \end{array}$$

with C and D α -bounded and g a $\mathcal{B}$ -epimorphism, u factorizes through g , i.e., $u = u' \cdot g$ for some u' .

3.3. Remark The following properties are easily established (compare with the properties of $\mathcal{B}$ -pure morphisms in [5]):

- (i) The composition of α - $\mathcal{B}$ -neat morphisms is an α - $\mathcal{B}$ -neat morphism.
- (ii) If $f \cdot g$ is α - $\mathcal{B}$ -neat then g is α - $\mathcal{B}$ -neat.
- (iii) Every γ - $\mathcal{B}$ -neat morphism is α - $\mathcal{B}$ -neat for $\gamma \geq \alpha$.
- (iv) All α - $\mathcal{B}$ -neat morphisms are monomorphisms; and every equalizer is an α - $\mathcal{B}$ -neat morphism.
- (v) If $\mathcal{B}$ is cogenerating in $\mathcal{A}$, then

$$StrongMono(\mathcal{A}) \subseteq \{\alpha\text{-}\mathcal{B}\text{-neat morphisms}\}.$$

The last statement follows from the fact that, in this case, every $\mathcal{B}$ -epimorphism is an epimorphism in $\mathcal{A}$.

3.4. Proposition Let $\mathcal{A}$ be a locally α -bounded category with respect to $\mathcal{M}$. Then any α -orthogonality class of $\mathcal{A}$ is a reflective subcategory of $\mathcal{A}$ which is

- (i) closed under α -directed $\mathcal{M}$ -unions;
- (ii) locally α -bounded with respect to $\mathcal{M}' = \mathcal{M} \cap Mor(\mathcal{B})$;
- (iii) closed under α - $\mathcal{B}$ -neat subobjects.

Proof Let $\mathcal{B} = \mathcal{N}^\perp$ for $\mathcal{N}$ a set of morphisms in $\mathcal{A}$ with α -bounded domains and codomains. From [10], we know that $\mathcal{B}$ is reflective and has an $(\mathcal{E}', \mathcal{M}')$ proper factorization system, with $\mathcal{E}' = (\mathcal{M}')^\dagger$. Moreover, cowellpoweredness of $\mathcal{A}$ with respect to $\mathcal{E}$ implies $\mathcal{E}'$ -cowellpoweredness of $\mathcal{B}$. Let $R : \mathcal{A} \rightarrow \mathcal{B}$ be the reflector.

- (i) Let

$$b_i : B_i \rightarrow Z \quad (i \in I)$$

be an α -directed $\mathcal{M}$ -union in $\mathcal{A}$ with all $B_i \in \mathcal{B}$. We want to show that $Z \in \mathcal{B} = \mathcal{N}^\perp$. Let $h : X \rightarrow Y$ be a morphism of $\mathcal{N}$ and let $f : X \rightarrow Z$. Since X is α -bounded there is some i and some $f' : X \rightarrow B_i$ such that $b_i \cdot f' = f$. The morphism f' factorizes through h , because $B_i \in \mathcal{B}$, and, hence, so does the morphism f . To show the uniqueness of the last factorization, let $y, y' : Y \rightarrow Z$ be such that $y \cdot h = y' \cdot h$. Since Y is α -bounded, we can find $k \in I$ and $t, t' : Y \rightarrow B_k$ such that $y = b_k \cdot t$ and $y' = b_k \cdot t'$. Now the equality $b_k \cdot t \cdot h = b_k \cdot t' \cdot h$, the orthogonality of B_k to h and the fact that $b_k \in \mathcal{M}$ imply that $t = t'$, thus $y = y'$.

(ii) Of course $\mathcal{B}$ is cocomplete. Moreover:

(a) If X is an α -bounded object of $\mathcal{A}$, then RX is an α -bounded object of $\mathcal{B}$. This is clear since, from (i), every α -directed $\mathcal{M}'$ -union in $\mathcal{B}$ is an α -directed $\mathcal{M}$ -union in $\mathcal{A}$.

(b) If $\mathcal{G}$ is an $\mathcal{E}$ -generator of $\mathcal{A}$ then it is well known that $R(\mathcal{G})$ is an $\mathcal{E}'$ -generator of $\mathcal{B}$ ([10]). In fact, let $A \in \mathcal{B}$, and let $e : \coprod_{i \in I} G_i \rightarrow A$ be a morphism of $\mathcal{E}$ with all G_i in $\mathcal{G}$. Then the morphism $Re : \coprod_{i \in I} RG_i \rightarrow A$ belongs to $\mathcal{E}'$ since, as it is easily seen, $R(\mathcal{E}) \subseteq (\mathcal{M}')^\perp$.

(iii) Let $m : Z \rightarrow B$ be an α - $\mathcal{B}$ -neat morphism with $B \in \mathcal{B}$. We want to show that $Z \in \mathcal{B}$. Let $h : X \rightarrow Y$ lay in $\mathcal{N}$. Given a morphism $f : X \rightarrow Z$, since $B \in \mathcal{N}^\perp$, we get f' such that $f' \cdot h = m \cdot f$. Because m is α - $\mathcal{B}$ -neat, there is f'' such that $f'' \cdot h = f$. The uniqueness of f'' follows from the fact that $m \cdot f$ factors uniquely through h and m is a monomorphism. $\square$

3.5. Remark Let $\mathcal{A}$ be a locally α -bounded category with respect to $\mathcal{M}$. Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$ which is locally α -bounded with respect to $\mathcal{M} \cap \text{Mor}(\mathcal{B})$ and closed under limits and under α -directed $\mathcal{M}$ -unions. Then $\mathcal{B}$ is reflective. In fact, the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ fulfils the solution set condition: Given $A \in \mathcal{A}$, there is some regular cardinal $\lambda \geq \alpha$ such that A is λ -bounded in $\mathcal{A}$ and $\mathcal{B}$ is a locally λ -bounded category. Consequently, there is a set $\{B_i, i \in I\}$ of λ -bounded objects of $\mathcal{B}$ such that every object of $\mathcal{B}$ is a λ -directed $\mathcal{M} \cap \text{Mor}(\mathcal{B})$ -union of B_i 's. But, being closed in $\mathcal{A}$ under α -directed unions, $\mathcal{B}$ is also closed under λ -directed unions. Then, any morphism $g : A \rightarrow B$ with codomain in $\mathcal{B}$ factorizes through some of the objects B_i .

Next we want to characterize the reflective subcategories of a locally bounded category which are small-orthogonality classes. We restrict ourselves to reflective subcategories whose reflector preserves $\mathcal{M}$ -monomorphisms. This kind of reflectors were studied by Ringel in [18], for $\mathcal{M} = \{\text{monomorphisms}\}$. $\mathbf{Top}_0$ and $\mathbf{Sob}$ are examples of subcategories of $\mathbf{Top}$ whose reflector preserves embeddings. Let $\mathbf{Sob}_\alpha$ denote the limit-closure in $\mathbf{Top}$ of the ordinal α regarded as a topological

space with the Alexandrov topology. Both **Top** and **Top**₀ have an {embeddings}-preserving reflector into **Sob**_α (see [19]). Also the inclusion functor of the category **Ban** of Banach spaces into the category **Norm** of normed spaces and linear contractions has a reflector which preserves embeddings.

3.6. Theorem *Let $\mathcal{A}$ be a locally α -bounded category with respect to $\mathcal{M}$. Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$ whose reflector preserves morphisms of $\mathcal{M}$. Then $\mathcal{B}$ is an α -orthogonality class in $\mathcal{A}$ iff it is closed under α -directed $\mathcal{M}$ -unions and α - $\mathcal{B}$ -neat subobjects.*

Proof The necessity was proved in 3.4.

In order to prove the sufficiency, we first show that the reflector $R : \mathcal{A} \rightarrow \mathcal{B}$ preserves α -directed $\mathcal{M}$ -unions. Given an α -directed $\mathcal{M}$ -union $m_i : X_i \rightarrow X$ ($i \in I$), we have commutative diagrams

$$\begin{array}{ccccc}
 X_i & \xrightarrow{r_{X_i}} & RX_i & & \\
 \downarrow m_i & \searrow \nu_i & \swarrow R\nu_i & & \downarrow Rm_i \\
 & \coprod_{i \in I} X_i & \xrightarrow{r} & \coprod_{i \in I} RX_i & \\
 & \swarrow e & & \searrow Re & \\
 X & \xrightarrow{r_X} & RX & &
 \end{array}$$

where $e \in \mathcal{E}$. But, as is easy to see, $R(\mathcal{E}) \subseteq \mathcal{E}' = (\mathcal{M}')^\dagger$ for $\mathcal{M}' = \mathcal{M} \cap \text{Mor}(\mathcal{B})$. Then the morphisms $Rm_i : RX_i \rightarrow RX$ form an $\mathcal{M}'$ -union in $\mathcal{B}$.

To finish the proof, we show that, for

$$\mathcal{N} = \{h : X \rightarrow Y \text{ in } \mathcal{A}, h \perp \mathcal{B}, X, Y \text{ } \alpha\text{-bounded}\},$$

$\mathcal{N}^\perp \subseteq \mathcal{B}$, and thus $\mathcal{B} = \mathcal{N}^\perp$. Let $X \in \mathcal{N}^\perp$. We show that the reflection $r_X : X \rightarrow RX$ of X in $\mathcal{B}$ is α - $\mathcal{B}$ -neat; consequently, as $\mathcal{B}$ is closed under α - $\mathcal{B}$ -subobjects, $X \in \mathcal{B}$. Let $f : Y \rightarrow Z$ be a $\mathcal{B}$ -epimorphism with Y and Z α -bounded. Given morphisms $s : Y \rightarrow X$ and $t : Z \rightarrow RX$ such that $t \cdot f = r_X \cdot s$, let $m_i : X_i \rightarrow X$ be an α -directed $\mathcal{M}$ -union in $\mathcal{A}$ with all X_i α -bounded. Then there is some $i \in I$ and $s' : Y \rightarrow X_i$ such that $m_i \cdot s' = s$. The closedness of $\mathcal{B}$ under α -directed $\mathcal{M}$ -unions and the fact that Z is α -bounded implies the existence of some $j \in I$ and a morphism $t' : Z \rightarrow RX_j$ such that $Rm_j \cdot t' = t$. Since I is α -directed, we can then find $k \in I$ and morphisms $\bar{s}$ and $\bar{t}$ such that the following diagram is commutative (the commutativity of the upper quadrilateral is derived from the fact that Rm_k is

monic):

$$\begin{array}{ccccc}
 Y & & \xrightarrow{f} & & Z \\
 & \searrow \bar{s} & & \swarrow \bar{t} & \\
 & X_k & \xrightarrow{r_{X_k}} & RX_k & \\
 s \downarrow & \swarrow m_k & & \searrow Rm_k & \downarrow t \\
 X & & \xrightarrow{r_X} & & RX
 \end{array}$$

Let $X_k \xrightarrow{f'} W \xleftarrow{s'} Z$ be the pushout of f along $\bar{s}$. Since $r_{X_k} \perp \mathcal{B}$, any morphism $g : X_k \rightarrow B$ with $B \in \mathcal{B}$ is factorizable through f' . Furthermore, as one easily sees, the pushout of a $\mathcal{B}$ -epimorphism is also a $\mathcal{B}$ -epimorphism. Hence $f' \perp \mathcal{B}$. The domain of f' is α -bounded, and from Lemma 2.7, also its codomain is α -bounded, then $f' \in \mathcal{N}$. Hence there is a morphism $n : W \rightarrow X$ such that $n \cdot f' = m_k$. Therefore, $n \cdot s'$ is the needed diagonal morphism, since $(n \cdot s') \cdot f = n \cdot f' \cdot \bar{s} = m_k \cdot \bar{s} = s$. $\square$

3.7. Examples (1) The category $\mathbf{Top}_0$ is an $\aleph_0$ -orthogonality class in $\mathbf{Top}$. In fact $\mathbf{Top}_0 = \{h\}^\perp$ where h is the map $h : \{0, 1\} \rightarrow \{0\}$, considering the two-elements set with the trivial topology.

(2) The category $\mathbf{Top}_1$ of T_1 topological spaces is an $\aleph_0$ -orthogonality class of $\mathbf{Top}$. It is just the subcategory of all objects orthogonal to the quotient $S \hookrightarrow \{0\}$, where S is the Sierpiński space. In this case, the reflector does not preserve embeddings.

(3) $\mathbf{Sob}$ is not an $\aleph_0$ -orthogonality class in $\mathbf{Top}_0$, and, consequently, it is not an $\aleph_0$ -orthogonality class in $\mathbf{Top}$. This follows from the above theorem taking into account that $\mathbf{Sob}$ is not closed under $\aleph_0$ - $\mathbf{Sob}$ -neat subobjects in $\mathbf{Top}_0$.

For that, we show that every $\mathbf{Sob}$ -epimorphism $e : X \rightarrow Y$ with X and Y finite is a surjection. (We recall that the $\mathbf{Sob}$ -epimorphisms of $\mathbf{Top}_0$ are the b -dense morphisms, see 3.1.) Let $y \in Y$, let $\{H_i, i \in I\}$ be the set of all open neighbourhoods of y , and put $H = \bigcap_{i \in I} H_i$. Since I is finite, H is an open containing y , and, then, $H \cap e(X) \cap \overline{\{y\}} \neq \emptyset$. Let y' be an element of that intersection. Thus $\overline{\{y'\}} \subseteq \overline{\{y\}}$. But for all H_i we have $y' \in H_i$, hence $\overline{\{y\}} = \overline{\{y'\}}$. Since $Y \in \mathbf{Top}_0$, we conclude that $y = y'$, then $y \in e(X)$.

As a consequence we have that

$$\{\text{embeddings}\} \subseteq \{\aleph_0\text{-}\mathbf{Sob}\text{-neat morphisms}\}.$$

But then, if $\mathbf{Sob}$ were closed under $\aleph_0$ - $\mathbf{Sob}$ -neat subobjects, it would also be closed under embeddings, what is obviously false (since the reflections are embeddings).

(4) The category **Norm** of normed (real or complex) vectorial spaces and linear contractions is a locally $\aleph_0$ -bounded category with respect to embeddings, and its $\aleph_0$ -bounded objects are the spaces with finite dimension. Analogously, all spaces with countable dimension are $\aleph_1$ -bounded. The subcategory **Ban** of all Banach spaces is an $\aleph_1$ -orthogonality class of **Norm**. In fact, it is easy to see that

$$\mathbf{Ban} = \mathcal{N}^\perp$$

where $\mathcal{N}$ is the class of all dense embeddings $X \hookrightarrow Y$ with X and Y with countable dimensions.

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